A UNIFIED THEORY

FOR

THE ANALYSIS OF SPATIAL MECHANISMS

BASED ON SPHERICAL TRIGONOMETRY

by

JOSEPH JOHN ROONEY, B.Sc.

A thesis presented at the Liverpool Polytechnic
in accordance with the regulations
for the degree of Doctor of Philosophy
of the Council for National Academic Awards.

LIVERPOOL POLYTECHNIC JANUARY, 1974.

BEST COPY

AVAILABLE

SUMMARY

One of the basic unsolved problems in the theory of spatial mechanisms is the algebraic displacement analysis of single-loop, one-degree-of-freedom linkages, of which the 7R (consisting of seven links connected by seven turning pairs, whose axes are arbitrarily orientated in space) is the most general. The required algebraic manipulation becomes complex for mechanisms having more than four links, and it is difficult to avoid obtaining extraneous roots in the derived displacement equations. The development of a unified theory for the analysis of spatial mechanisms, however, is a significant and worthwhile aim since an algebraic (as opposed to a numerical) approach can illuminate many basic aspects of linkage behaviour, such as, for example, the necessary proportions for overclosure, the criteria for rotability, the determination of type, the transmission characteristics, etc.

Thus, the major objective is the derivation of input-output displacement equations free of extraneous or unwanted roots and this becomes the central problem. Generally, for a particular mechanism, it is necessary first to derive loop equations and then to eliminate unknown angular displacements. Usually, performing more than one elimination procedure produces a final eliminant which contains the required input-output equation multiplied by an extraneous factor (which is, practically, impossible to find and extract).

The theory presented in Part I of this thesis has been the foundation for the development of unified procedures for the analyses of spatial five-link 3R-2C (Chapter 6) and six-link 4R-P-C (Chapters 7, 8 and 9) mechanisms (R, C and P denote respectively revolute, cylindric and prismatic pairs). The input-output equations for these linkages must be derived by eliminating a single angular displacement from two simultaneous equations in one operation. In addition to this, the theory is used to obtain sixteenth degree input-output equations for spatial six-link 5R-C mechanisms (Chapter 10), by eliminating

two unknown angular displacements, from four simultaneous equations, in a single operation. These results are particularly significant as they hold promise for the eventual solution of the general 7R linkage.

A contributing factor to the difficulties encountered in the derivation of input-output equations appears to be the presentation of the loop equations themselves. Little attention has been given to devising a general, efficient and compact notation. In the past, it has not been understood which loop equations can be considered to be basic or fundamental, how they are inter-related and how they may be classified.

In this dissertation, loop equations are derived for spherical polygons simply by adding a series of spherical triangles together in succession and using the existing trigonometrical laws for the triangle (Chapter 4). A natural unified notation is evolved, and it is established that, in analogy with the case of the triangle, there are only three fundamental loop equations for any spherical polygon, namely, the sine, sine-cosine and cosine laws. All other loop equations (which fall naturally into groups of subsidiary sine, subsidiary sine-cosine and subsidiary cosine laws) may be derived from these three basic laws. In addition, novel half-tangent laws (Chapter 5) are derived which are expressions for the common root of pairs of sine and sine-cosine laws and are linear in the half-tangent of at least one angular displacement.

These real spherical equations are extended to corresponding dual loop equations (applicable to spatial polygons), without further derivation, by means of the important Principle of Transference (Chapter 3), and in Part II the appropriate spatial polygons and dual equations are used to model and describe various five and six-link spatial linkages. The unified notation, the grouping of loop equations into sine, sine-cosine and cosine laws and especially the existence of the half-tangent laws, greatly facilitate the formation of proper sets of simultaneous algebraic equations for these linkages, from which input-output equations must then be derived in a single operation.

Finally, an intuitive geometrical procedure is described (Chapter 2) which enables one to attach a physical significance to the number of closures of a given spatial mechanism, and hence to predict the correct degree of its input-output equation in advance. The preliminary results obtained in this way, for the mechanisms dealt with here, are verified algebraically in Part II.

ACKNOWLEDGEMENTS

I wish to thank my academic supervisor, Dr. J. Duffy, for suggesting this work and for his invaluable guidance. I also express my gratitude to Mr. P. Potts for his counsel and to the staff of the computer department, at the Polytechnic, for the time and facilities they have allowed me for the running of lengthy computer programs.

The financial assistance of the Science Research Council (Grant No. BS/SR/8420), and the Liverpool Education Authority is gratefully acknowledged.

In addition, I wish to express my special thanks to Mrs. Anne Delves for her patience and perseverance in the typing of such a difficult manuscript, and to Mr. Robert Delves for his help in the preparation of the diagrams.

Finally, I wish to emphasise how much more difficult all this would have been but for the encouragement given me by my wife, Cath.

CONTENTS

Page No.

PART I

Page No.

CHAPTER 3

CHAPTER 4

Page No.

CHAPTER 5

PART II

CHAPTER 6

CHAPTER 7

CHAPTER 8

CHAPTER 9

Page No.

CHAPTER 10

CHAPTER 11

APPENDICES

APPENDIX I

Page No.

APPENDIX II

APPENDIX III

APPENDIX IV

APPENDIX V

APPENDIX VI

Page No.

LIST OF TABLES AND FIGURES

Page No.

TABLES

FIGURES

Page No.

Page No.

Page No.

PART I

DEVELOPMENT
OF THE
BASIC THEORY

CHAPTER 1

DEFINITIONS

AND

NOMENCLATURE

1.1 Introduction.

For many years now a number of researchers have applied themselves to the problems encountered in the field of mechanisms. This is especially true in the case of planar mechanisms, where great inroads have been made in both the analysis and synthesis of such devices. However, the same cannot be said for spatial mechanisms, especially in the area of analysis where only recently have any significant results been obtained.

The aim of this dissertation is to present a unified theory for the displacement analysis of spatial mechanisms and, to this end, it would be advantageous to define certain basic concepts.

1.2 Mechanisms.

For the purposes of kinematics a mechanism may be defined as an assemblage of rigid bodies mutually connected by non-rigid connections, in such a way as to transform one motion into another. Of all such possible devices there exists a class which deserves special attention, as its members are able to convert uniform motion into non-uniform motion. In particular, the set of linkages belongs to this class.

1.3 Linkages.

A linkage is a collection of rigid members (also known as links) connected together by a number of lower-pair connectors that permit a restricted relative motion between adjacent links, and in such a way that all pairs are complete, thereby ensuring that the kinematic chain be closed. Henceforth in this work the words 'linkage' and 'mechanism' will be considered as being synonymous and as having the meaning of 'linkage'.

Linkages may be classified into three basic types:- planar, spherical and spatial linkages. They are categorized thus according as their constituent parts (links) move in parallel planes, on the surface of concentric spheres, or exhibit a more general spatial motion.

In order to identify and distinguish linkages in each class, it is necessary to understand the concepts of connectivity and mobility.

1.4 Connectivity.

When two rigid bodies are connected by a pair connector or joint, the resultant number of degrees of freedom which one of the bodies has relative to the other is defined as the connectivity of the joint (see Waldron [39]). If this connectivity is denoted by the integer, f, then it is clear that, for spatial motion, $0 \leqslant f \leqslant 6$. When $f = 0$, the pair forms a rigid coupling between the bodies and no relative motion can take place, whilst if $f = 6$ unconstrained relative spatial motion exists. For the pairs considered here the strict inequalities hold and $0 < f < 6$.

1.5 Mobility.

The mobility, or overall number of degrees of freedom, of a linkage is defined as the number of independent variables required to specify completely the position of each link relative to a fixed reference link called the frame. It may be denoted by the integer, F, so that for linkages with one degree of freedom (referred to as constrained mechanisms), $F = 1$.

1.6 Mobility Criteria.

There exist various criteria by which the mobility of a linkage or mechanism may be determined. In general terms, the mobility depends on several factors which include the connectivities of the joints in the linkage, the number of joints and constraints and the number of links involved.

Now for an assemblage of n unconnected links there are $k.n$ degrees of freedom, where k (assumed to be the same for each link) is the number of degrees of freedom of a single free link. However, in the case of a mechanism, one link is held fixed, as the frame, thereby reducing the number of degrees of freedom to $k.(n - 1)$. Furthermore, since each joint removes $(k - f)$ degrees of freedom, it is possible to express the mobility of a mechanism by the following formula:-

$$F = k(n - 1) - \sum_{i=1}^{i=g} (k - f_i) \qquad (1.1)$$

where: F = mobility of the mechanism.
n = number of links.
g = number of joints.
f_i = connectivity of the i^{th} joint.
k = number of degrees of freedom of a single free link.

For planar linkages each link is restricted to move in a plane and hence two position co-ordinates and one angular co-ordinate are required to define the position and orientation of a free link. There are thus three degrees of freedom for a single free link and hence k = 3. Equation (1.1) then reduces to the Grübler criterion [18]:-

$$F = 3(n - 1) - \sum_{i=1}^{i=g} (3 - f_i) \tag{1.2}$$

In a similar manner one obtains the Kutzbach criterion [27], for spatial mechanisms (where k = 6):-

$$F = 6(n - 1) - \sum_{i=1}^{i=g} (6 - f_i) \tag{1.3}$$

For spherical linkages it is clear that k = 3 and equation (1.2) applies.

1.7 Single-Loop Linkages.

A single-loop mechanism is a mechanism, each link of which is connected to two other links (i.e. it is a binary link), and each joint of which connects two links (i.e. it is a second degree joint). For single-loop planar or spherical mechanisms of mobility one, it is apparent that the number of links and joints must be equal (i.e. n = g) and this implies from equation (1.2) that:-

$$\sum_{i=1}^{i=n} f_i = F + 3 = 4 \tag{1.4}$$

Now from the definition of f_i it is clear that, mathematically, $f_i \geq 1$ and hence equation (1.4) implies that $n \leq 4$. Physically, however $f_i = 1$ for planar and spherical pairs and so n = 4.

We are thus restricted to a few four-link mechanisms which satisfy equation (1.4).

For single-loop spatial mechanisms with mobility one, however, equation (1.3) becomes:-

$$\sum_{i=1}^{n} f_i = F + 6 = 7 \qquad (1.5)$$

For $f_i = 1$ this implies that the maximum permissible number of links, n, must be seven. However, for spatial mechanisms, it is possible to have joints for which $f_i > 1$ and consequently a large variety of single loop mechanisms, of mobility one, exists, in contrast with the planar and spherical cases. This variety stems from the existence of six types of lower pair as defined by Reuleaux [30], a great many combinations of which will yield mobility one spatial mechanisms. These six pairs are listed in Table I. with their symbols and connectivities.

Throughout the remainder of this dissertation these symbols will be used without further reference.

1.8 Inversion.

A particular linkage will be derived from a closed kinematic chain by fixing the reference co-ordinate system to a chosen link and hence treating all motions as being absolute, relative to this 'stationary' link, which is designated as the frame. Now, for a given arrangement and combination of joints in a kinematic chain, it is possible, in general, to obtain a number of distinct linkages by selecting a succession of different links as the frame.

This process of fixing different links of a kinematic chain to create different mechanisms is referred to as kinematic inversion and the distinct mechanisms obtained are termed inversions of one another.

1.9 Notation.

Of the many notations which have been employed for the description and specification of linkages, that of Yang [44] will be employed throughout this work, being the most appropriate for present purposes. This dissertation will deal almost exclusively with single loop spatial mechanisms of unit mobility which may be treated, therefore, as continuously deformable spatial polygons.

The latter are defined by a number of pair axes (these, in general, will be non-parallel and non-intersecting), together with a set of common perpendiculars between adjacent axes.

Figure 1.1 illustrates a six-sided spatial polygon, where the six axes are labelled $\hat{\underline{S}}_1,\ldots,\hat{\underline{S}}_6$, the common perpendicular distance and projected angle between two adjacent axes, $\hat{\underline{S}}_i$ and $\hat{\underline{S}}_j$, are denoted by a_{ij} and α_{ij}, and the offset distance and projected angle between two adjacent common perpendiculars, $\hat{\underline{a}}_{ij}$ and $\hat{\underline{a}}_{jk}$, are denoted by S_j and θ_j respectively.

Following Yang [44], when representing a spatial mechanism by a spatial polygon, it is necessary to choose certain parameters (the α_{ij}'s and a_{ij}'s) as being constant together with some of the S_j's and/or θ_j's in order to accurately model the required joints. When this is done, it will prove convenient to label those S_j's and θ_j's that are constant with double subscripts (i.e. S_{jj} or θ_{jj}). Thus all constant parameters will be represented by a symbol with two suffices, whilst variables will have a single suffix.

The above system of notation is quite adequate for the purposes of this work since only three of the possible six lower pairs will appear in various combinations in the spatial mechanisms analysed. These are the revolute (R), prismatic (P) and cylindric (C) pairs.

Henceforth, it will prove convenient to represent a linkage by a sequence of such symbols denoting the order of the constituent joints in the kinematic chain, starting with that pair between the frame and input link and finishing with that between the output link and the frame. Thus the spatial four-link mechanism having one revolute and three cylindric pairs would be represented as RCCC, where the input angular displacement takes place about the revolute pair axis. Sometimes this notation is shortened even further to RC^3 or R-3C, but in certain circumstances this may lead to confusion over which inversion is being considered, particularly in the case of spatial mechanisms with more than four links.

It is clear from equation (1.5) that any such combination of joints is possible provided that the connectivities of the pairs total seven. Mechanisms which satisfy this condition and which contain only R, P and C joints, include the five-link RCRCR and RRCCR, the six-link RCRPRR and RRRRCR, and the seven-link RPRRPRR and RRRRRRR linkages.

Finally it must be noted that two similar groupings of such symbols may either represent two inversions of a particular mechanism or else two completely different mechanisms. Thus the RCRRC is an inversion of RCRCR, whereas neither of these is an inversion of the RRCCR, although all three mechanisms contain three revolute and two cylindric pairs and are sometimes classified collectively as 3R-2C mechanisms. Consequently, it can be seen that the order of the symbols is normally very important in the description of a mechanism.

1.10 Displacement Relationships.

For a single loop spatial mechanism of mobility one, there are, in general, seven variables remaining, once the fixed parameters have been selected. Of these, only one can be independent, since the linkage has but a single degree of freedom, and hence algebraic relationships must exist between each of the other six variables and the chosen independent one. Thus, once the latter has been specified, there can only be a finite number of values for each of the other six.

Such relationships are referred to as displacement relationships or equations and it is clear that there must exist six equations of this type for each variable chosen as input. For the purposes of a displacement analysis of a particular linkage, two adjacent pairs are designated as input and output, respectively, and a variable associated with the input pair (usually chosen as a revolute) is selected as the independent variable. The link between the input and output pair axes is then the fixed link or frame.

The relationship that exists between the input and output variables is referred to as the input-output relationship or equation. It is possible to express this equation as a polynomial in the output variable, whose coefficients depends on all of the constant parameters and the input variable. The degree of this polynomial plays an important role in the analysis since it represents the number of closures that a mechanism has. In other words, if the input variable is specified the degree of the input-output polynomial must be equal to the maximum possible number of real values that the output variable can assume. This topic will be pursued further in Chapters 2 and 5.

Now, although it is possible to reach certain general conclusions about the form and complexity of input-output relationships, they are extremely difficult to obtain for the great majority of spatial mechanisms, and no general method has yet been presented, to the author's knowledge, that can be applied successfully to the solving of this problem.

The main difficulty that one is presented with is that of the elimination of all but the input and output variables from the loop equations selected, without obtaining a final eliminant (i.e. input-output equation) of too high a degree.

1.11 Elimination.

In the analysis of spatial mechanisms one is often able to derive numerous loop equations relating two or more linkage variables. The major problem is then to derive the unique input-output relationship from these sets of loop equations, and this must relate only the input and output variables. Furthermore, this eliminant must not contain any extraneous or unwanted roots, for it is a relatively simple task to obtain a relationship of too high a degree, but difficult to obtain the correct eliminant, in general.

A relationship involving only the input and output variables can be derived in many ways from combinations of the loop equations, but usually such a relationship will not be the desired eliminant and will possess extraneous solutions.

In the past several researchers have evolved various diverse methods for obtaining the correct eliminant in certain special cases but most of these methods lack any sort of generality and only work well for particular spatial mechanisms. It is the intention of the present author in this investigation to develop a general unified theory for the analysis of spatial mechanisms which may be applied to the problem of determining the input-output relationship of all single-loop spatial mechanisms of mobility one. The problem of elimination will be examined in greater detail in Chapter 5.

However, it was considered desirable to devise some means of predicting the degree of the input-output equations initially for the various spatial five, six and seven-link mechanisms. In Chapter 2 a geometric method is developed for this purpose.

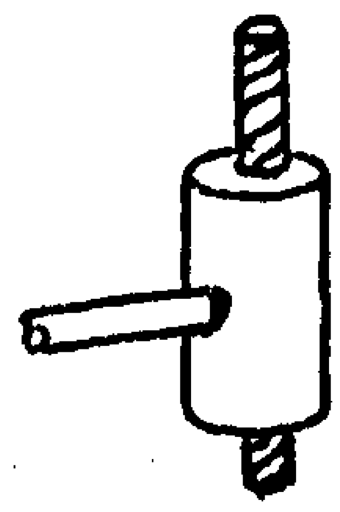
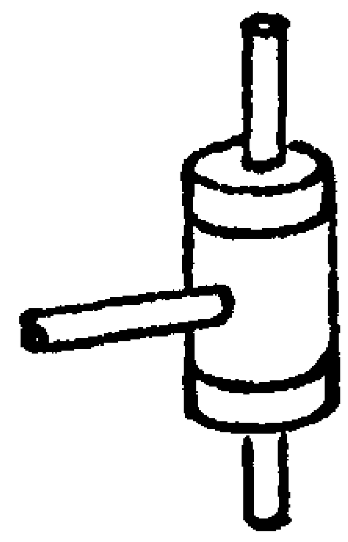
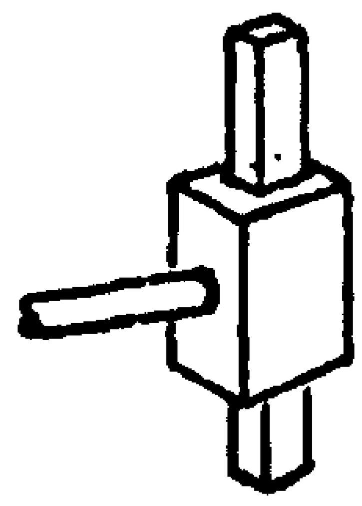
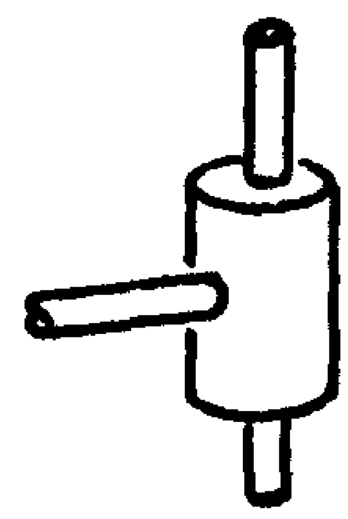
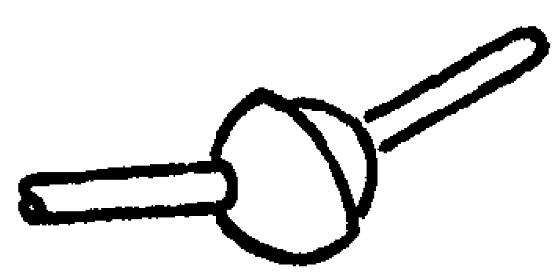
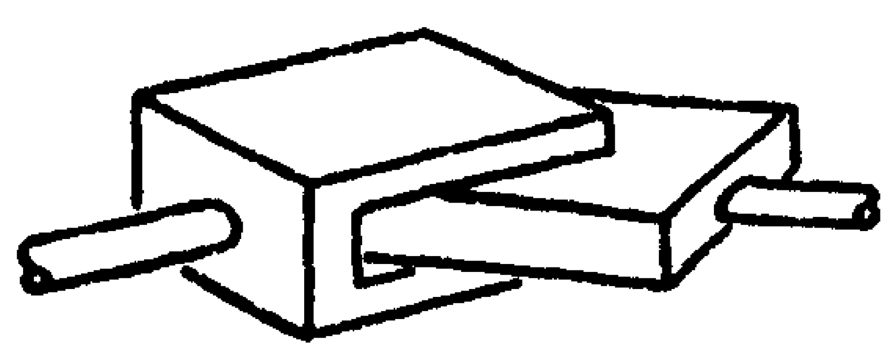
KINEMATIC PAIR	SYMBOL	CONNECTIVITY	REPRESENTATION
SCREW	H	1	
REVOLUTE	R	1	
PRISMATIC	P	1	
CYLINDRIC	C	2	
SPHERICAL	S	3	
PLANAR	E	3	

Table I. The six lower pairs as defined by Reuleaux.

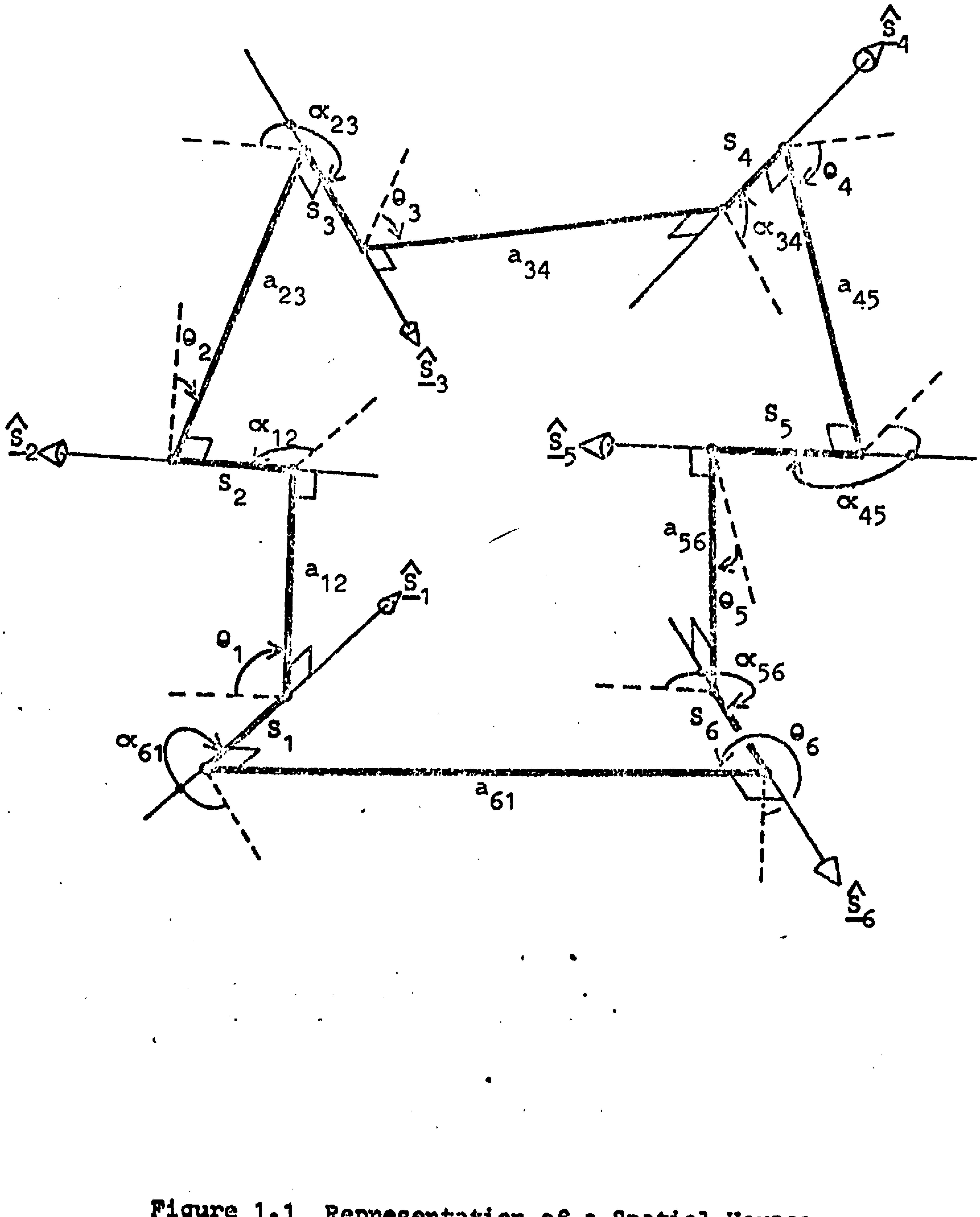


Figure 1.1 Representation of a Spatial Hexagon.

CHAPTER 2

PRELIMINARY GEOMETRICAL CONSIDERATIONS

2.1 Introduction.

In recent years many researchers have attempted to solve some of the difficult problems associated with the derivation of closed-form displacement equations for spatial mechanisms with more than four links. The results appearing in the literature are discussed in detail later in this Chapter. The central problem has been one of algebraic elimination and the avoidance of the introduction of extraneous or unwanted roots into the analysis. This problem is enhanced by the fact that at the outset the correct degree of the input-output equation for any particular mechanism is not known. Hence, at present, it is necessary to devise and perform an elimination procedure in order to derive an input-output equation, without prior knowledge of its correct degree.

This requires a considerable amount of algebraic manipulation and, in addition, the final equation must be tested for extraneous roots by numerical computation. The latter procedure is in itself a problem since it is difficult to select a set of mechanism proportions for which a range of values of the input variable will yield all real solutions to the input-output equation.

Thus, for example, the correct sixteenth degree polynomials for the RRRRCR and RRCRRR mechanisms are derived in Chapter 10, whereas in [16] a degree 128 polynomial is presented for the latter mechanism, without any verification of this result. Clearly, this polynomial is of little use since it contains 112 extraneous roots which have no direct physical significance. It is necessary, therefore, to determine which real roots of a particular input-output equation lead to closure of the mechanism, in addition to performing the excessive labour involved in the algebra, programming and computer running time.

The aim of this chapter is to present an intuitive geometrical approach to the problem of predicting the correct degree of input-output equations.

2.2 Basic Considerations.

The mobility criterion for single loop spatial mechanisms has been discussed in Chapter 1. (see equation (1.5)). Here, spatial structures are examined, for which:-

$$\sum_{i=1}^{i=n} f_i = 6 \tag{2.1}$$

Thus in the present context a spatial structure is an immovable or rigid assemblage of links and kinematic pairs, forming a single closed loop in three-dimensional space, for which the total connectivity is six. Although a structure is not continuously movable, in general, as is a mechanism, it can be assembled in a finite number of distinct configurations without altering the constant parameters defining the relative orientations of its adjacent skew axes. Each distinct configuration will be termed an assembly configuration or assembly, and a structure must be disconnected at a pair axis and reassembled in a different orientation in order to change from one such assembly configuration to another. In general, the angular displacements at a revolute or cylindric pair and the sliding displacements at a cylindric or prismatic pair, assume distinct values for different assemblies.

Now clearly, by definition, a mechanism is a continuously deformable polygon and hence the term "assembly configuration" is not directly applicable. Nevertheless, there is an analogous concept which is applicable to mechanisms and this is the idea of closure.

Thus the number of closures of a particular mechanism is defined as the maximum possible number of real values that the output variable can assume for a given value of the input variable, and it will now be demonstrated that the closures of a mechanism can be directly related to the assembly configurations of a corresponding structure.

Consider the planar four-link RRRR mechanism shown in Figure 2.1(a). For any arbitrary specified value of the input angular displacement, θ_1, the "input frame triangle", 124, is completely defined and for convenience may be replaced by a single rigid link, a_{24}, in the plane. The original four-link mechanism has now been effectively reduced to a three-link RRR planar structure which can be assembled in two distinct positions (234 and 23'4). These two assembly configurations for the RRR structure give the two distinct values, θ_4 and θ'_4, for the output angular displacement. Consequently, the number of closures of the planar four-link RRRR mechanism is in direct one-one correspondence with the number of assembly configurations of the related three-link RRR planar structure.

A similar procedure can be applied to single loop spatial mechanisms with revolute input joints, since the "input frame spatial triangle" is again uniquely defined in space for any specified value of the input angle, in this case. The purpose of this chapter is to verify and utilize the following hypothesis:-

The maximum number of <u>closures</u> of a single loop n-link mechanism with unit mobility, and with a revolute pair connecting the input link to the frame, is equal to the maximum number of <u>assembly configurations</u> of a corresponding (n - 1)-link structure obtained by holding the input angular displacement constant, and replacing the input link and frame (referred to as the input frame triangle) with a single link.

Thus, consider the spatial four-link RCCC mechanism illustrated by Figure 2.2, which shows the constant mechanism proportions and the seven mechanism variables in accordance with the notation introduced in Chapter 1. The input frame spatial triangle defined by the three skew pair axes, $\hat{\underline{S}}_1$, $\hat{\underline{S}}_2$ and $\hat{\underline{S}}_4$ is uniquely defined in space for any specified value of the input angle, θ_1 (see also Yang [44]). For convenience this spatial triangle can be replaced by a single rigid link, of length, a_{42}, (the common perpendicular distance between $\hat{\underline{S}}_4$ and $\hat{\underline{S}}_2$) and relative twist angle α_{42}.

The number of closures of the four-link RCCC mechanism is now identical to the number of assembly configurations of the resulting spatial three-link CCC structure (defined by $\hat{\underline{S}}_2$, $\hat{\underline{S}}_3$ and $\hat{\underline{S}}_4$).

This situation is analogous to the planar case described above, and the procedure is applicable to any single loop, mobility-one spatial mechanism, provided that the input joint is a revolute pair.

The determination of the maximum number of closures (and hence the degree of the input-output equation) for a single loop spatial mechanism has now been reduced to the somewhat easier problem of determining the number of assembly configurations for a structure with fewer links.

2.3 Procedure for Determining Assembly Configurations.

The number of assembly configurations for the structures (with $\sum_{i=1}^{i=n} f_i = 6$) listed in column 1 of Table II. will now be determined, since they are related to the corresponding mechanisms listed in column 3. For each mechanism considered the input pair is a revolute joint and hence the hypothesis stated in the previous section is valid.

However, before proceeding with the CCC structure, it is useful to consider briefly how the two assembly configurations mentioned above for the planar RRR triangle (i.e. 234 and 23'4, see Figure 2.1) are obtained. The approach is illustrated by Figure 2.1(b). (the triangle 234, from Figure 2.1(a), is considered here in isolation). Disconnecting the revolute joint at vertex 3, reduces the structure to two cranks, a_{23} and a_{34}, which clearly generate circular arcs. In general, the latter intersect in two points, 3 and 3', and it is only at these points that reassembly of the RRR structure may be achieved. Thus the three-link planar structure has two assembly configurations.

The above simple example illustrates and suggests the following general procedure for determining the assembly configurations of any structure:-

(i) Disconnect the structure at a suitable pair axis.
(ii) Examine the motion of the two open kinematic chains thus produced.
(iii) Reconnect the pair in as many positions as possible (the number of such possible positions will be dictated by the nature of the motions produced at (ii)).

Clearly the motions of open spatial chains, for various combinations of links and pairs, is more complex than the planar case dealt with above and, in addition, the method of reassembly is not as straightforward since one must reconnect at a pair axis rather than at a point. Thus in order to fully appreciate the difficulties involved, it is advantageous to examine the spatial CCC structure and obtain its assembly configurations. It is then the author's intention to determine the motions of those spatial chains relevant to the structures listed in Column 1. of Table II.

2.4 The Spatial CCC Three-Link Structure.

In Figure 2.3, which is a representation of the spatial CCC three-link structure (i.e. spatial triangle), the directions and positions in space, of the three skew axes of the spatial triangle are denoted by the unit line vectors $\hat{\underline{S}}_1$, $\hat{\underline{S}}_2$ and $\hat{\underline{S}}_3$, which represent the pair axes of three cylindric pairs. The common perpendicular distances between the adjacent pair axes are a_{12}, a_{23} and a_{31} (measured along the unit line vectors $\hat{\underline{a}}_{12}$, $\hat{\underline{a}}_{23}$ and $\hat{\underline{a}}_{31}$, between $\hat{\underline{S}}_1$ and $\hat{\underline{S}}_2$, $\hat{\underline{S}}_2$ and $\hat{\underline{S}}_3$, and $\hat{\underline{S}}_3$ and $\hat{\underline{S}}_1$ respectively). These are taken as the link lengths. The length of the offset distance between the adjacent links a_{ij} and a_{jk} (taken as positive in the direction of $\hat{\underline{S}}_j$) is denoted by S_j. The orientation of say $\hat{\underline{S}}_2$ relative to $\hat{\underline{S}}_1$ is given by the twist angle or angle of rotation, α_{12}, required by a right-handed screw (advancing along the common perpendicular $\hat{\underline{a}}_{12}$ from $\hat{\underline{S}}_1$ to $\hat{\underline{S}}_2$) to align $\hat{\underline{S}}_1$ with $\hat{\underline{S}}_2$. In a similar manner the orientation of say $\hat{\underline{a}}_{12}$ relative to $\hat{\underline{a}}_{23}$ is defined, and denoted by θ_2.

The maximum number of assembly configurations for this structure is determined by disconnecting the second cylindric pair $\hat{\underline{S}}_2$ and labelling the pair elements attached to $\hat{\underline{a}}_{12}$ and $\hat{\underline{a}}_{23}$ as $\hat{\underline{S}}_2$ and $\hat{\underline{S}}'_2$ respectively. It is clear that the link a_{12} (a_{23}) can now both slide along and rotate about $\hat{\underline{S}}_1$ ($\hat{\underline{S}}_3$) and therefore generates a right circular cylinder. This cylinder has a tangent unit line vector defined at every point of its surface by $\hat{\underline{S}}_2$ ($\hat{\underline{S}}'_2$), and Figure 2.4 illustrates the two cylinders, and their attendant line vector fields, thus produced.

One must now reassemble the two open spatial chains in as many ways as possible. Unlike the planar case, however, it is not sufficient to consider just the coincidence of the end points of the two free chains as determining an assembly. Here an assembly configuration must be defined by the coincidence of the pair elements $\hat{\underline{S}}_2$ and $\hat{\underline{S}}'_2$. In other words the two open spatial chains may only be reconnected at positions where $\hat{\underline{S}}_2$ and $\hat{\underline{S}}'_2$ are equal. (Unit line vectors are <u>equal</u> when they are collinear and have the same sense. They are <u>opposite</u> when they are collinear and have opposite senses).

From Figure 2.5 it can be seen that, in general, for the CCC structure the line vectors $\hat{\underline{S}}_2$ and $\hat{\underline{S}}'_2$ are equal in two positions (axes 2 and 4) and opposite in two positions (axes 1 and 3). (Note:- The proportions $\alpha_{12} = \alpha_{23}$ = 90 deg. have been selected in Figure 2.5 and the link a_{31} has been omitted for greater clarity, without loss in generality). The magnitudes of the offsets along the assembly axes 2 and 4 need not be specified in order to determine a configuration since the structure has been disconnected at a cylindric pair. In particular these offsets clearly need not be the same for each assembly configuration. However, if a spatial structure were disconnected at a revolute pair then the offsets at each assembly axis must all have the same specified value.

It is interesting to note that for each assembly axis along which $\hat{\underline{S}}_2$ and $\hat{\underline{S}}'_2$ are equal, there is a corresponding axis along which these line vectors are opposite and this phenomenon occurred for each structure considered here. However, in subsequent illustrations those axes along which the pair elements are opposite will be omitted since they would tend to obscure the acceptable assembly axes.

Finally, one may list the following two observations which are relevant in determining the number of assembly configurations for structures that can be disconnected at a cylindric pair:-

(i) In general an open spatial chain generates a surface (this surface may, for certain cases, degenerate into a space curve or alternatively the chain may describe a volume). However, the important factor is that a unit line vector is defined at each point of the surface by one element of the disconnected pair, and it is this family of directed lines that is of greatest importance when considering the reassembly of two chains.

(ii) The acceptable assembly configurations are defined by those lines <u>common</u> to the two open chains, but along which the line vectors, representing the two pair elements (for example $\hat{\underline{S}}_2$ and $\hat{\underline{S}}'_2$ in Figure 2.5), are <u>equal</u>.

With these considerations in mind, it is now possible to investigate in detail the line ensembles generated by various relevant spatial chains.

2.5 Open Spatial Chains.

Six spatial chains are of interest here, and they may be listed as follows:- Rc, Cc, RRc, CRc, PRc and RPc. Here, the cylindric pair at the end of the chain is denoted by a lower case letter to suggest that it is only one element of a disconnected pair from a spatial structure. Chains such as, RRr, which terminate in a revolute pair element, will be discussed later since they pose difficult problems of reassembly.

2.5.1 The Rc Chain.

The Rc spatial chain is illustrated by Figure 2.6(a). The single link, a_{12}, is free to rotate about the line vector $\hat{\underline{S}}_1$ representing a revolute joint and hence its free end will describe a circle in a plane perpendicular to $\hat{\underline{S}}_1$. Attached to the free end of a_{12} is the line vector $\hat{\underline{S}}_2$ representing an element of a disconnected cylindric pair, and inclined at a fixed angle, α_{12}, relative to $\hat{\underline{S}}_1$. Consequently, there is a line-vector field defined on the circumference of the circle, and this system of lines generates a ruled surface (the hyperboloid of one sheet) in this case.

Now if α_{12} were chosen as 0 or π radians the hyperboloid would degenerate to the special case of a cylinder, as shown in Figure 2.6(b). Similarly with the choice $\alpha_{12} = \pi/2$ or $3\pi/2$ the situation illustrated by Figure 2.6(c) would exist where the line vectors all lie in the plane of the circle.

2.5.2 The Cc Chain.

The Cc spatial chain is illustrated by Figure 2.7., where the link a_{12} is now free to both slide along and rotate about the line vector $\hat{\underline{S}}_1$. Hence the free end of a_{12} will generate a right circular cylinder, attached to every point of which will be a single tangent line vector $\hat{\underline{S}}_2$ inclined at a fixed twist angle, α_{12}, relative to $\hat{\underline{S}}_1$. In other words there exists a helical tangent vector field of line vectors enveloping a cylindrical surface with axis $\hat{\underline{S}}_1$.

If α_{12} is chosen as $\pi/2$ or $3\pi/2$ the tangent line vectors lie in planes perpendicular to the central axis $\hat{\underline{S}}_1$ as shown in Figure 2.5.

2.5.3 The RRc Chain.

The RRc spatial chain is (unlike the previous two considered) a spatial dyad consisting of two spatial links in series. Figure 2.8 is a representation of the chain. The link a_{12} can rotate about the line vector $\hat{\underline{S}}_1$ and hence the point, A_1, will describe a circle as shown. The point, A_2, attached to the line vector $\hat{\underline{S}}_2$ will also describe a circle in a plane perpendicular to $\hat{\underline{S}}_1$, but of larger radius than a_{12}. Now the link a_{23} is free to rotate about $\hat{\underline{S}}_2$ in a circular path and hence the point, A_3, at the free end of a_{23} will generate the surface of a skew torus in general, with central axis $\hat{\underline{S}}_1$, (possibly intersecting itself). Attached to every point of this surface will be a line vector $\hat{\underline{S}}_3$ as shown. (The line vector does not lie in a tangent plane in general).

If the parameters are chosen as, $\alpha_{12} = \alpha_{23} = \pi/2$ or $3\pi/2$, $a_{23} < a_{12}$ and $S_{22} = 0$, the torus becomes right circular of circular cross-section and the vector field on its surface, defined by $\hat{\underline{S}}_3$, becomes tangential. Each line vector, $\hat{\underline{S}}_3$, then also lies in the same plane as $\hat{\underline{S}}_1$. This is illustrated by Figure 2.9.

2.5.4 The CRc Chain.

This dyad is illustrated by Figure 2.10(a). The situation is identical with that for the previous RRc chain except that the link, a_{12}, has the additional freedom of being able to slide along $\hat{\underline{S}}_1$ as well as rotating about it. Hence the free end of a_{23} generates a volume which is that of a hollow right circular cylinder (central axis $\hat{\underline{S}}_1$) with finite wall thickness, as shown. A line vector, $\hat{\underline{S}}_3$, is defined at every point throughout this volume.

If $\alpha_{23} = \pi/2$ or $3\pi/2$ and $S_{22} = 0$, the volume may be thought of as being swept out by a torus of elliptical cross-section, sliding along the central axis $\hat{\underline{S}}_1$. This is illustrated by Figure 2.10(b).

2.5.5 The PRc Chain.

The PRc spatial dyad is illustrated by Figure 2.11. Here the link a_{12} is free to slide along the line vector $\hat{\underline{S}}_1$ whilst the link a_{23} is able to rotate about $\hat{\underline{S}}_2$. The free end of the chain therefore generates a cylinder with elliptical cross-section in general. Again, there is a line vector $\hat{\underline{S}}_3$ defined at each point of this surface.

If the parameters are selected as $\alpha_{12} = 0$ or π, and $\alpha_{23} = \pi/2$ or $3\pi/2$ then the cylinder becomes right circular and $\hat{\underline{S}}_3$ will lie in a plane perpendicular to $\hat{\underline{S}}_2$ (and $\hat{\underline{S}}_1$) at all points on the surface. This is illustrated by Figure 2.5.

2.5.6 The RPc Chain.

Figure 2.12 is a representation of the spatial RPc dyad. The link a_{12} is free to rotate about $\hat{\underline{S}}_1$, and the link a_{23} is free to slide along $\hat{\underline{S}}_2$. Thus the free end of the chain generates a hyperboloid of one sheet as shown, and $\hat{\underline{S}}_3$ defines a line vector at every point of the surface. The line vectors, $\hat{\underline{S}}_3$, at the free end of the chain do not generate the hyperboloid, as is the case for the Rc chain discussed above, but the surface does have a line vector associated with each of its points.

If one chooses the parameters to be $\alpha_{12} = 0$ or π and $\alpha_{23} = \pi/2$ or $3\pi/2$ then the hyperboloid degenerates into a right circular cylinder with central axis $\hat{\underline{S}}_1$ and with $\hat{\underline{S}}_3$ lying in a plane perpendicular to $\hat{\underline{S}}_1$, at each point on its surface. This is again illustrated by Figure 2.5.

The systems of lines generated by the above six open spatial chains reduce to the envelopes of either a cylinder, a torus or a plane circle for a suitable choice of parameters, without any loss of generality (in the sense that there is no reduction in the number of common lines between various pairs of chains). These simpler models are used for greater ease of visualisation in the following sections.

2.6 Spatial Structures and their Assembly Configurations.

The six structures that are of interest may be listed:- CCC, RCRC, RRCC, RRCRR, RRCRP and RPCRR.

2.6.1 The CCC Three-Link Spatial Structure.

It has already been demonstrated above that the number of assembly configurations for the CCC structure is two. This is because the two Cc chains, produced after disconnecting a cylindric pair, generate right circular cylinders (Figure 2.4) and these have two line vectors in common. Figure 2.5 illustrates the assemblies and has been drawn for the choice of proportions $\alpha_{12} = \alpha_{23} = \pi/2$.

2.6.2 The RCRC Four-Link Spatial Structure.

The number of assembly configurations for the RCRC structure, which is illustrated by Figure 2.13 may be determined for a given set of parameters by disconnecting the cylindric pair $\hat{\underline{S}}_2$, thereby producing the two free chains Rc and CRc. Selecting the proportions $\alpha_{12} = \alpha_{23} = \pi/2$, $\alpha_{34} = \alpha_{41}$, $a_{12} = a_{23}$ and $S_{33} = 0$, the chains generate respectively a circle and a torus which is free to slide along the cylindric pair axis $\hat{\underline{S}}_4$. Generally, there exist four common line vectors (i.e. $\hat{\underline{S}}_2$ and $\hat{\underline{S}}'_2$ become equal in four positions) to the two systems, and this is shown in Figure 2.14, where the torus adopts three positions (labelled A, B, C) along $\hat{\underline{S}}_4$. The broken lines each represent two assemblies, as shown.

Thus the RCRC spatial structure has a maximum of four assembly configurations.

2.6.3 The RRCC Four-Link Spatial Structure.

The number of assemblies of the RRCC structure shown in Figure 2.15 can be determined by disconnecting the structure at the cylindric pair $\hat{\underline{S}}_3$. This reduces the structure to the two free chains RRc and Cc, which, for the choice of parameters $\alpha_{12} = \alpha_{23} = \alpha_{34} = \pi/2$, $a_{23} < a_{12}$ and $S_{22} = 0$, generate respectively a right circular torus with circular cross-section and a right circular cylinder as shown in Figure 2.16.

Generally, the line vectors $\hat{\underline{S}}_3$ and $\hat{\underline{S}}'_3$ are equal in eight positions (shown in Figure 2.17) and opposite in eight positions (not shown but symmetrically positioned). Therefore the RRCC structure has a maximum of eight assembly configurations.

2.6.4 The RRCRR Five-Link Spatial Structure.

To determine the number of assemblies for the RRCRR structure, illustrated by Figure 2.18, the cylindric pair, $\hat{\underline{S}}_3$, is disconnected, producing two RRc open chains, which generate skew torii, in general, with central axes $\hat{\underline{S}}_1$ and $\hat{\underline{S}}_5$.

For ease of visualisation it is desirable to select the proportions,

$\alpha_{12} = \alpha_{23} = \alpha_{34} = \alpha_{45} = \alpha_{51} = \pi/2$, $a_{23} = a_{34} < a_{12} = a_{45}$, and $S_{22} = S_{44} = a_{51} = 0$. For these proportions the two chains generate identical right circular torii whose central axes ($\underline{\hat{S}}_1$ and $\underline{\hat{S}}_5$) intersect at right angles, as shown by Figure 2.19.

Generally, the line vectors $\underline{\hat{S}}_3$ and $\underline{\hat{S}}'_3$ are equal in sixteen positions (shown by broken lines in Figure 2.20) and opposite in sixteen positions (not shown). The two broken lines numbered 9-12 and 13-16 in Figure 2.20 each represent four possible assemblies since $\underline{\hat{S}}_3$ and $\underline{\hat{S}}'_3$ may be either both at the same point or at opposite points of the torii (giving four combinations).

Hence the RRCRR structure has a maximum of sixteen assembly configurations.

2.6.5 <u>The RRCRP Five-Link Spatial Structure.</u>

This structure is illustrated by Figure 2.21. By disconnecting at the cylindric pair $\underline{\hat{S}}_3$ the two open chains RRc and PRc are produced. Upon selecting the proportions $\alpha_{12} = \alpha_{23} = \alpha_{34} = \alpha_{51} = \pi/2$, $a_{23} < a_{12}$, $\alpha_{45} = 0$ and $S_{22} = 0$, the chains generate respectively a right circular torus and a right circular cylinder (Figure 2.16) and hence the RRCRP structure has eight assembly configurations in general. (Figure 2.17 illustrates these for a cylinder and torus).

2.6.6 <u>The RPCRR Five-Link Spatial Structure.</u>

This structure is illustrated by Figure 2.22. Again disconnecting at the cylindric pair $\underline{\hat{S}}_3$, produces the two open chains RPc and RRc which generate respectively a right circular cylinder and right circular torus with the choice of proportions $\alpha_{12} = 0$, $\alpha_{23} = \alpha_{34} = \alpha_{45} = \alpha_{51} = \pi/2$, $a_{34} < a_{45}$ and $S_{44} = 0$. (Figure 2.16). Thus the RPCRR structure has eight assembly configurations in general. (Figure 2.17 illustrates these for a cylinder and a torus).

The number of assembly configurations for the six structures analysed above is summarised in column 2 of Table II. With the exception of the RCRC structure the problem was reduced to that of determining the number of common directed lines between various combinations of cylinder and torus. Hence, in general, two cylinders have two such lines in common, a cylinder and a torus have eight common lines, whilst two torii have sixteen directed lines in common.

2.7 The Closures of Spatial Mechanisms.

It is evident by comparing columns 1 and 3 of Table II. that a large number of spatial mechanisms can be reduced to the above structures, for the purpose of deriving the number of closures. The derivation of the two closures for the spatial four-link RCCC mechanism (which reduces to the CCC spatial triangle) has already been given in a previous section and this explains the quadratic input-output equation obtained by Yang and Freudenstein [46] (see also Chapter 6) for this mechanism.

In the following sections the number of closures for various spatial five-link 3R-2C mechanisms is derived. These mechanisms exhibit the interesting property that certain kinematic inversions differ from one another in the number of their closures. Also the number of closures of those spatial six-link 4R-P-C (Chapters 7, 8 and 9) and 5R-C (Chapter 10) mechanisms analysed by the author are derived. (The derivation for other six-link mechanisms is similar). Finally, the problems of determining the number of closures of spatial seven-link mechanisms are discussed.

2.7.1 The 3R-2C Five-Link Spatial Mechanisms.

The RRCCR five-link spatial mechanism with frame, a_{51}, and input and output angular displacements, θ_1 and θ_5, respectively, is illustrated by Figure 2.23. For the RRRCC mechanism (the only other inversion of the RRCCR with a revolute input), the frame is a_{23} and the input and output angular displacements are θ_2 and θ_3 respectively.

In both these cases, holding the respective input angles constant, reduces the mechanism to the RRCC structure as shown. Thus the RRCCR and RRRCC mechanisms both have a maximum of eight closures, corresponding to the eight assemblies of the RRCC structure, and hence their input-output polynomials are both of the eighth degree, in the half-tangents of their respective output angular displacements. In addition they are also of degree eight in the input angles since, for a given value of the output angular displacement, the RRCCR again reduces to the RRCC structure, whilst the RRRCC reduces to the five-link RRRCP structure (which has eight assemblies).

The RCRCR mechanism (representing the other class of 3R-2C mechanisms which have the two cylindric pairs separated by a revolute pair) is illustrated by Figure 2.24. There are three distinct inversions in this class. RCRCR has frame a_{51} and input and output angular displacements θ_1 and θ_5 respectively. For the RRCRC the frame is a_{12}, the input is θ_1 and the output is θ_2, whilst the third inversion, the RCRRC, has frame a_{23} and input and output angular displacements θ_3 and θ_2 respectively.

In the case of the RCRCR and RRCRC, holding the input angular displacement (θ_1 for both mechanisms) constant, reduces the mechanisms to the RCRC four-link structure (Figure 2.13). Thus both of these inversions of the RCRCR have a maximum of four closures (the number of assembly configurations for the RCRC structure). However, for the third inversion, the RCRRC, holding the input, θ_3, constant, reduces the mechanism to the RRCC structure which has eight assembly configurations. Thus the RCRRC has eight closures and it can be seen that two inversions of the same spatial mechanism need not have the same number of closures. This is an unexpected result which has been verified algebraically [13].

Finally it must be noted that, for a constant output angle the RCRCR, RRCRC and RCRRC mechanisms reduce to the RCRC, RRCRP and RCRRP structures respectively. The first of these has four assemblies whilst the other two each have eight. Thus the input-output equations for the RCRCR, RRCRC and RCRRC are of degrees four, eight and eight respectively in their input angular displacements. (Notice that the RRCRC input-output equation is of degree eight in the input, but of degree four in the output).

2.7.2 The 4R-P-C Six-Link Spatial Mechanisms.

The three distinct six-link RCRPRR, RCRRPR and RRRPCR spatial mechanisms are illustrated by Figures 2.25, 2.26, and 2.27, respectively. Holding either their respective input angular displacements (θ_1 in each case) or their output angles (θ_6 in each case) at a constant value, reduces these three mechanisms to the RRCRP, RRCRP and RPCRR five-link structures, respectively. Since the latter all have eight assemblies, the three 4R-P-C mechanisms each have eight closures. Consequently one would expect to obtain input-output equations of degree eight in input and output variables for each of these six-link mechanisms. Furthermore, any inversion of the latter, with a revolute input, will reduce to either the five-link RRCRP or RPCRR structure and hence will have eight closures.

2.7.3 The 5R-C Six-Link Spatial Mechanisms.

The 5R-C spatial mechanisms are illustrated by Figure 2.28. There are three inversions of these mechanisms, with revolute input. Thus the RRRRCR mechanism has frame a_{61}, input angle θ_1 and output angle θ_6. For the RRCRRR the frame is a_{12} and the input and output angular displacements are θ_1 and θ_2 respectively. Finally, the third inversion, the RRRRRC mechanism, has frame a_{56}, input angle θ_6 output angle θ_5 and output sliding displacement S_5.

Holding the respective input angles constant, in all three cases, reduces the mechanisms to the RRCRR five-link structure, which has sixteen assembly configurations. Hence the three 5R-C mechanisms with revolute inputs all have a maximum of sixteen closures and hence input-output polynomials of degree sixteen in the output angular displacements. In addition, if the output is held fixed, for the two inversions with revolute output (RRRRCR and RRCRRR), one again obtains the RRCRR structure. Consequently, for these two inversions, one would expect input-output equations of degree sixteen in both the input and the output angular displacements, and this is a novel result. It is confirmed algebraically in Chapter 10.

2.7.4 The 5R-2P, 6R-P and 7R Seven-Link Spatial Mechanisms.

There are three distinct 5R-2P seven-link mechanisms:- the RPPRRRR (Figure 2.29) the RPRPRRR (Figure 2.30) and the RPRRPRR (Figure 2.31). Holding the input angular displacement constant for each of these, or any of their inversions (with revolute inputs), reduces the mechanism to one of the three 4R-2P six-link structures shown in Table II. (i.e. RRRRPP, RRRPRP, RRPRRP).

Similarly the RRRRRPR seven-link mechanism shown in Figure 2.32, and all of its inversions, reduces to the six-link RRRRRP structure if its input angular displacement is held constant. Finally the RRRRRRR mechanism illustrated by Figure 2.33 reduces to the RRRRRR (or 6R) six-link spatial structure for a constant input value.

All of the structures discussed so far have been analysed by disconnecting at a cylindric pair and determining the number of line vectors which are common to two vector fields. In order to analyse 4R-2P, 5R-P and 6R six-link spatial structures, it is necessary to disconnect at a revolute or at a prismatic pair, since there are no cylindric pairs present. Thus, in addition to obtaining common line vectors between the free chains produced, it is necessary to introduce into the analysis either the magnitude of the fixed offset distance, S_{ii}, of a revolute pair, or the magnitude of the constant angular displacement, Θ_{ii}, of a prismatic pair.

The introduction of these additional constraints presents a formidable problem which may be well suited to solution by the sophisticated analogue techniques developed by Torfasson and Crossley [37] .

Nevertheless it is possible to categorise all seven-link mechanisms into sets which have the same number of closures, as is illustrated by Table II. Thus if the RRRRPP six-link structure has n_1 (a positive integer) assembly configurations then the group of four seven-link mechanisms listed in column 3 of Table II. (i.e. RPPRRRR, RRRRRPP, RRPPRRR and RPRRRRP), must have n_1 closures, from the hypothesis, since they all reduce to the RRRRPP structure for a fixed value of their input angular displacements. A similar situation occurs for the remaining structures and mechanisms in Table II.

2.8 Predicting the Closures of Spatial Mechanisms Algebraically.

The geometrical procedure developed in this chapter has proved to be an invaluable aid in the derivation of input-output equations. In addition to providing the analyst with the number of closures of a multitude of spatial mechanisms, the method has led to the discovery of certain interesting and unexpected phenomena (for example, two inversions of the RCRCR have differing numbers of closures). However, the author considers that it is desirable to devise a more rigorous approach (based on a firm mathematical foundation) for predicting the number of closures of spatial mechanisms. It is suggested that an investigation, based on algebraic and projective geometry and, in particular, of line ensembles, may produce the desired results. A preliminary outline of a possibly fruitful approach is given in Appendix I.

2.9 Discussion and Comparison of Results.

The spatial mechanisms considered in this chapter have been grouped in column 3 of Table II. according to the basic spatial structure from which their maximum number of closures has been derived. Mechanisms labelled with the same superscript (e.g. RCRCR[1] and RCRRC[1]) are inversions of one another although they do not necessarily occur in the same group (since they may not have the same number of closures).

The Table is divided into two main columns. The first contains the physical results and predictions of this chapter whilst the second contains the results obtained algebraically by various authors (referenced). Columns 2 and 4 must contain the same number for each group of mechanisms and its related structure, since the hypothesis discussed earlier establishes such a direct correlation.

The maximum number of closures of each mechanism is listed in column 4 and this should compare directly with the degree of its input-output equation, as obtained algebraically by various researches [6, 7, 11, 12, 13, 14, 16, 17, 22, 33, 45, 46, 47, 48, 49] and presented in column 5. There is considerable agreement although, in a number of cases, there are discrepancies which imply that the input-output displacement equations, presented by a number of researchers, contain extraneous or unwanted roots.

2.9.1 R-3C Mechanism.

It has been well established by Denavit [6] and Yang and Freudenstein [46] that the input-output equation for the RCCC four-link spatial mechanism is quadratic in the half-tangents of the input and output angular displacements.

2.9.2 3R-2C Mechanisms.

Dimentberg [7] derived an eighth degree input-output equation for the RCRCR[1] mechanism using the algebra of screws. However, a quartic equation was subsequently first derived by Yang [45] using matrices with dual-number elements. This quartic equation was corrected by Yuan [47] using line geometry and by Duffy and Habibolahi [11] using spherical trigonometry. The latter authors also derived eighth degree and then quartic input-output equations for the RRCRC[1] mechanism [12, 14], an inversion of the RCRCR[1]. Again, a degree-eight equation was derived by Duffy and Habibolahi [13] for the RCRRC[1] mechanism, which one would expect at the outset to have a quartic input-output equation, since it is an inversion of both the RCRCR[1] and RRCRC[1] mechanisms. As explained in this chapter however, this is the first example of the phenomenon that common inversions of the same spatial mechanism can have input-output equations of a different degree. Table II. illustrates that this is not, after all, an unexpected result, since the maximum number of closures of the RCRCR[1] and RRCRC[1] mechanisms is equal to the maximum number of assembly configurations of the RCRC structure (4 assemblies), whilst the number of closures for the RCRRC[1] is given by the number of assemblies for the RRCC structure (8 assemblies). Thus they reduce to different structures.

2.9.3 <u>4R-P-C Mechanisms.</u>

There are three distinct 4R-P-C six-link mechanisms labelled with superscripts 4, 5, 6. Table II. illustrates that the maximum number of closures and, therefore, the degree of the input-output equation of any 4R-P-C mechanism with revolute input must be eight, since they all reduce to either the RRCRP or the RPCRR five-link structures (which both have eight assemblies) for fixed input angles. An inversion of each of the three types (i.e. the RCRPRR[4], RCRRPR[5] and RRRPCR[6] mechanisms) will be analysed in detail, in Chapters 7, 8 and 9 of this dissertation, where degree eight, input-output polynomials are obtained algebraically (agreeing with the physical results shown). Table II. also suggests that the sixteenth degree equation derived by Yuan [49] for the RPRCRR[4] mechanism must contain an extraneous factor of the eighth degree. This was in fact, suggested by Yuan, since he failed to obtain all real solutions to the polynomial he had derived.

2.9.4 5R-C Mechanisms.

Table II. illustrates that all three 5R-C mechanisms with revolute inputs have input-output equations of degree sixteen in the half-tangent of the output angular displacement and this result will be derived algebraically in Chapter 10 for the RRRRCR and RRCRRR mechanisms. The input-output equations for the latter two inversions must be also of degree sixteen in the half-tangent of the input angular displacement since they both have revolute output joints and, for a fixed output angle, again reduce to the RRCRR five-link structure.

Recently, however, Dukkipati and Soni [16] derived a degree 128 equation for the six-link RRRCRR and RCRRRR mechanisms and, most recently, the same authors obtained a degree 64 equation for the six-link RRRCRR and RRRRRC mechanisms [17] with general proportions. It now seems clear from this chapter and Chapter 10 that these equations must contain 112 and 48 extraneous roots respectively. The above authors did in fact fail to obtain more than six real solutions in reference [17].

2.9.5 5R-2P Mechanisms.

Although it has not been possible to derive the maximum number of assembly configurations for 4R-2P structures, using simple models, much useful and valuable information can be obtained from Table II. There are clearly three distinct 5R-2P mechanisms, which are labelled with the superscripts 7, 8 and 9, and the various inversions are grouped according to the three different basic structures (RRRRPP, RRRPRP and RRPRRP) to which they reduce for fixed values of their input angular displacements. It follows that all the $(5R\text{-}2P)^7$ mechanisms and one $(5R\text{-}2P)^8$ mechanism have the same degree, n_1, input-output equations, whilst the other three $(5R\text{-}2P)^8$ and two $(5R\text{-}2P)^9$ mechanisms also have the same degree, n_2, input-output equations. The remaining two $(5R\text{-}2P)^9$ mechanisms have input-output equations of degree n_3. (It is possible that $n_1 = n_2 = n_3$ but this need not necessarily be so).

Recently Keen [22] obtained input-output equations of degree eight, twelve and sixteen for spatial seven-link $RPPRRRR^7$, $RPRPRRR^8$ and $RPRRPRR^9$ mechanisms, respectively. These results, which are obtained algebraically without prior knowledge of Table II., imply that $n_1 = 8$, $n_2 = 12$ and $n_3 = 16$. Furthermore, as for the case of some five-link 3R-2C mechanisms presented earlier, certain common inversions (labelled with superscripts 8 and 9) must have input-output equations with different degrees.

2.9.6 6R-P, 7R and Other Seven-Link Mechanisms.

As far as the author is aware, there is no information available on the 6R-P mechanisms. However, Wallace and Freudenstein [41] succeeded in obtaining quartic displacement equations for spatial five-link RRSRR and RRERR mechanisms which may be considered to be special cases of spatial $RR\overline{RRR}RR$ and $RR\overline{PPR}RR^{7}$ (or $RR\overline{PRP}RR^{8}$) mechanisms, respectively, obtained by amalgamating combinations of either three revolute pairs or two prismatics and a revolute. He achieved this by developing a novel geometric configuration method and in addition explained some of the difficulties of obtaining a displacement equation for the spatial 7R mechanism by using dual-number methods.

Recently Duffy and Keen [15, 23] have derived quartic equations for the RERRR and RSRRR spatial mechanisms using spherical trigonometry and dual numbers. Other contributions, incorporating dual numbers, have been made by Wörle [43] and Keler [24, 25].

Finally various reasearchers [14, 22, 48, 49] have reported difficulties in employing computer-aided search techniques to find mechanism proportions which will give all-real solutions to input-output equations. Clearly, simple geometrical models (with the proportions given), such as those described in this chapter, can be used to overcome this problem, since, if the mechanism can be physically assembled, there must exist a corresponding real solution to the input-output equation. Keler [25] has, in fact given some consideration to the checking of closures by subdividing spatial mechanisms into spatial triangles. (see Yang [44])

2.10 Summary of Results.

Thus, in summary, one may list the following four points:-

(i) It has been established that:- The maximum number of closures of a single loop n-link mechanism with unit mobility, and with a revolute pair connecting the input link to the frame, is equal to the maximum number of assembly configurations of a corresponding (n - 1)-link structure.

(ii) The maximum number of closures has been derived for spatial four-link R-3C, five-link 3R-2C, six-link 4R-P-C, and six-link 5R-C mechanisms.

(iii) A physical explanation has been given of the unexpected phenomenon that two distinct inversions of the same spatial mechanism may have input-output displacement equations of a different degree.

(iv) By using mechanism proportions based on those used here for ease of visualisation, it is possible to obtain all-real solutions to input-output equations, i.e. to design mechanisms with all-real closures for ranges of values of their input angular displacements.

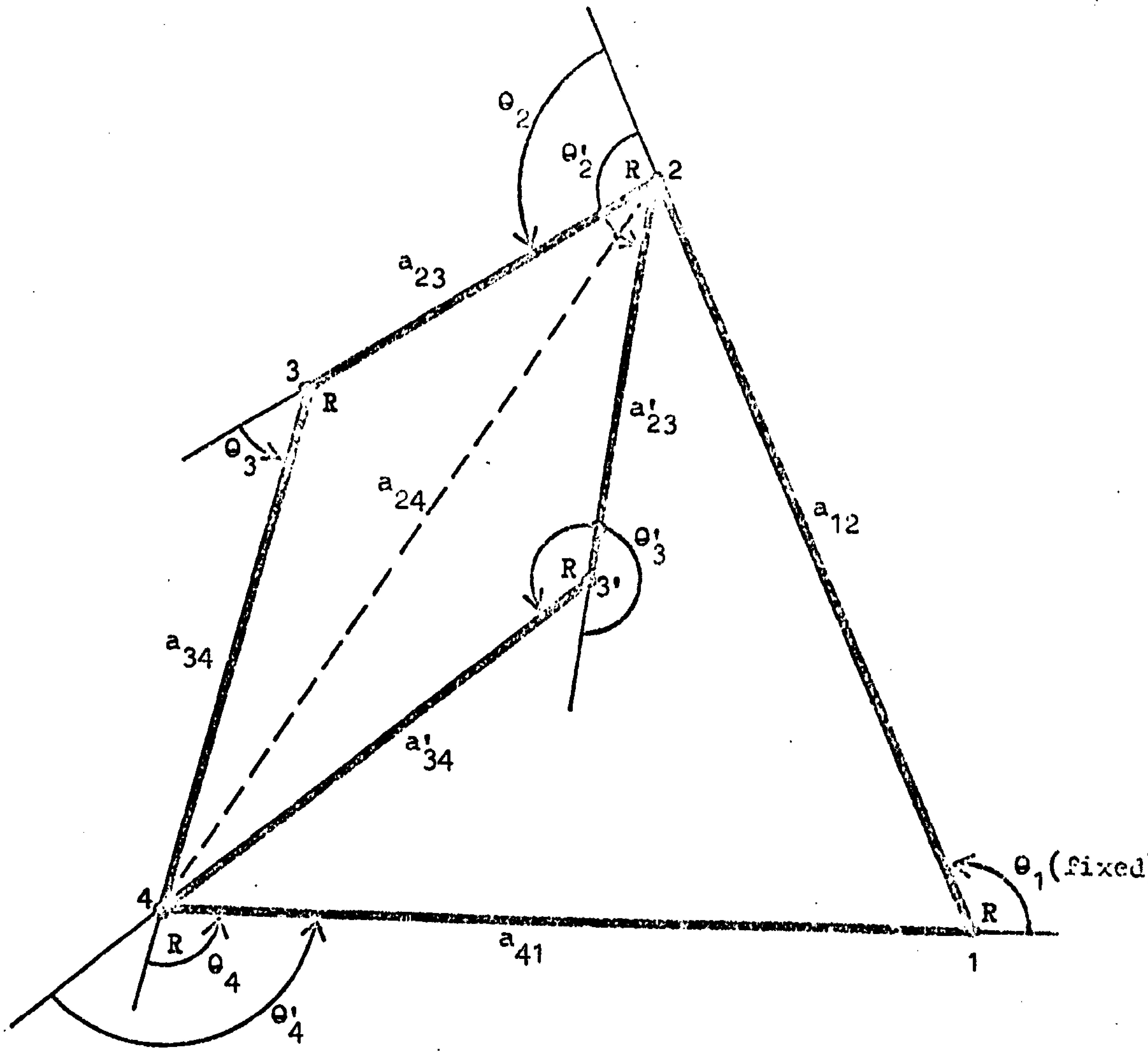
In the following three chapters (3, 4 and 5) a unified theory for analysing spatial mechanisms is developed using spherical trigonometry and dual numbers.

PHYSICAL RESULTS				ALGEBRAIC RESULTS	
Basic Structure	Maximum number of Assembly Configurations	Mechanism	Maximum number of Closures	Degree of Input/ Output Equation	Reference
CCC	2	RCCC	2	2	[6, 45]
RCRC	4	RCRCR[1]	4	8	[7]
				4	[46, 47, 11]
		RRCRC[1]	4	8	[12]
				4	[14]
RRCC	8	RRRCC[2]	8	8	[14]
		RRCCR[2]	8	8	[48, 33]
		RCRRC[1]	8	8	[13]
RRCRR	16	RRRCRR[3]	16	128	[16]
				64	[17]
				16	Chapter 10.
		RRRRCR[3]	16	128	[16]
				16	Chapter 10.
		RRRRRC[3]	16	64	[17]
RRCRP	8	RPRCRR[4]	8	16	[49]
		RCRPRR[4]	8	8	Chapter 7.
		RCRRPR[5]	8	8	Chapter 8.
		RRRCRP[4]	8		
		RRRPRC[4]	8		
		RRPRRC[5]	8		
		RRCRRP[5]	8		

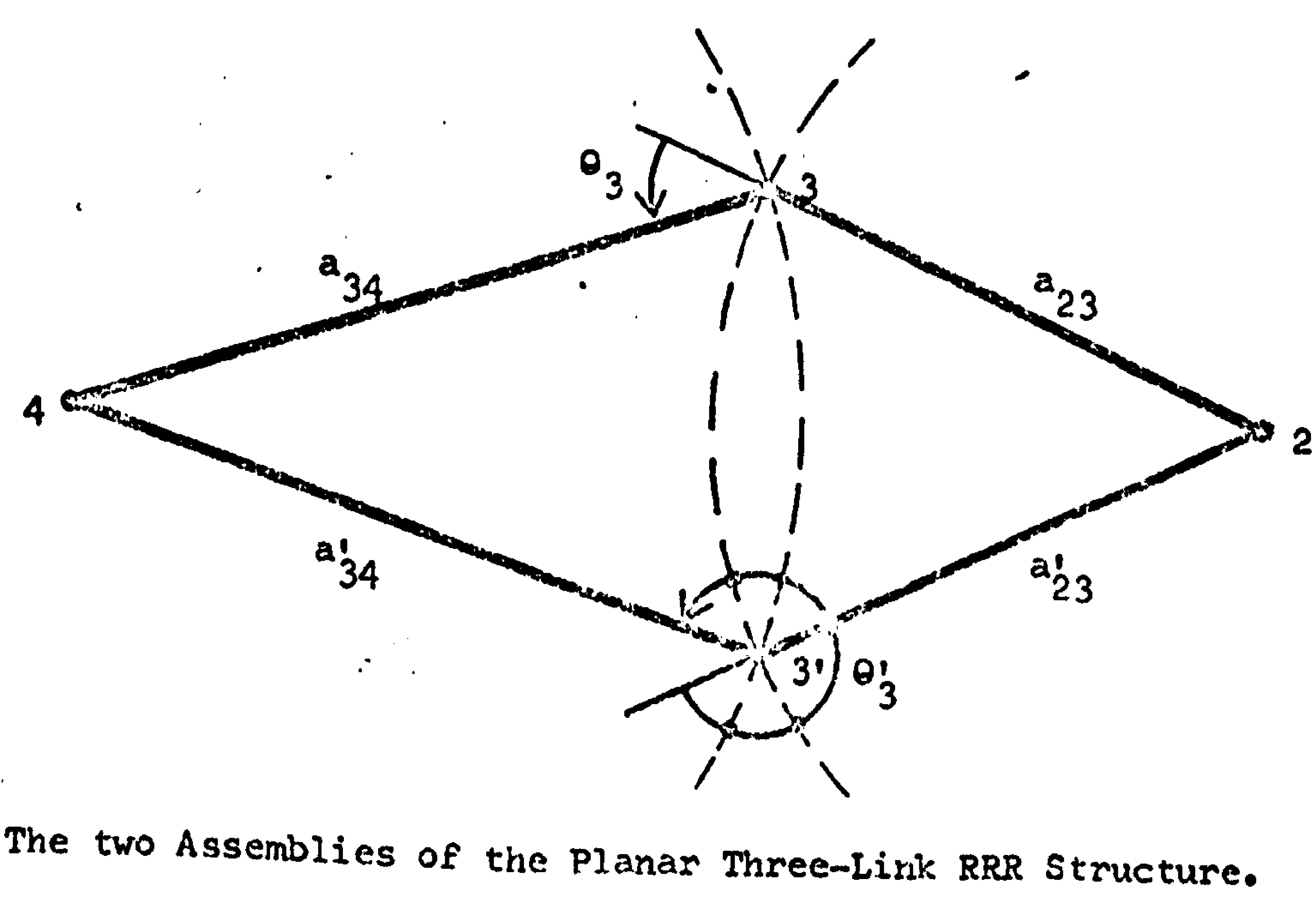
Table II. Comparison of Physical and Algebraic Results obtained for various Spatial Mechanisms.

PHYSICAL RESULTS				ALGEBRAIC RESULTS	
Basic Structure	Maximum number of Assembly Configurations	Mechanism	Maximum number of Closures	Degree of Input/ Output Equation	Reference
RPCRR	8	$RRRPCR^6$ $RRCPRR^6$ $RRRCPR^6$ $RRRRCP^6$ $RPRRRC^4$ $RCRRRP^4$ $RRRRPC^6$	8 8 8 8 8 8 8	8	Chapter 9.
RRRRPP	n_1	$RPPRRRR^7$ $RRRRRPP^7$ $RRPPRRR^7$ $RPRRRRP^8$	n_1 n_1 n_1 n_1	8	[22]
RRRPRP	n_2	$RPRPRRR^8$ $RRRRPRP^8$ $RRPRPRR^8$ $RPRRRPR^9$ $RRPRRRP^9$	n_2 n_2 n_2 n_2 n_2	12	[22]
RRPRRP	n_3	$RPRRPRR^9$ $RRRPRRP^9$	n_3 n_3	16	[22]
RRRRRP	n_4	$RRRRRRP^{10}$ $RPRRRRR^{10}$ $RRPRRRR^{10}$ $RRRPRRR^{10}$	n_4 n_4 n_4 n_4		
RRRRRR	n_5	RRRRRRR	n_5		

Table II. (Continued).



(a) Representation of the Planar Four-Link RRRR Mechanism.



(b) The two Assemblies of the Planar Three-Link RRR Structure.

Figure 2.1 The Closures of the Planar Four-Link RRRR Mechanism.

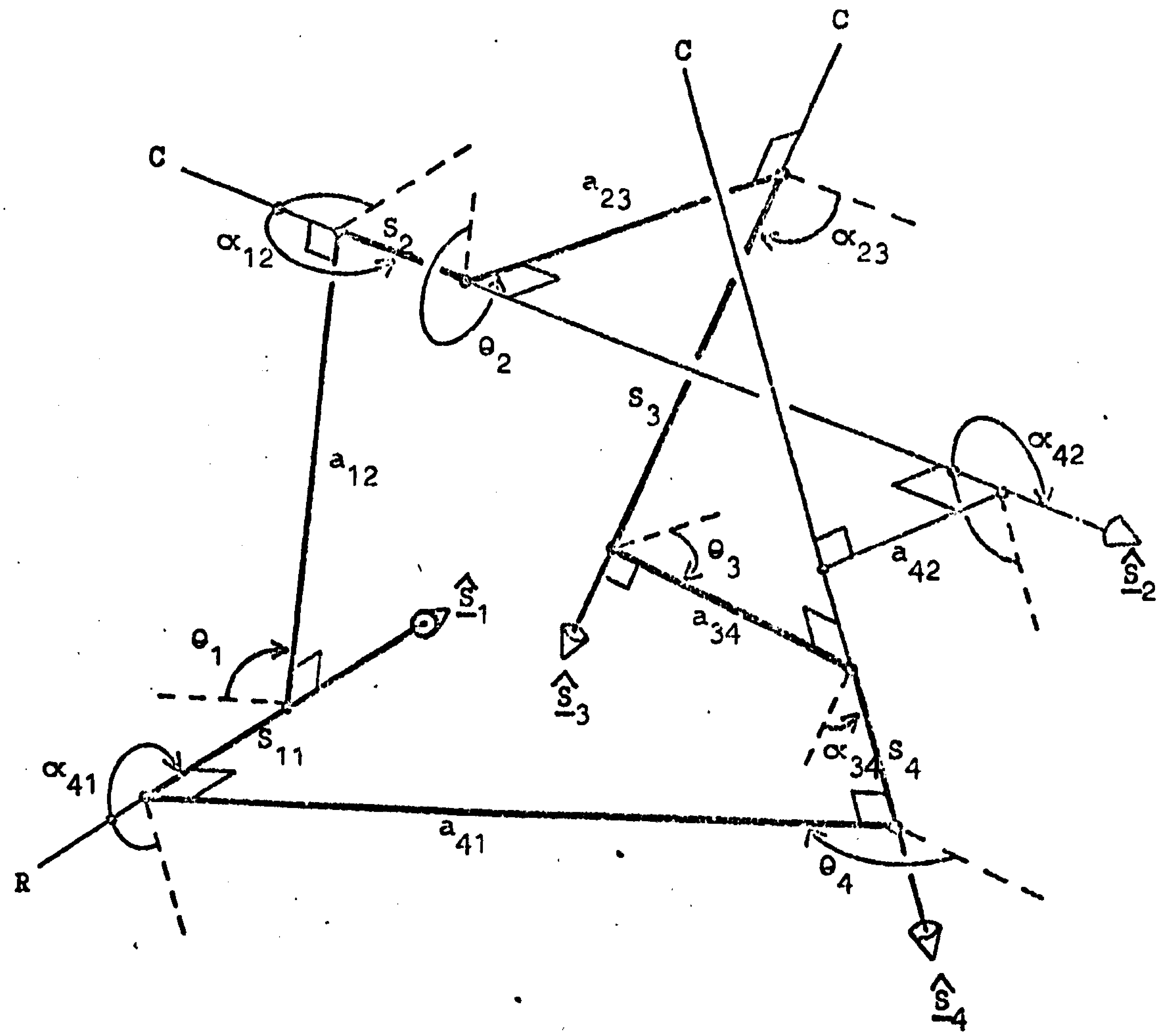


Figure 2.2 Representation of the Four-Link RCCC Mechanism.

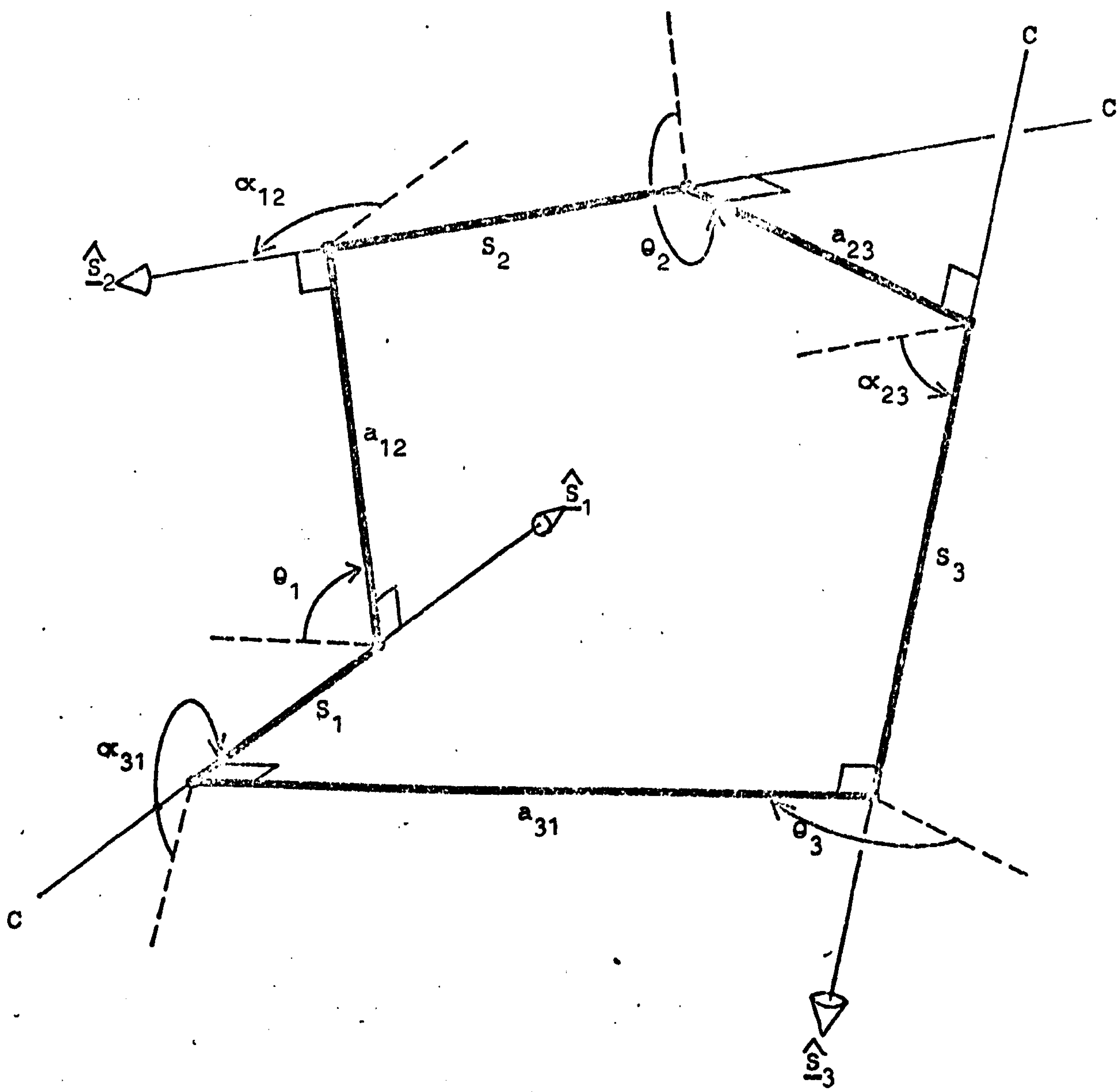


Figure 2.3 Representation of the Three-Link CCC Structure.

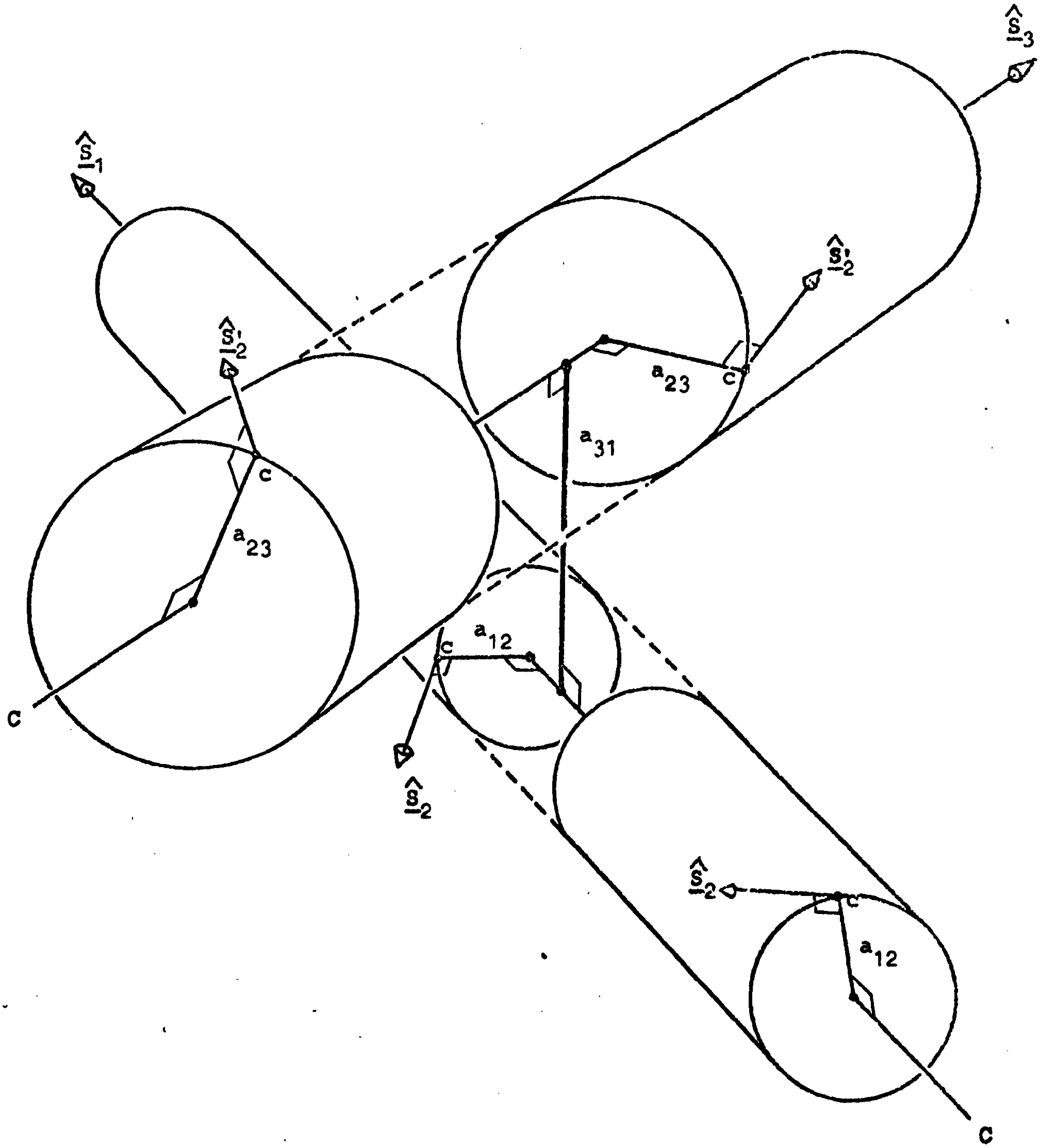


Figure 2.4 The Two Right Circular Cylinders Produced by the CCC Spatial Structure.

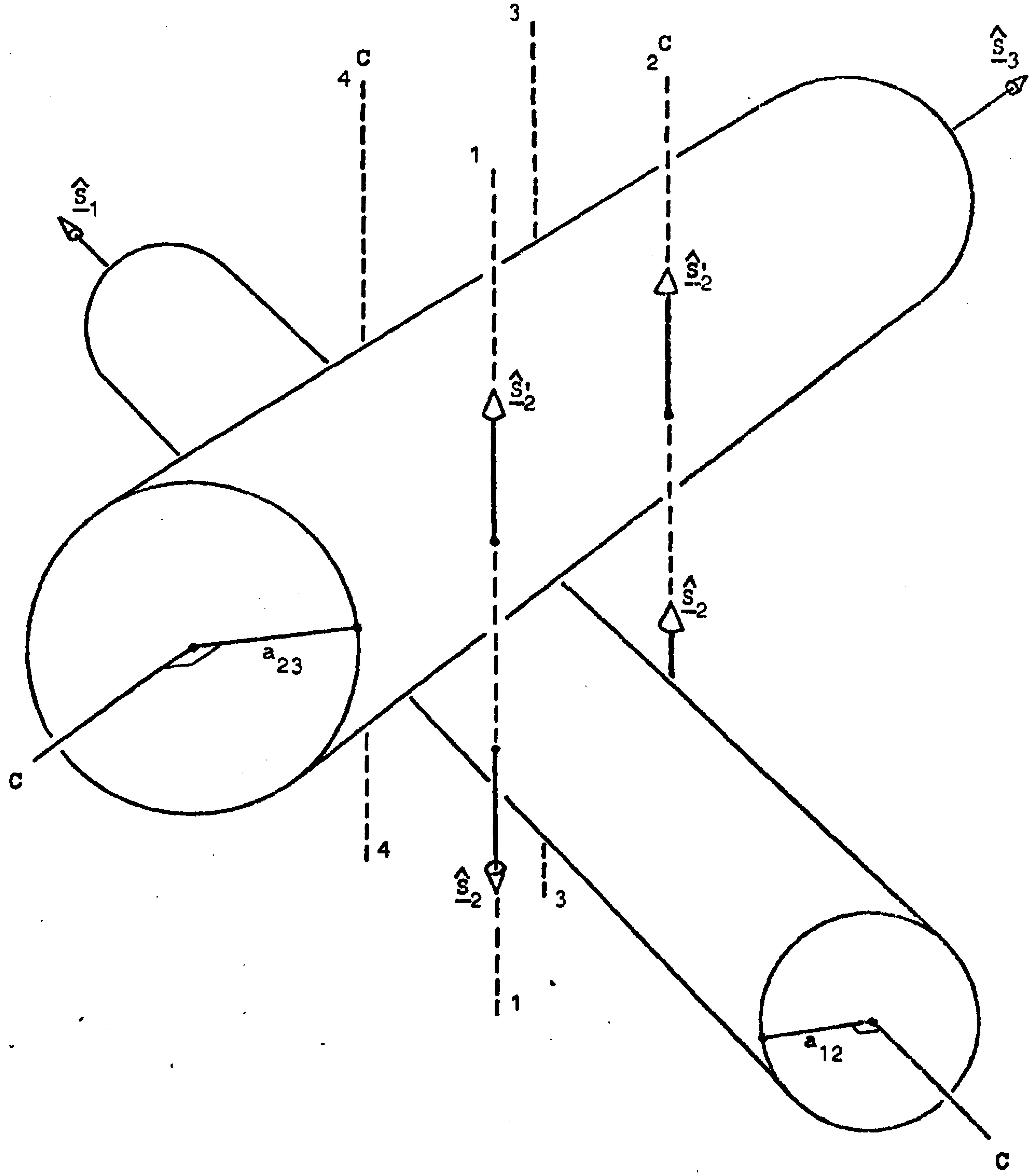
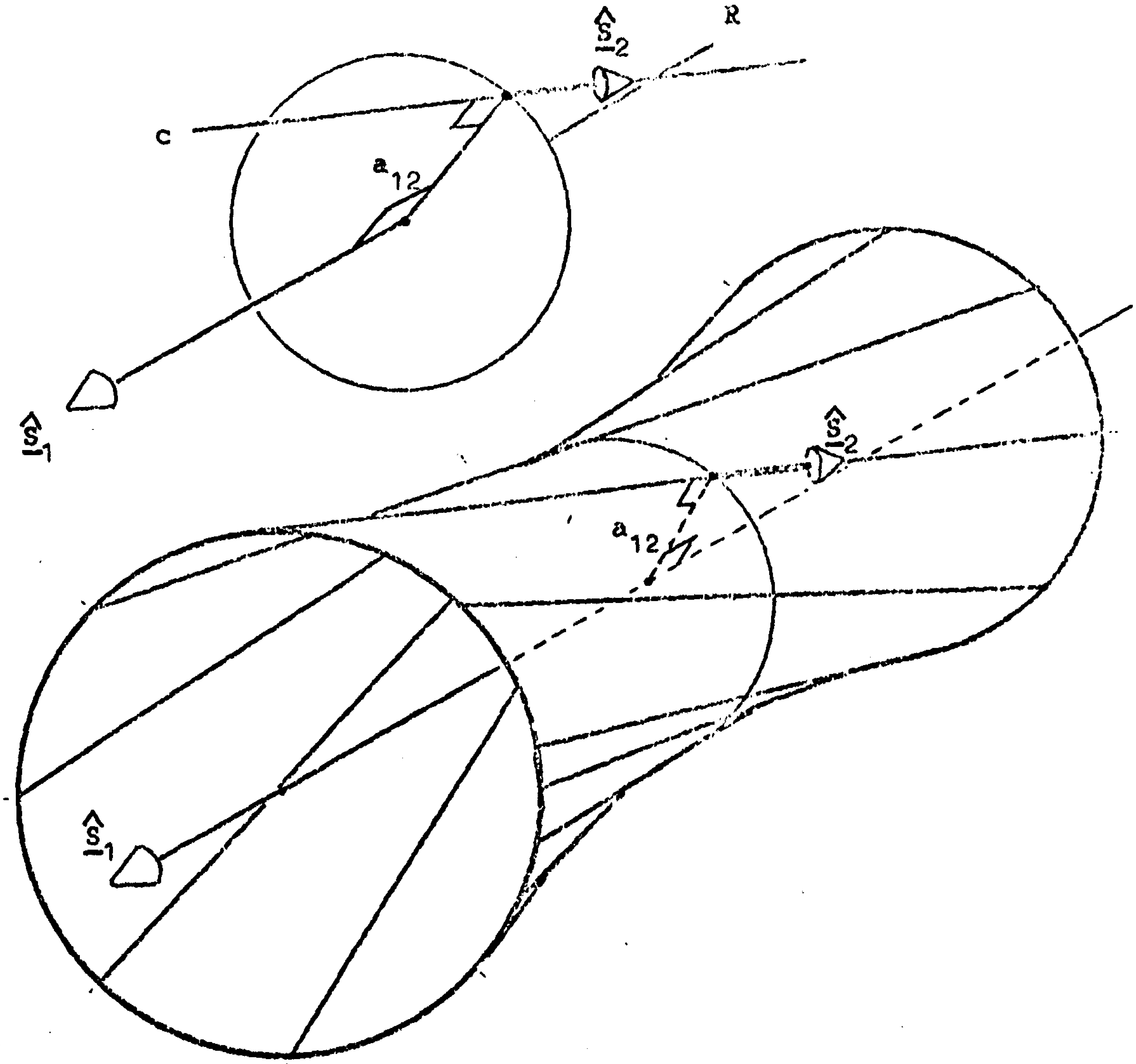
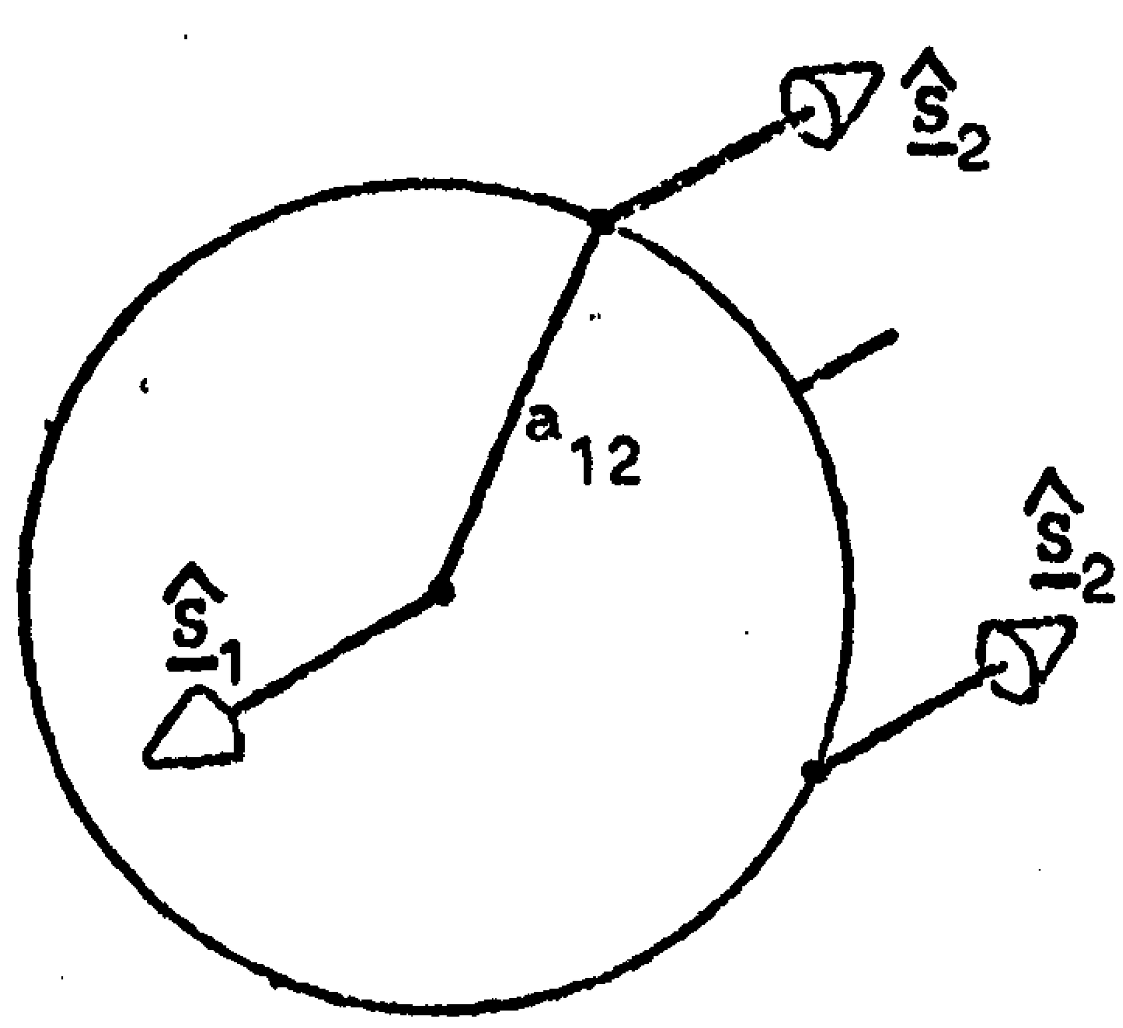
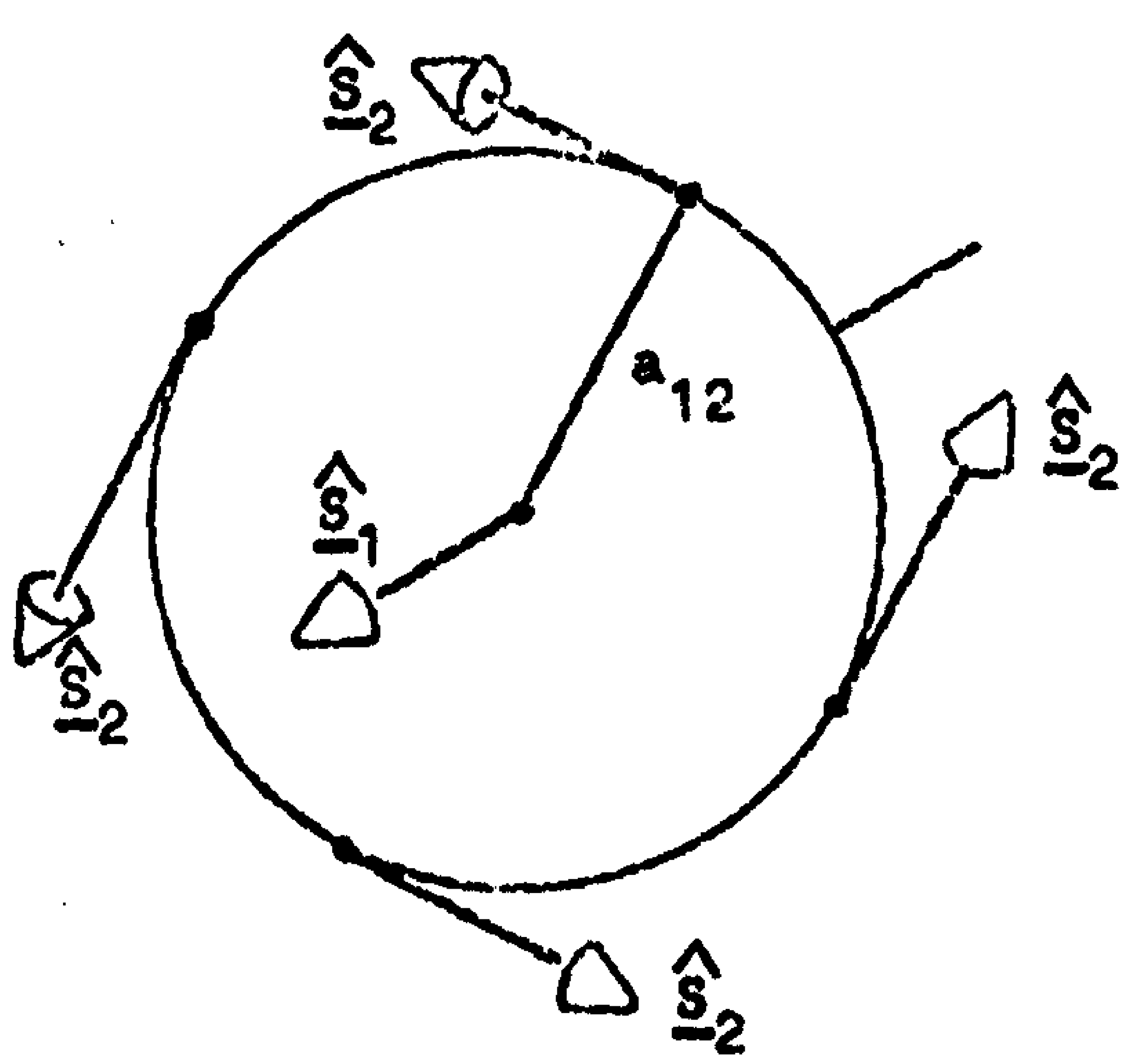


Figure 2.5 The Two Acceptable Assemblies (defined by the broken lines labelled 2 and 4) and the Two Unacceptable Assemblies (broken lines 1 and 3) Derived from the Four Common Lines between the Two Cylindrical Line Vector Fields Produced by the CCC Spatial Structure.



(a) The Hyperboloid of one Sheet Generated by the Rc Open Spatial Chain.





(b) Reduction to a Cylinder. (c) Reduction to a Plane.

Figure 2.6 Representation of the Rc Open Spatial Chain.

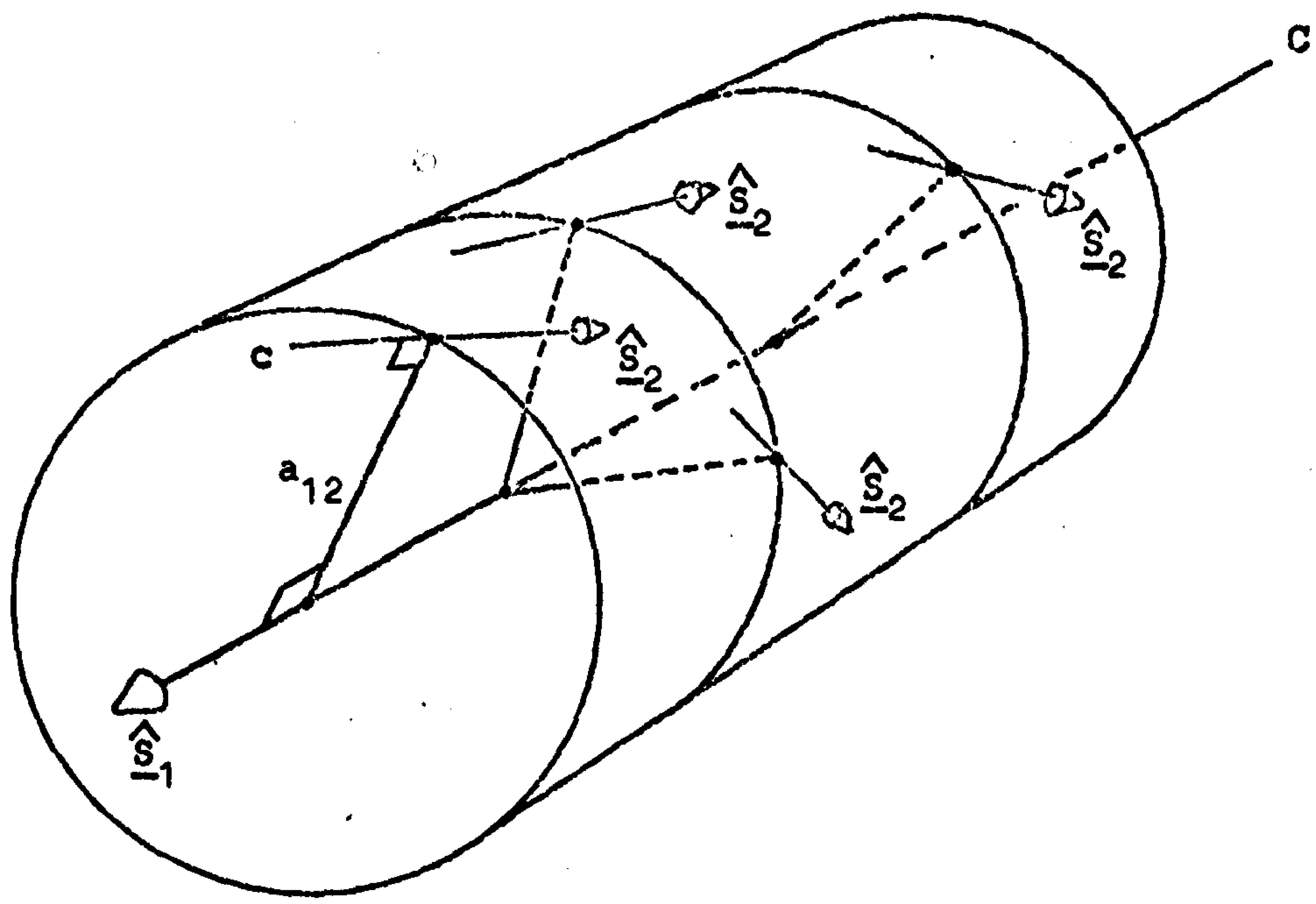


Figure 2.7 Representation of the Cc Open Spatial Chain and the Right Circular Cylinder that it Generates.

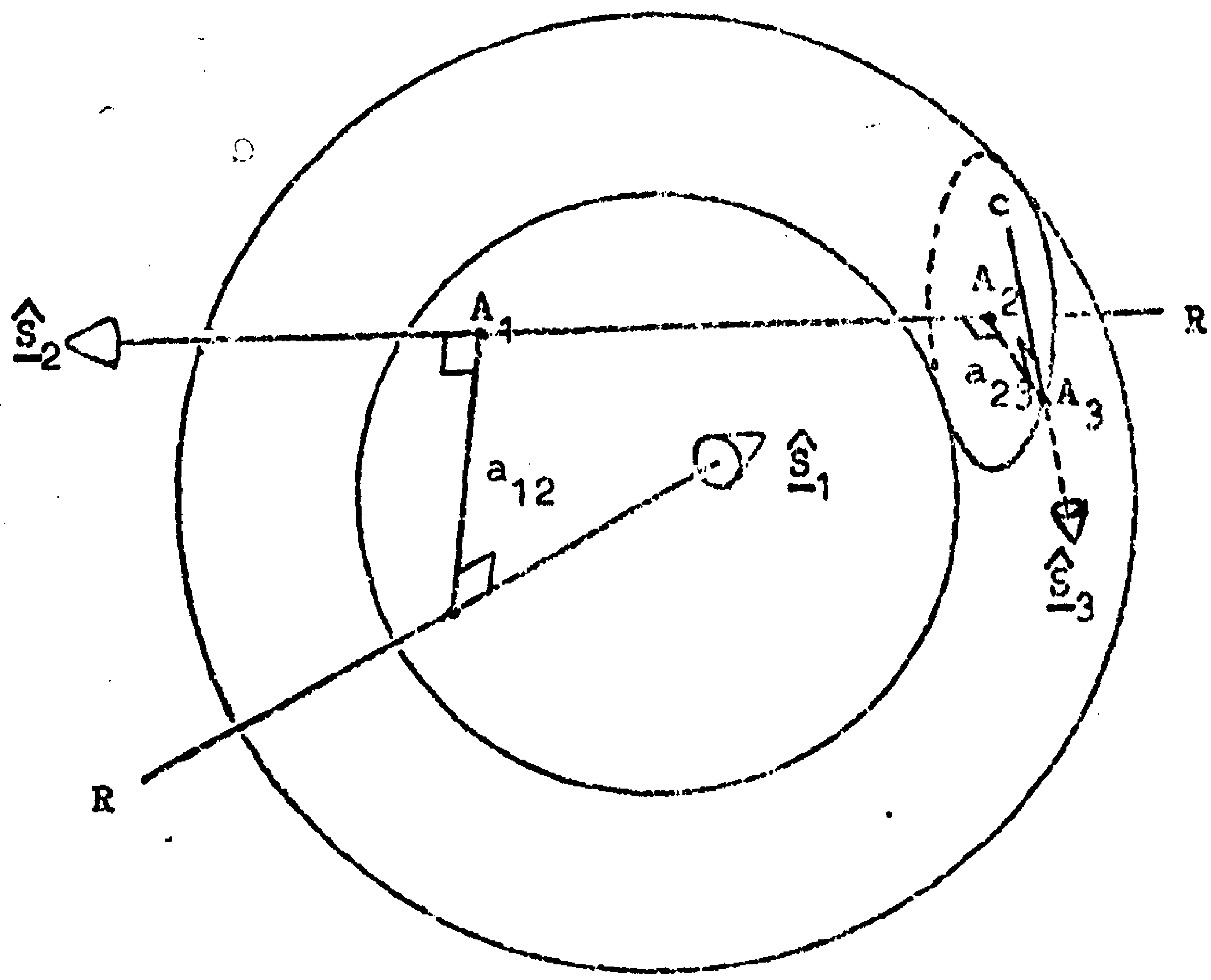


Figure 2.8 Representation of the RRc Open Spatial Chain and the Skew Torus that it Generates.

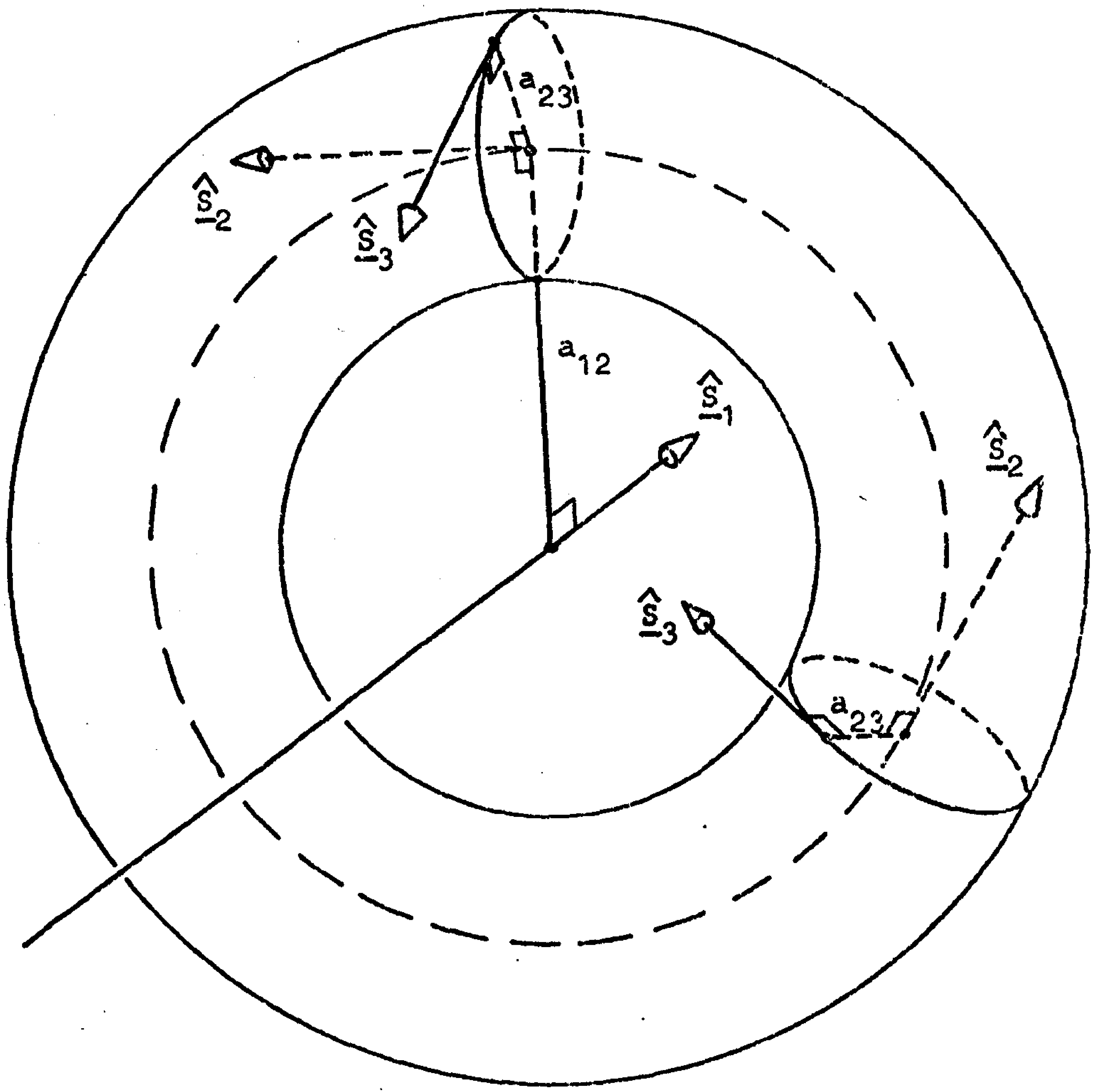
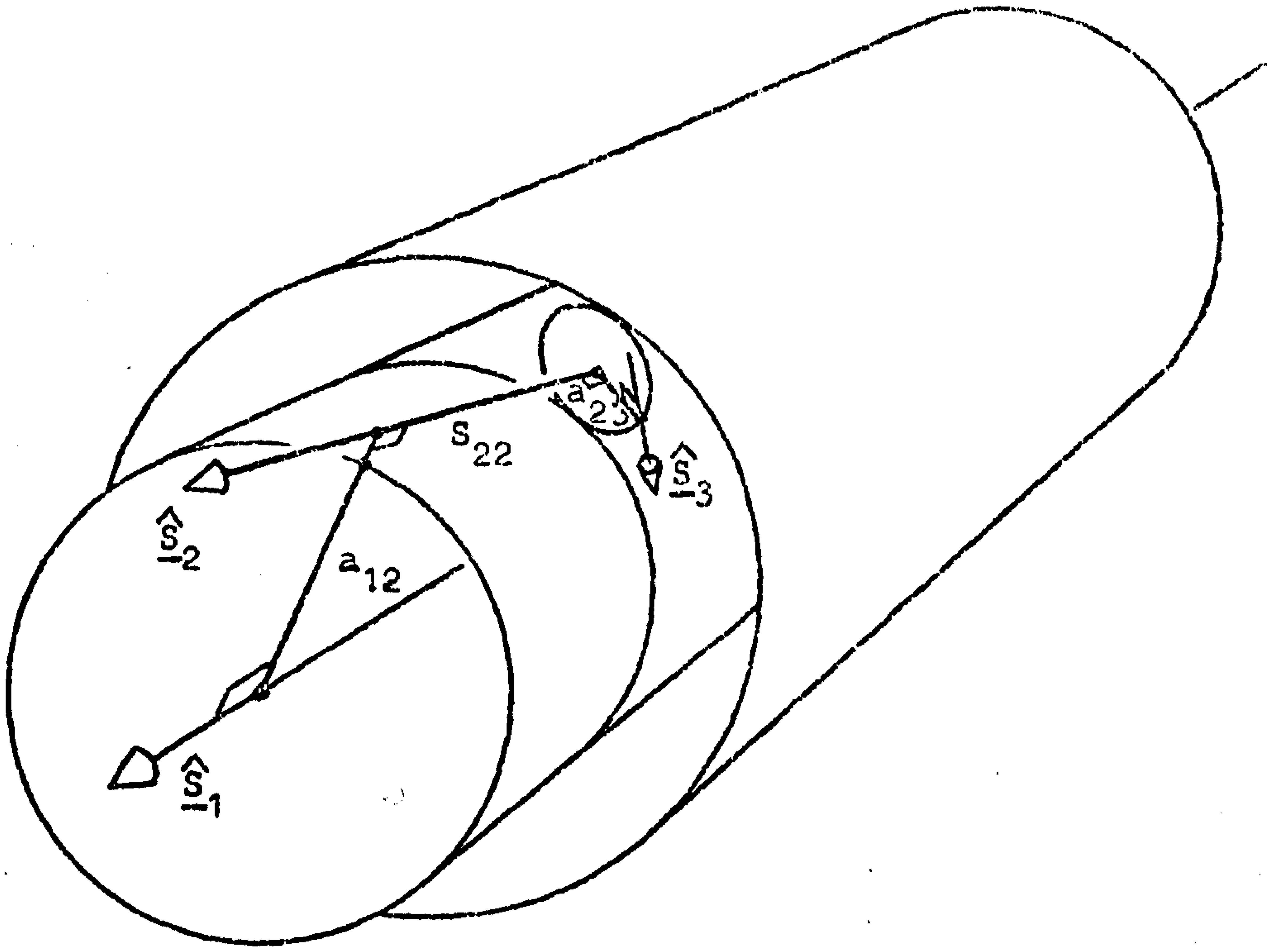
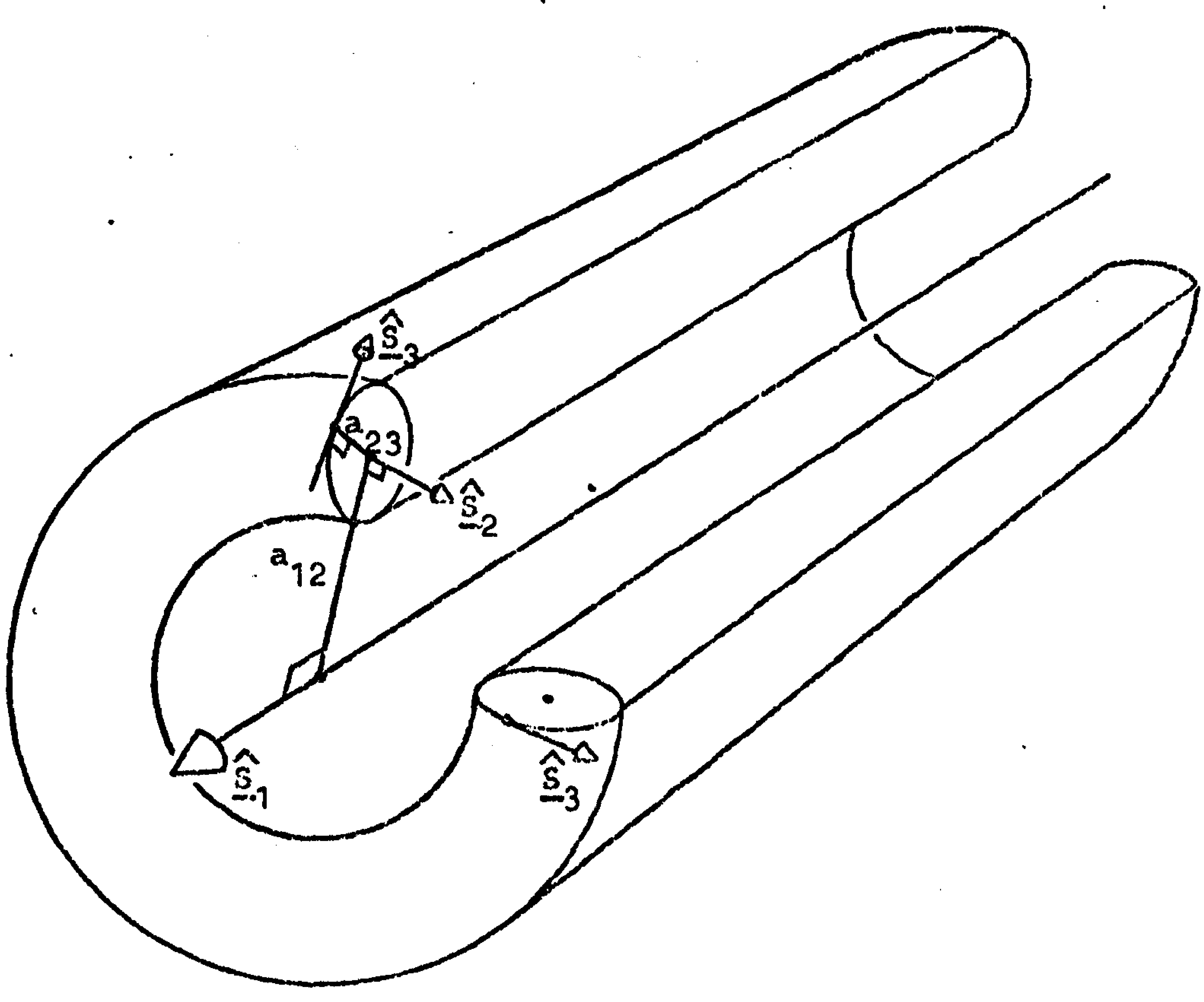


Figure 2.9 The Right Circular Torus with Circular Cross-Section Produced by the RRc Open Spatial Chain with Special Proportions.



(a) Volume Generated by the CRc Chain.



(b) Volume Swept out by Torus of Elliptical Cross-Section.

Figure 2.10 Representation of the CRc Open Spatial Chain.

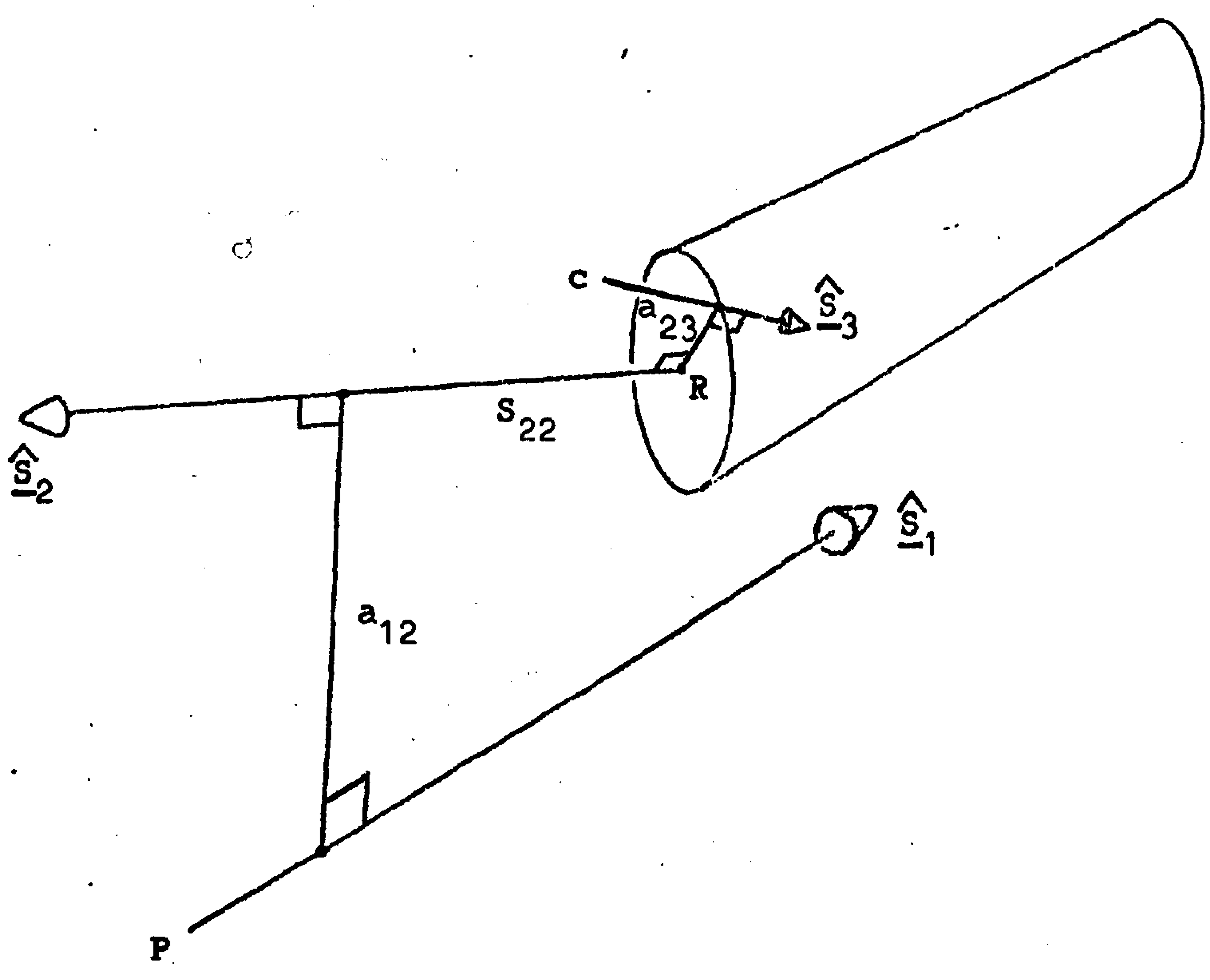


Figure 2.11 Representation of the PRc Open Spatial Chain and the Cylinder with Elliptical Cross-Section that it Generates.

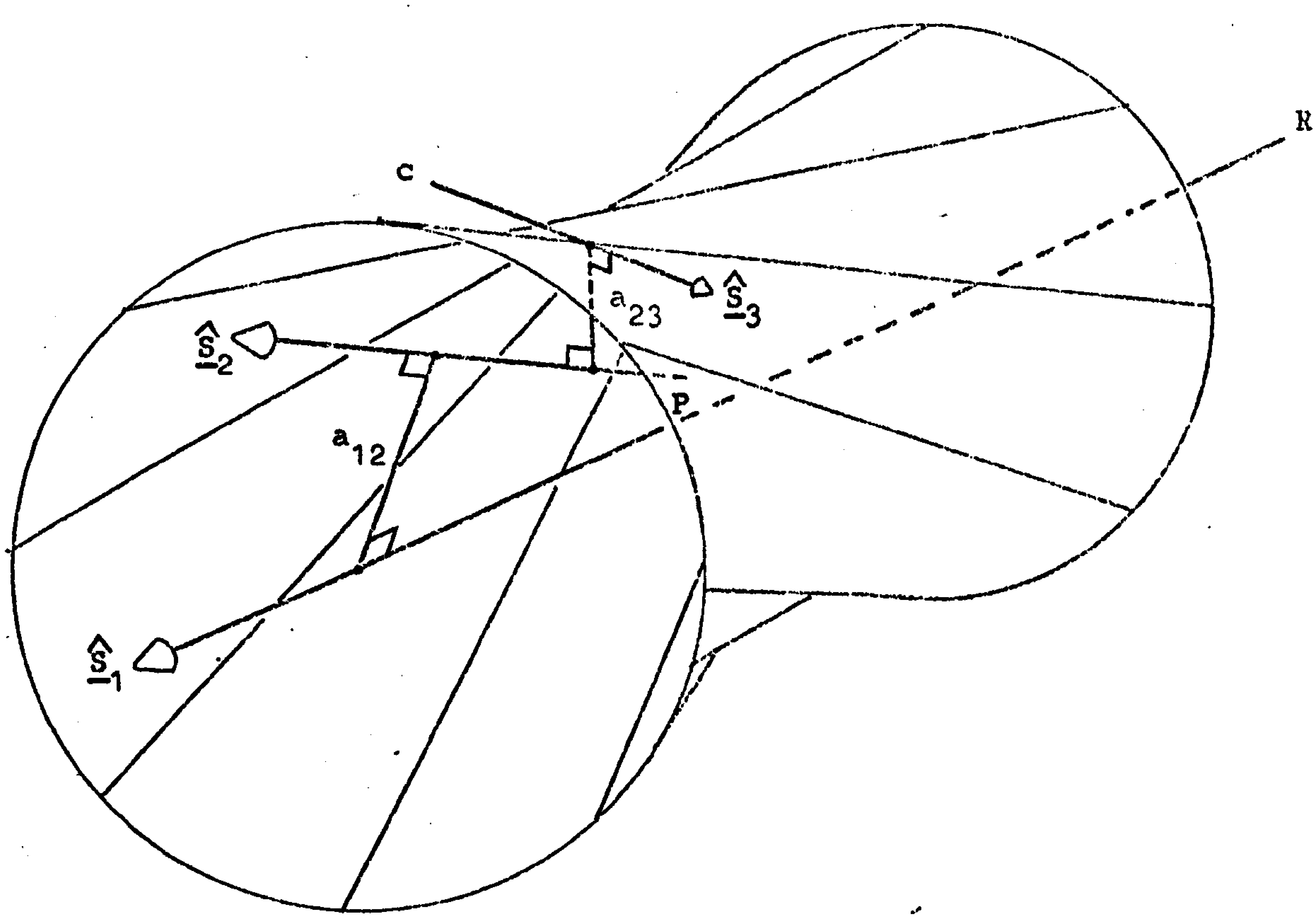


Figure 2.12 Representation of the RPc Open Spatial Chain and the Hyperboloid of One Sheet that it Generates.

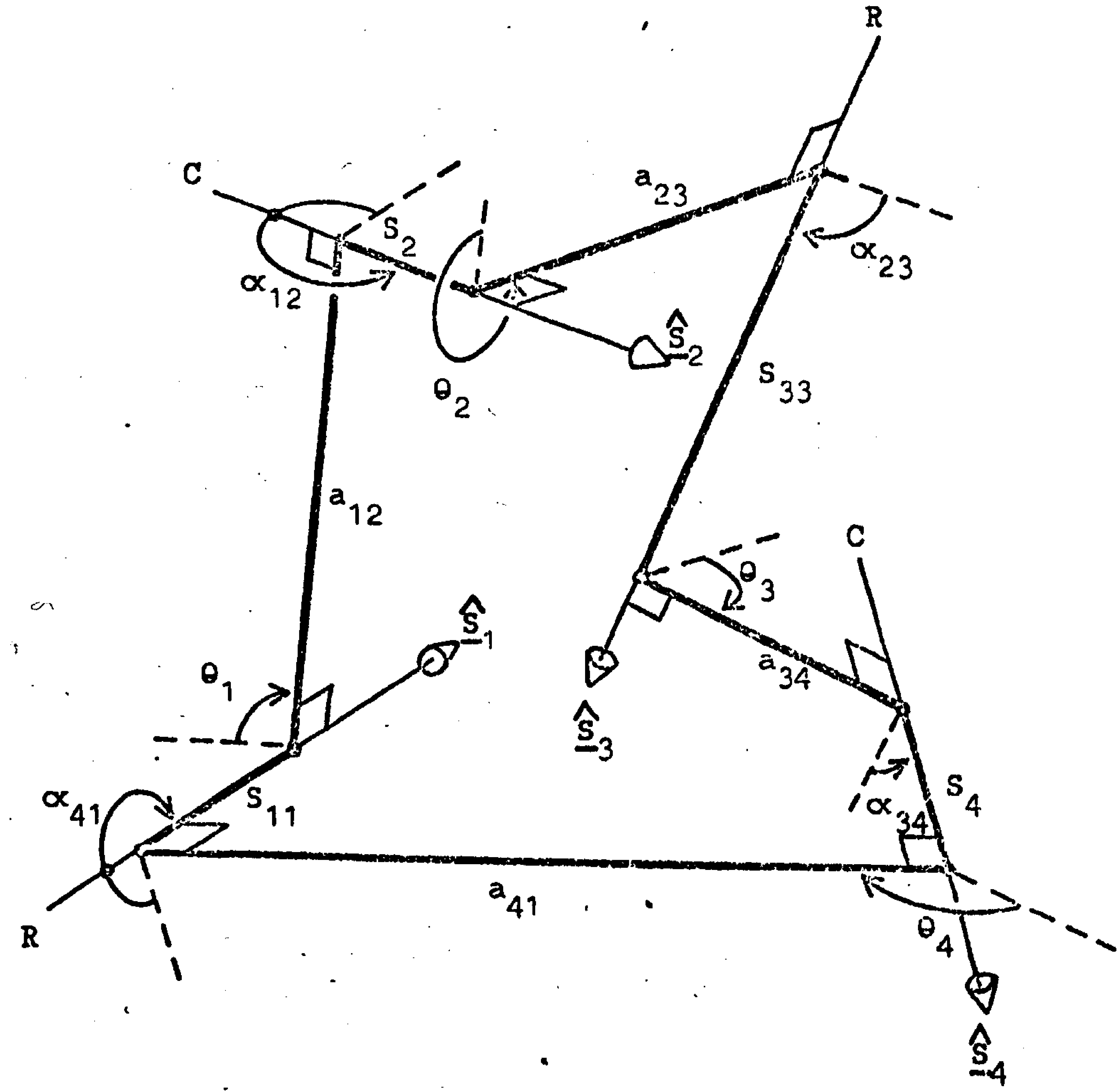
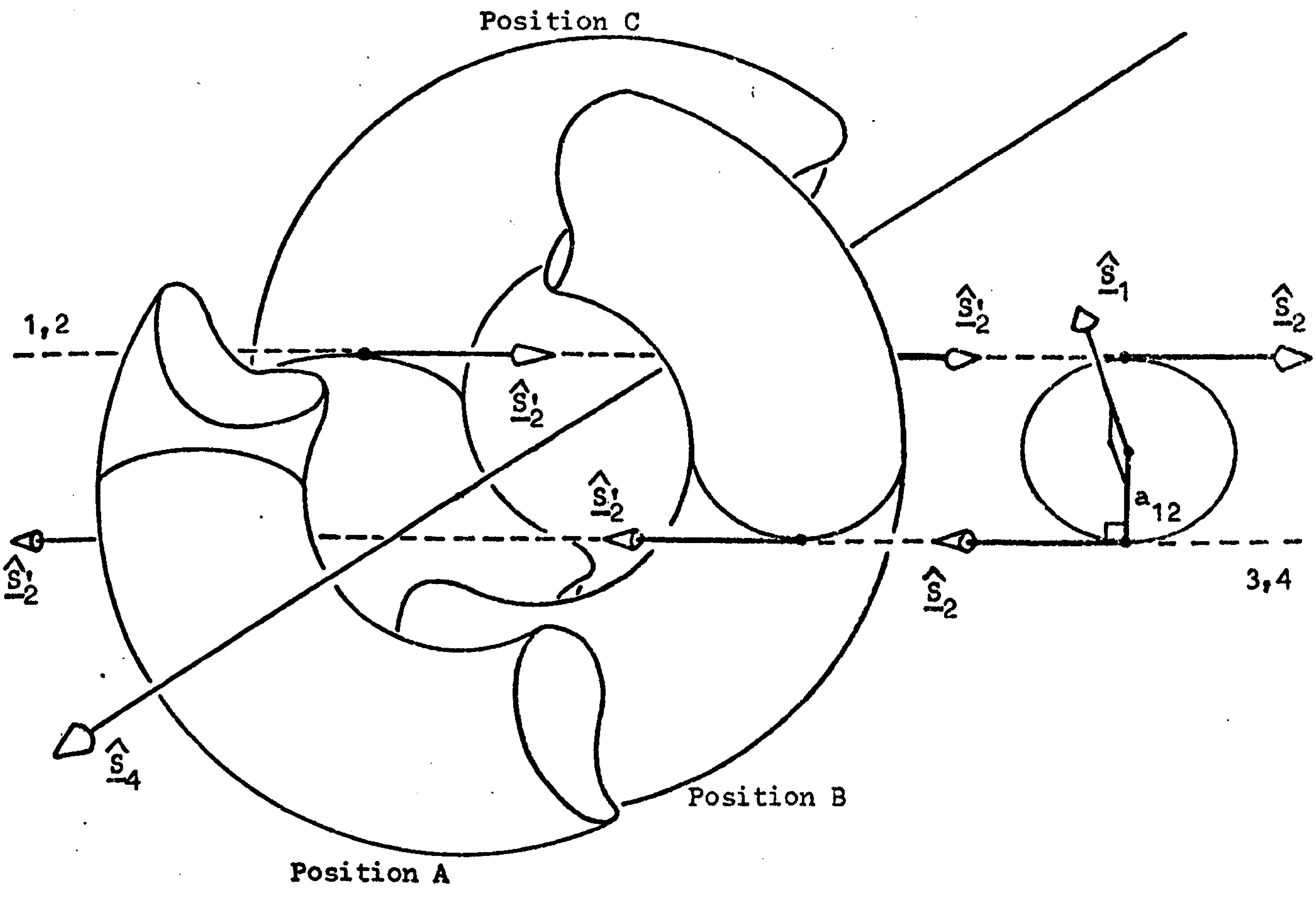


Figure 2.13 Representation of the Four-Link RCRC Structure.



The Torii are of Elliptical Cross-Section

Figure 2.14 The Four Assemblies (broken lines 1-4) of the RCRC Spatial Structure.

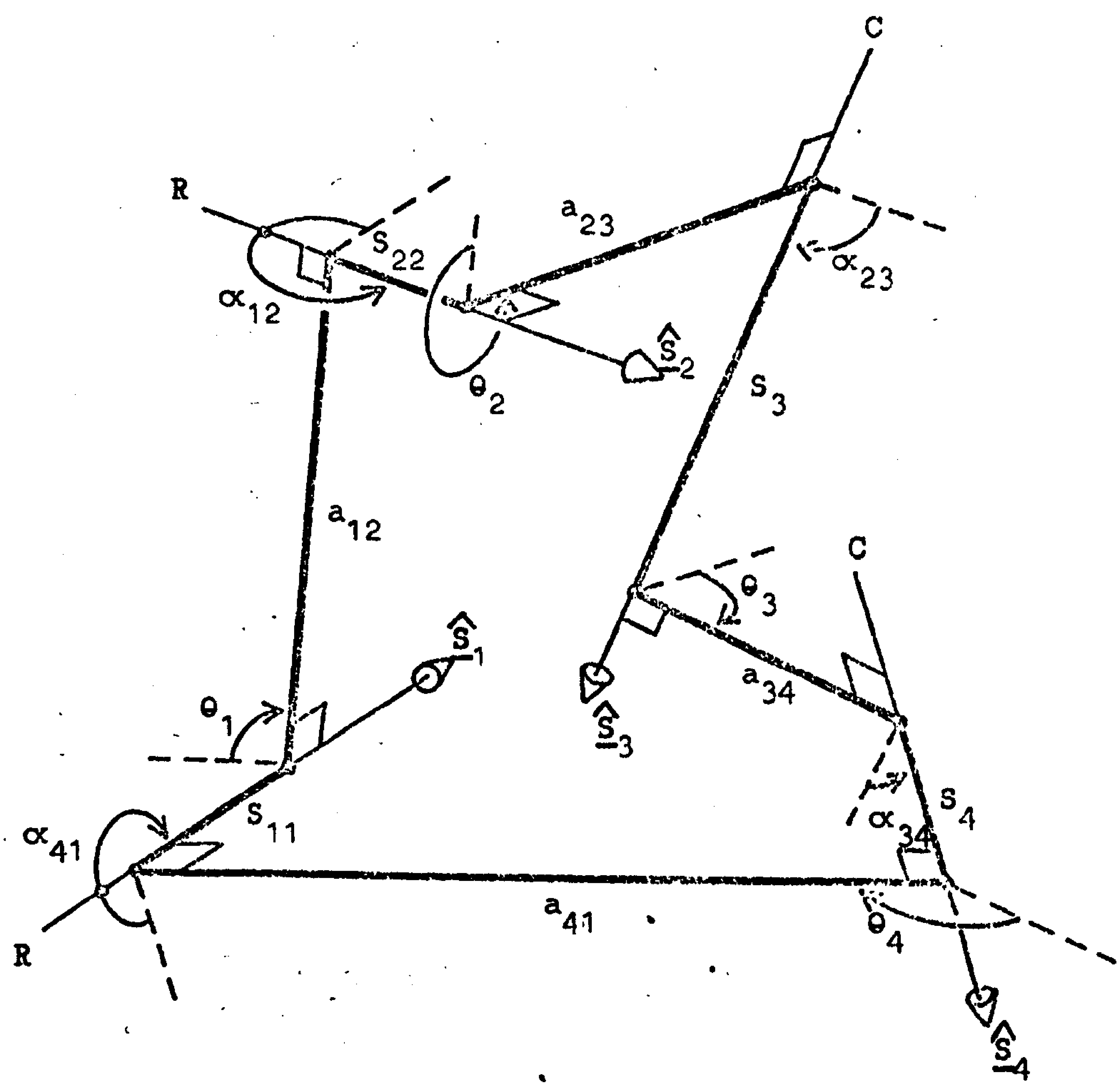
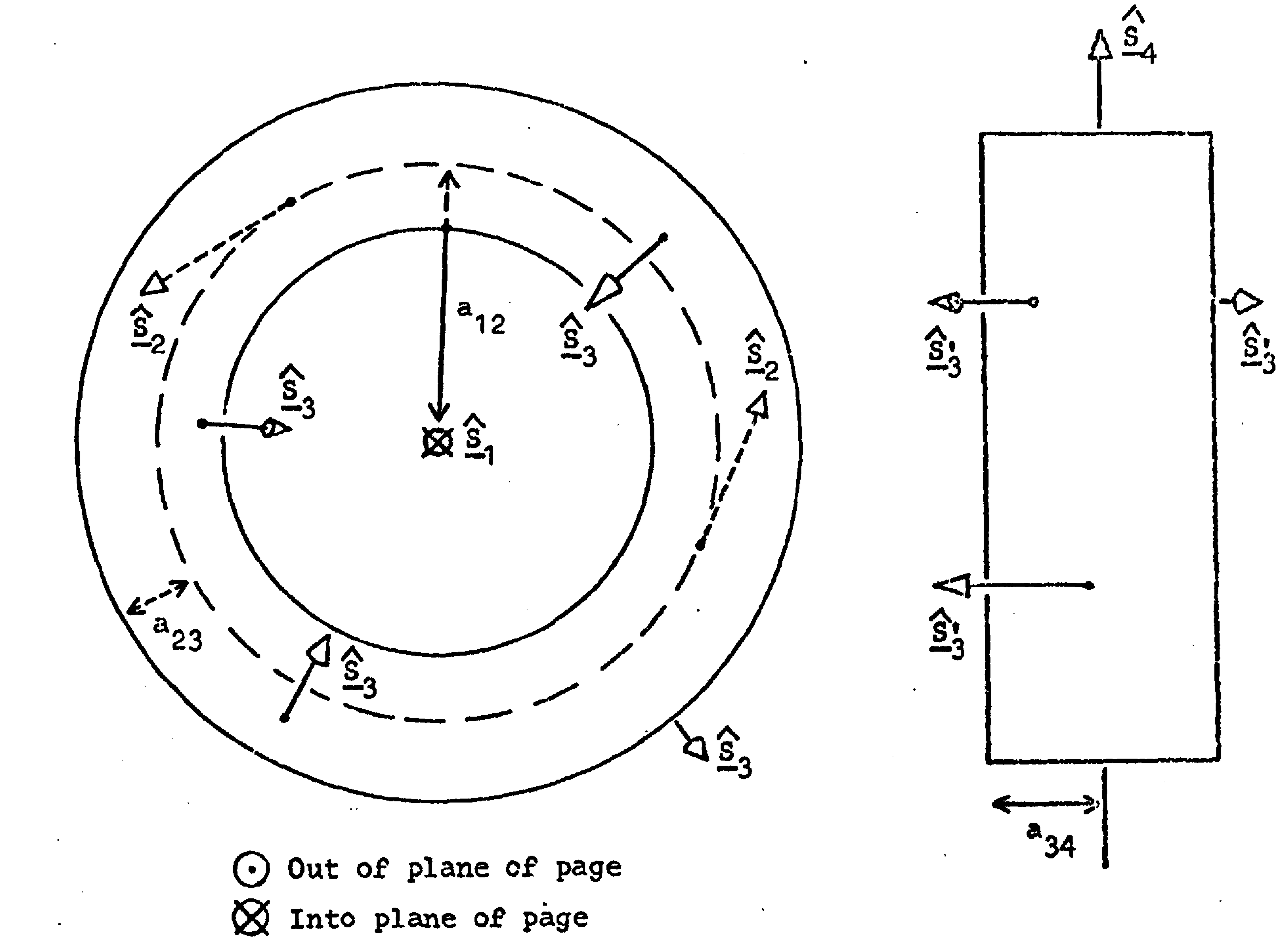
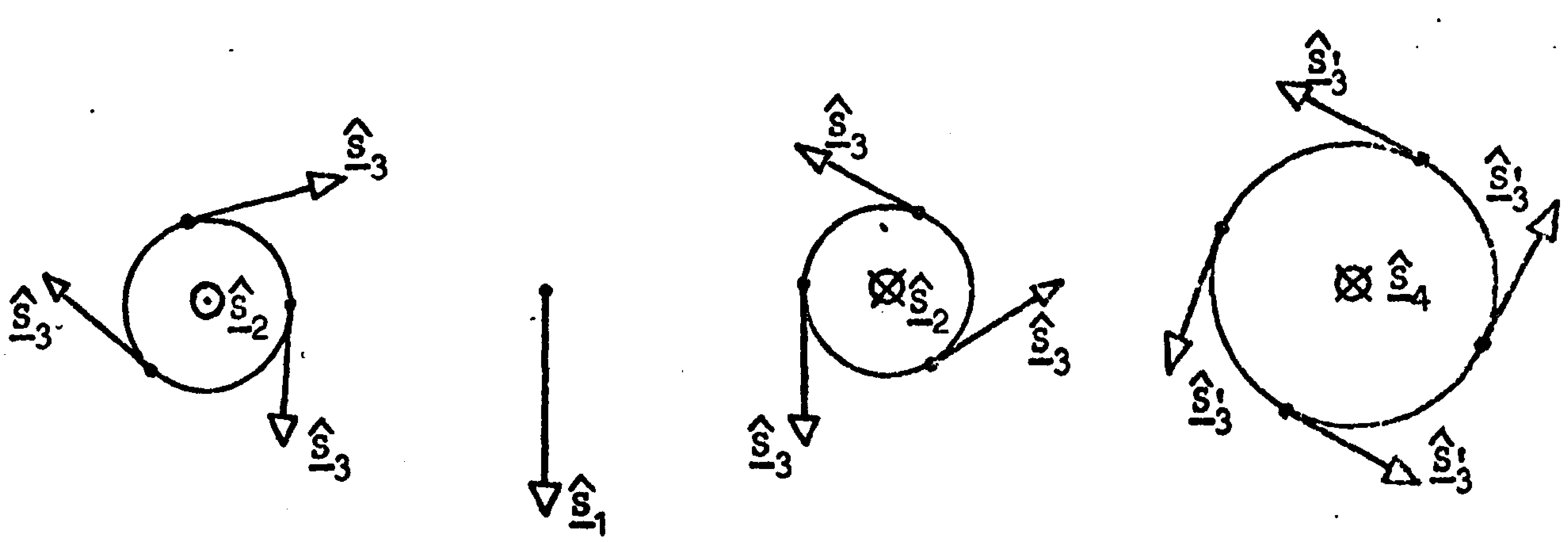


Figure 2.15 Representation of the Four-Link RRCC Structure.

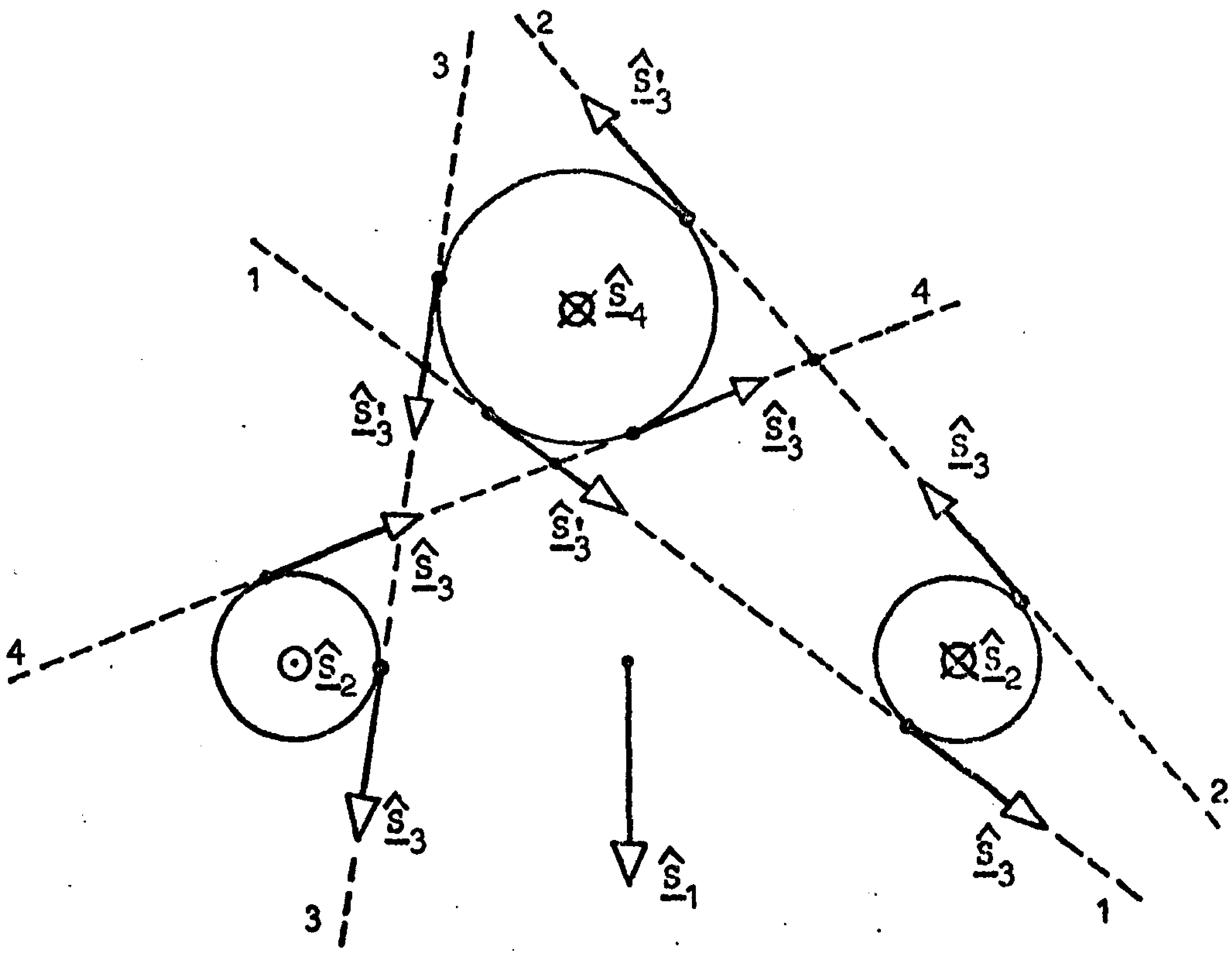


(a) Top View.

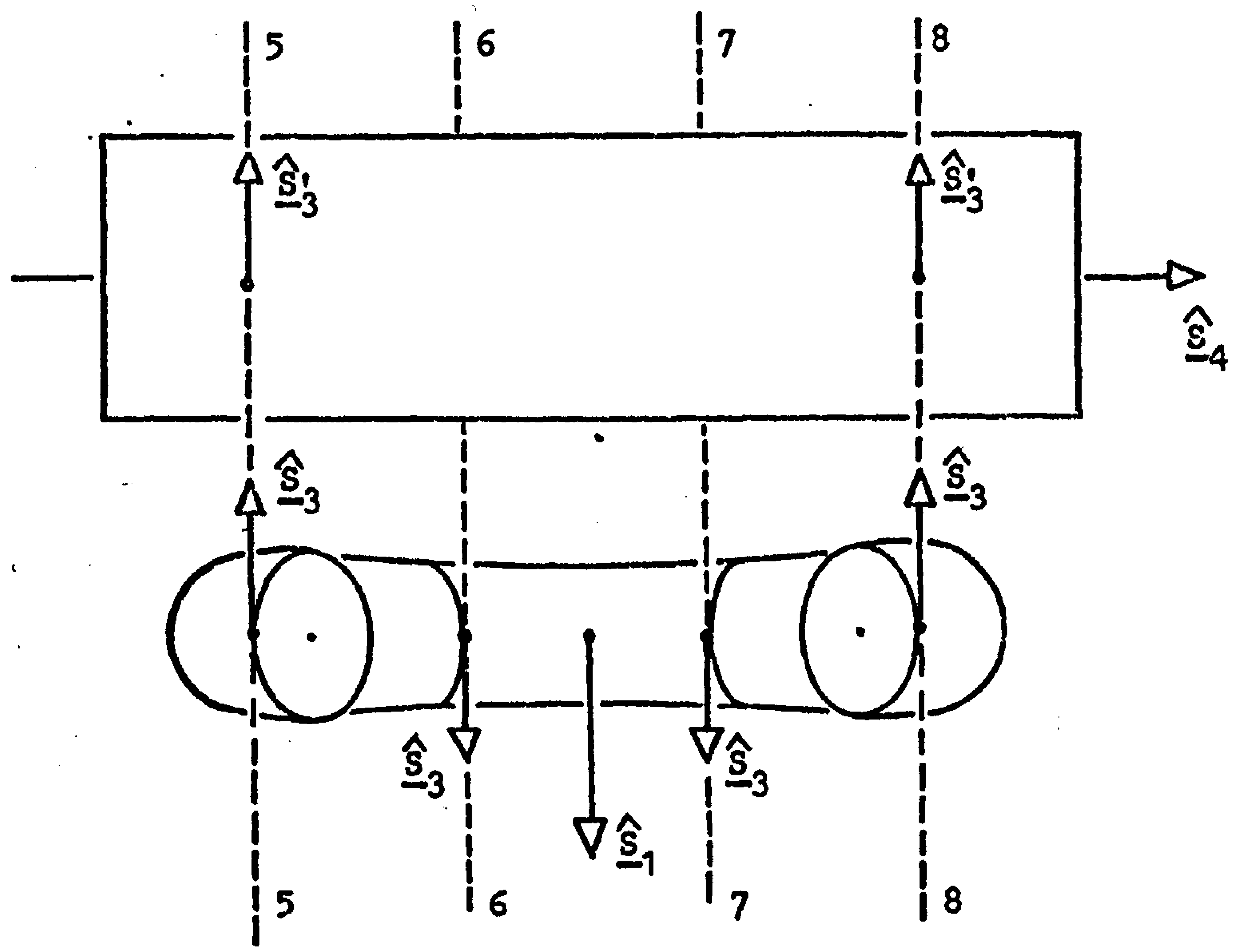


(b) Side Cross-Section.

Figure 2.16 The Right Circular Torus with Circular Cross-Section and the Right Circular Cylinder Produced by the RRCC Spatial Structure.



(a) End Cross-Section Showing First Four Assemblies.



(b) Side View Showing Second Four Assemblies. (Part of the torus is cut away for clarity).

Figure 2.17 The Eight Assemblies (broken lines 1-8) Derived from the Eight Common Directed Lines between the Cylindrical and the Toroidal Line Vector Fields Produced by the RRCC Spatial Structure.

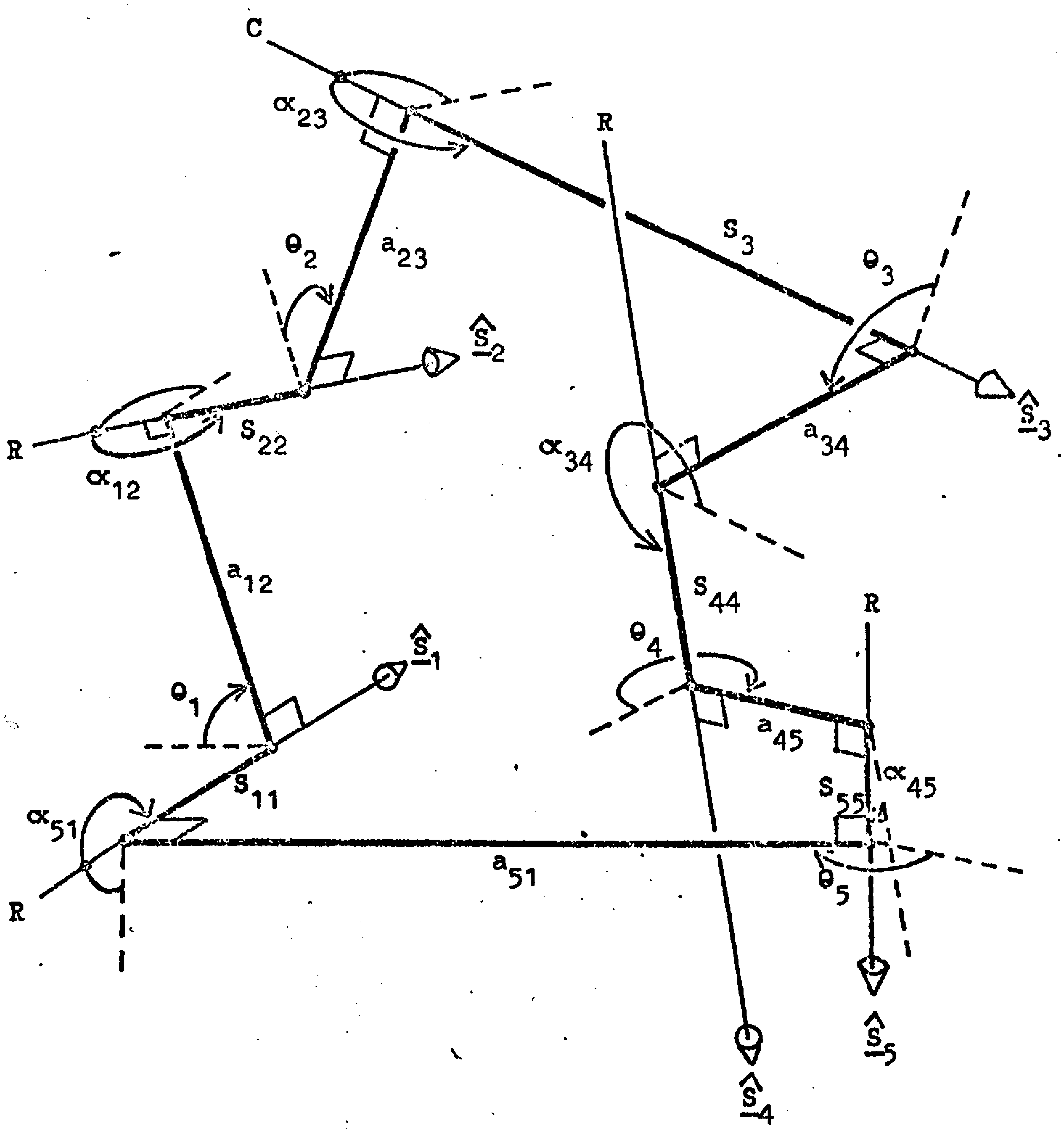
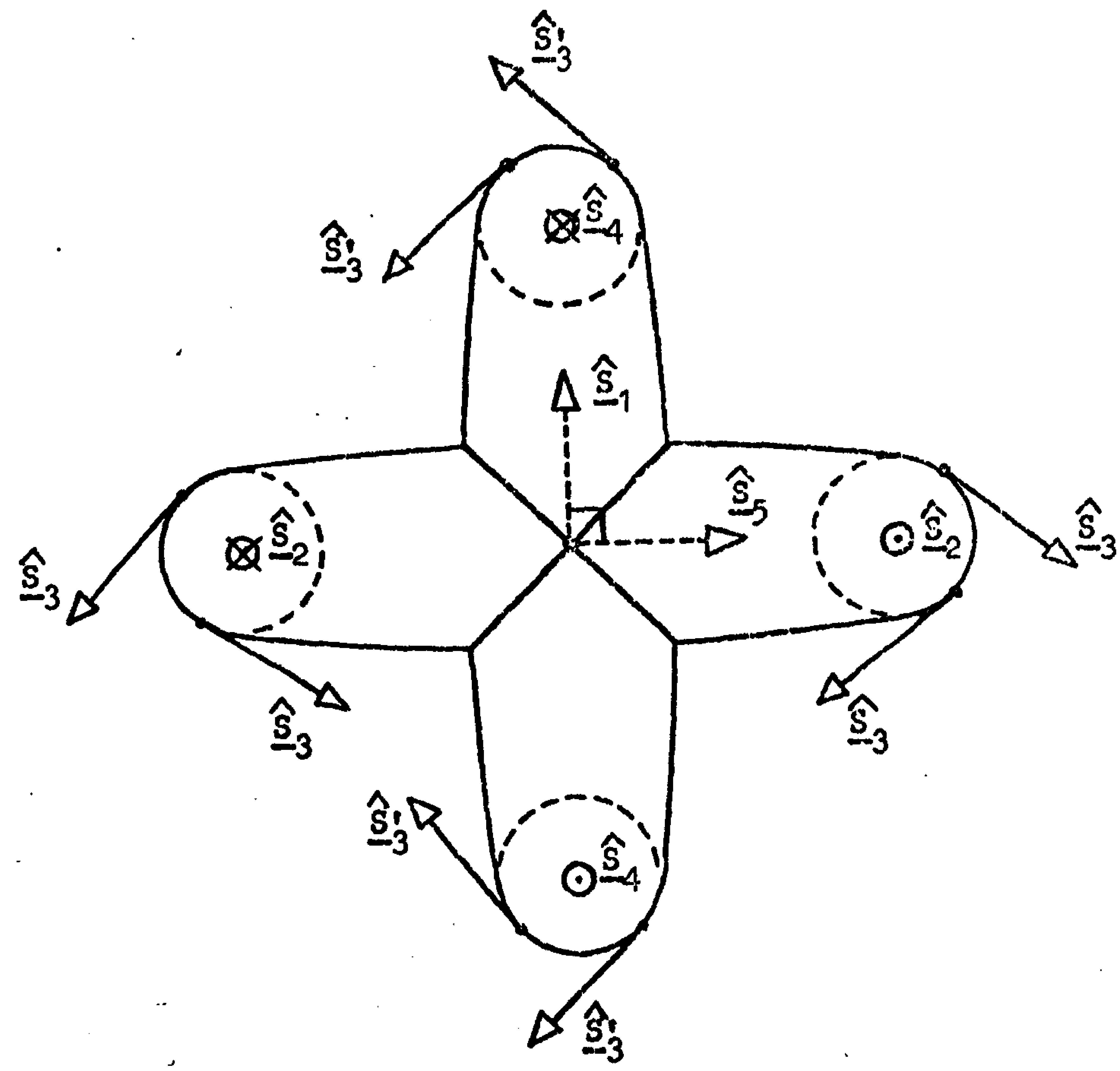
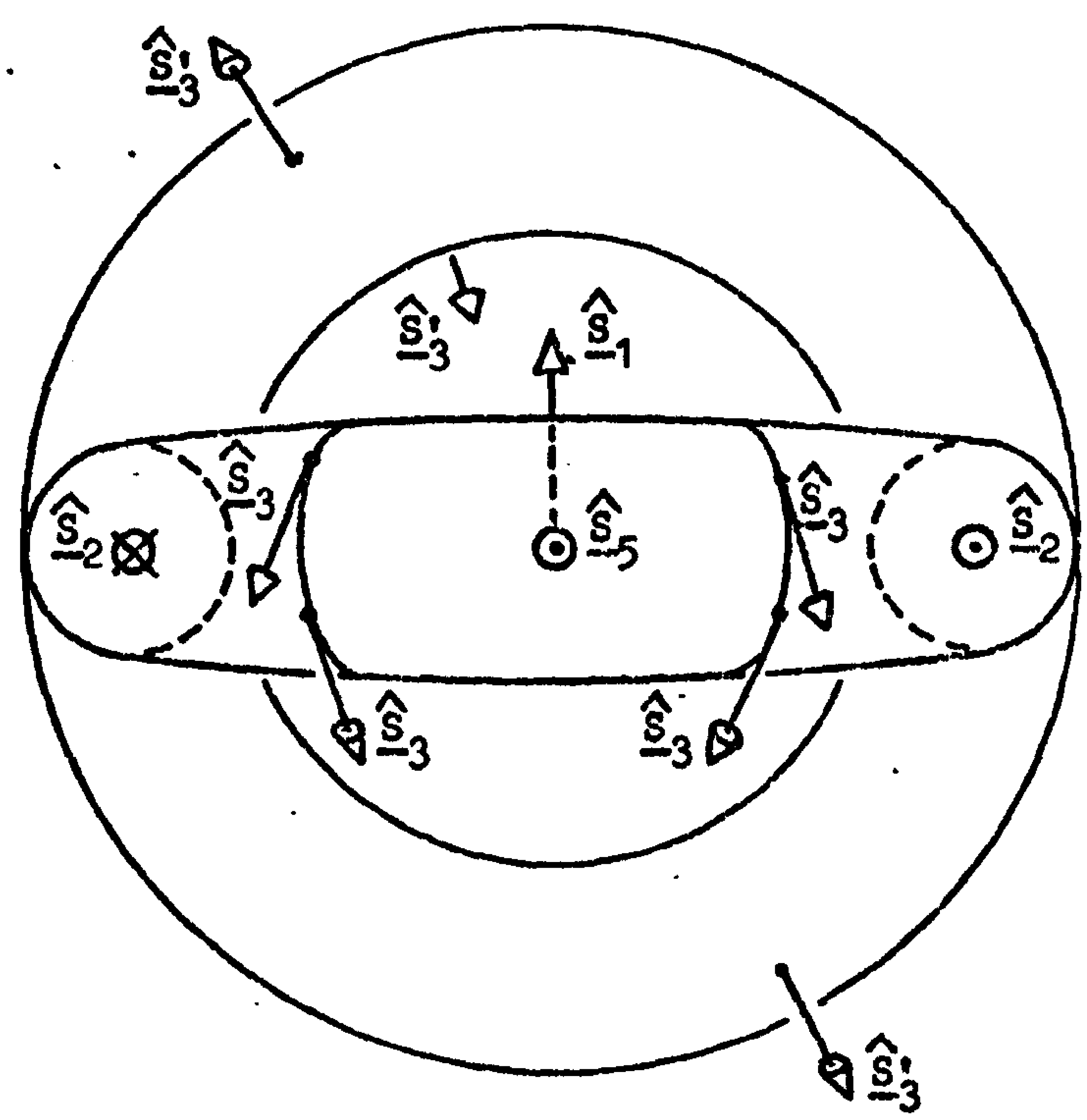


Figure 2.18 Representation of the Five-Link RRCRR Structure.

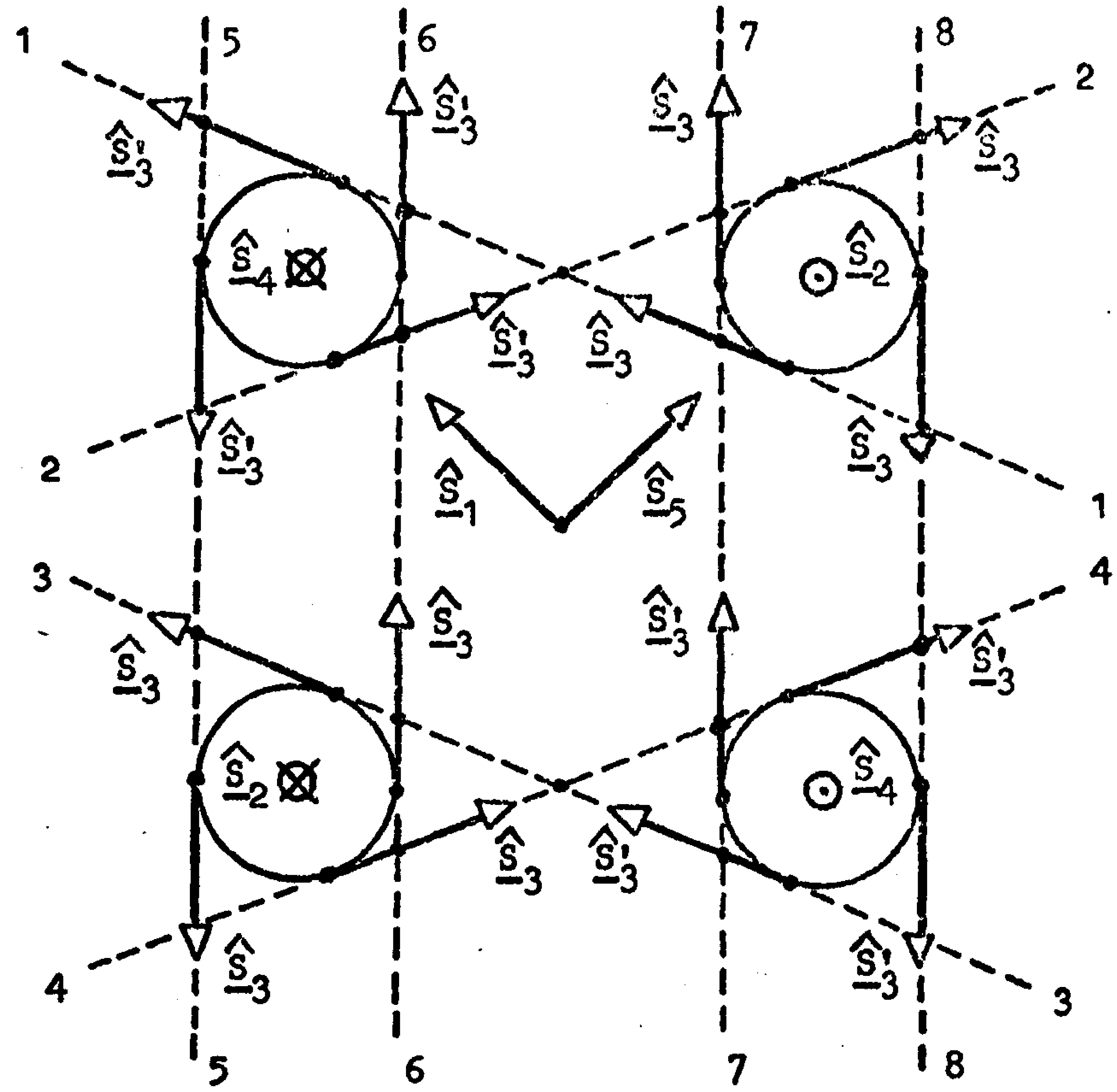


(a) End View.

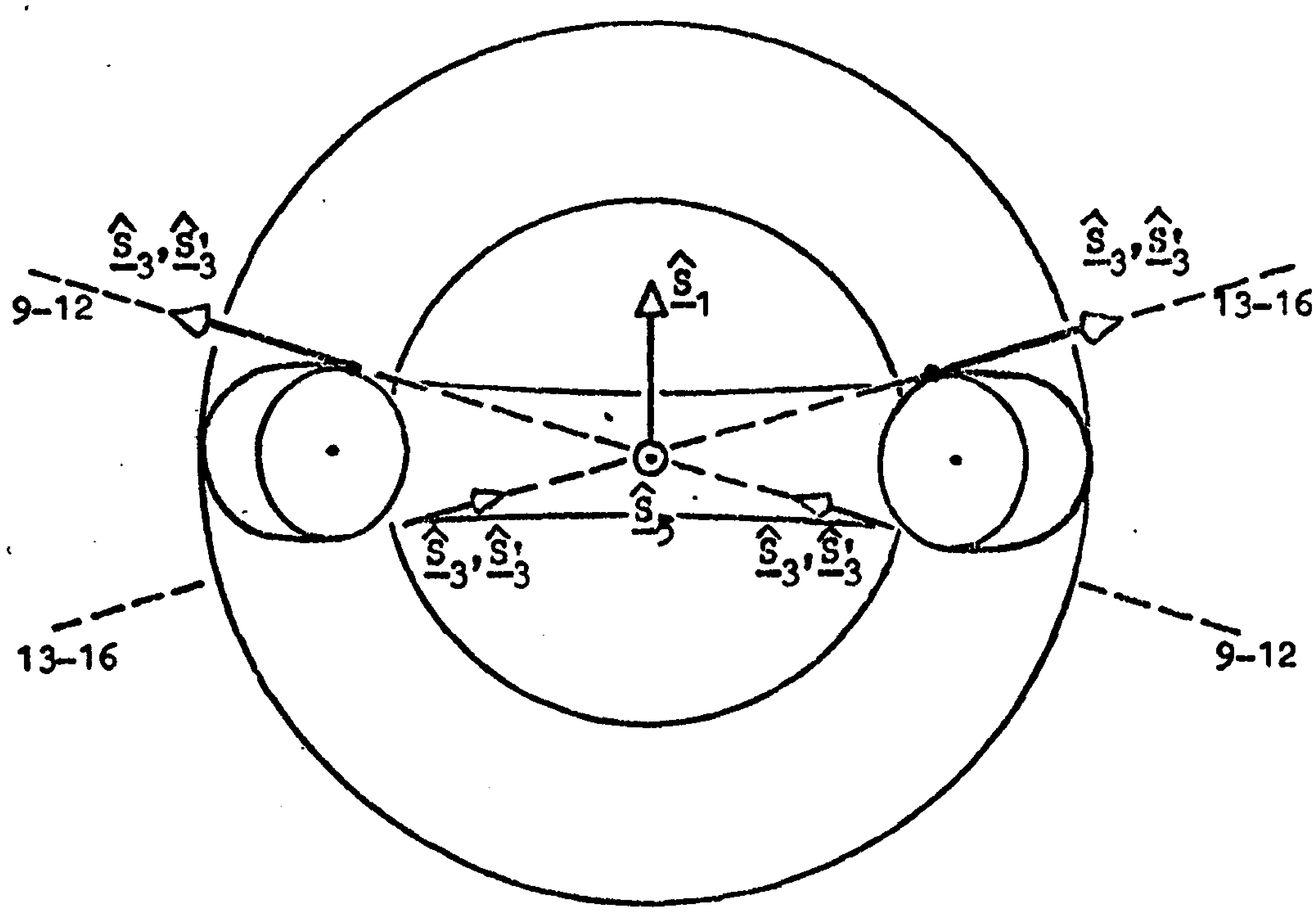


(b) Side View.

Figure 2.19 The Two Right Circular Torii with Circular Cross-Sections Intersecting at Right Angles Generated by the RRCRR Spatial Structure.



(a) Central Cross-Section Showing First Eight Assemblies.



(b) Side View Showing Second Eight Assemblies.
(Part of one torus is cut away for clarity).

Figure 2.20 The Sixteen Assemblies (broken lines 1-16) of the RRCRR Spatial Structure Derived from the Sixteen Common Directed Lines between Two Torii.

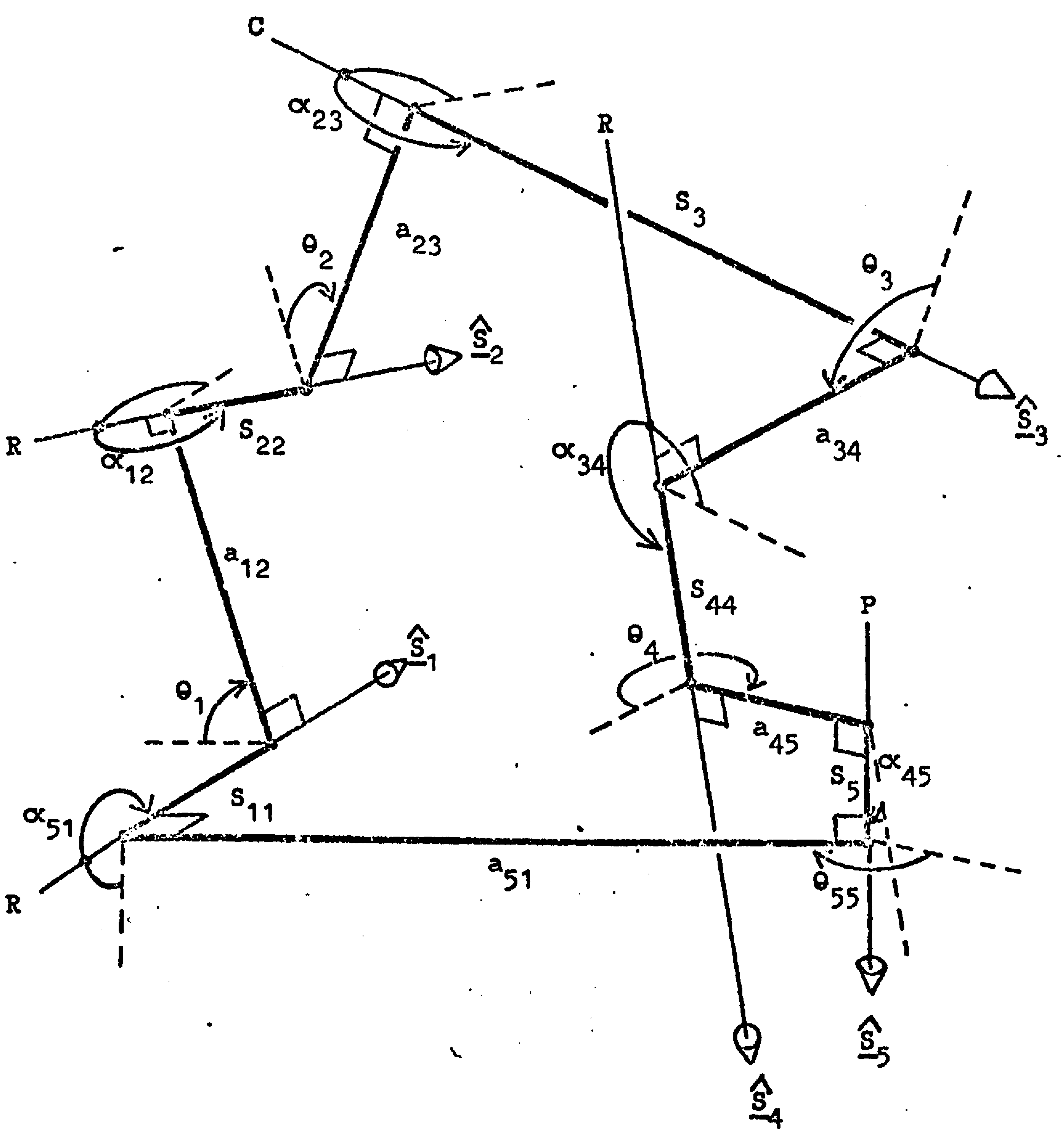


Figure 2.21 Representation of the Five-Link RRCRP Structure.

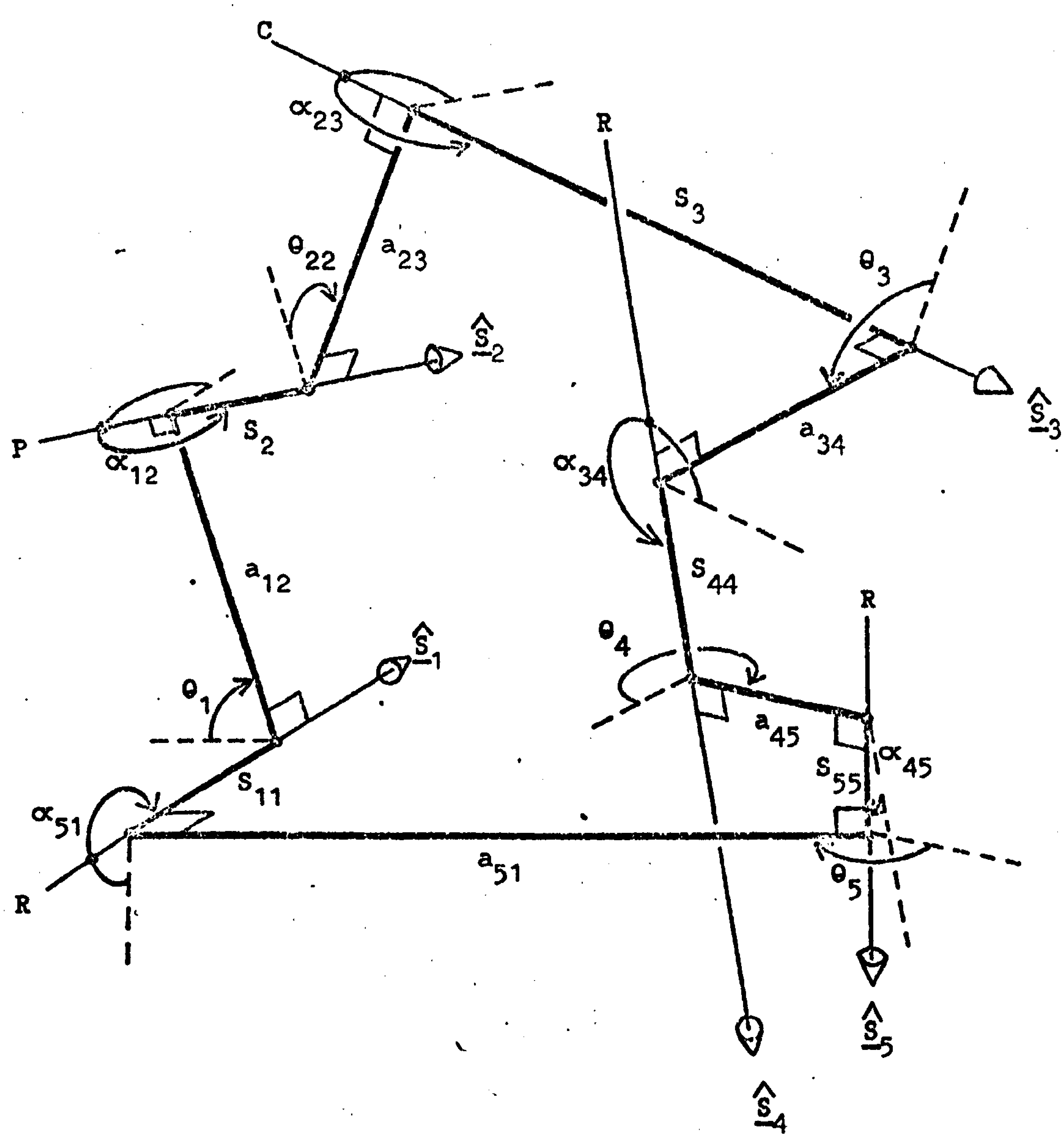


Figure 2.22 Representation of the Five-Link RPCRR Structure.

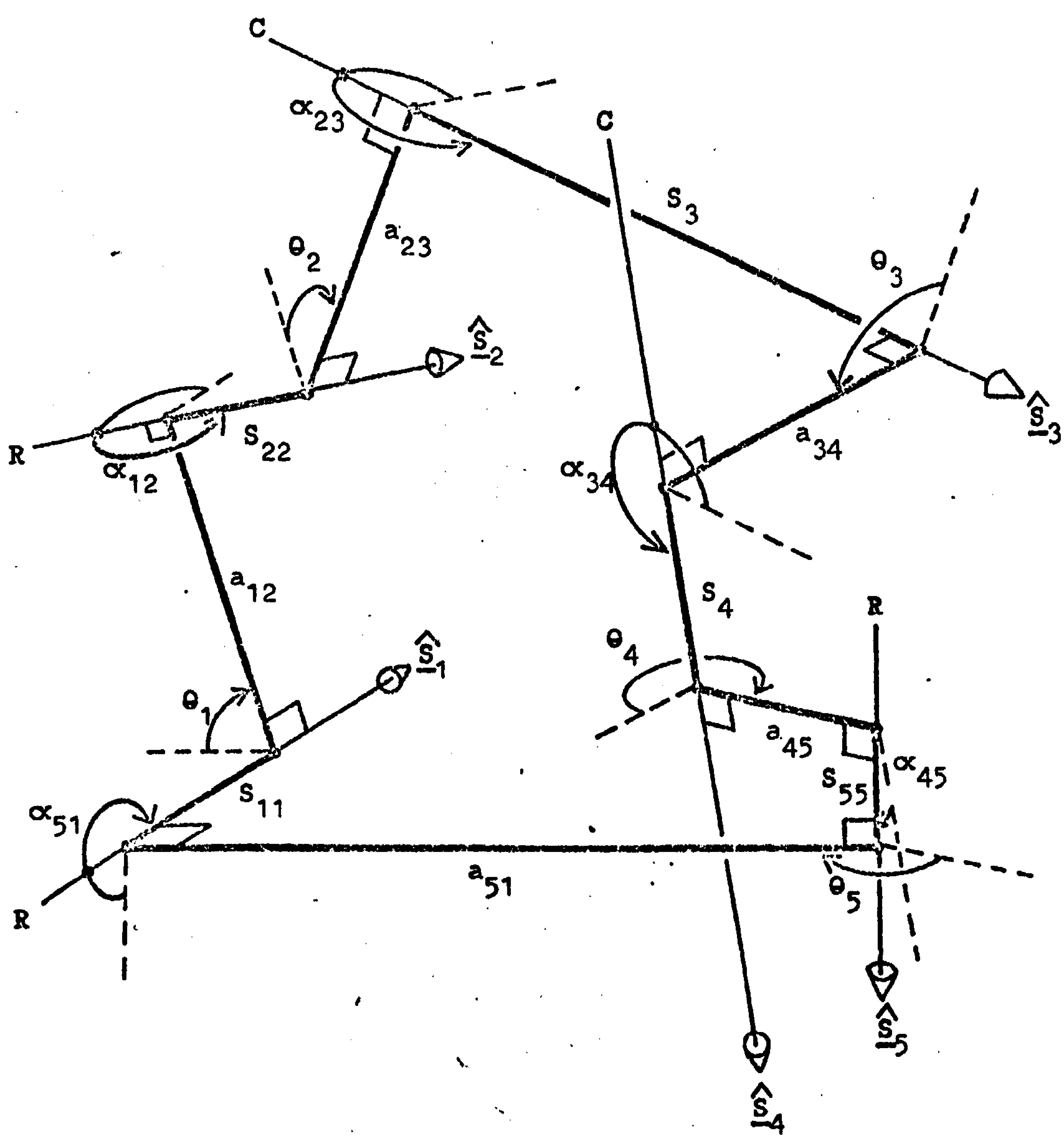


Figure 2.23 Representation of the Five-Link RRCCR and RRRCC Mechanisms.

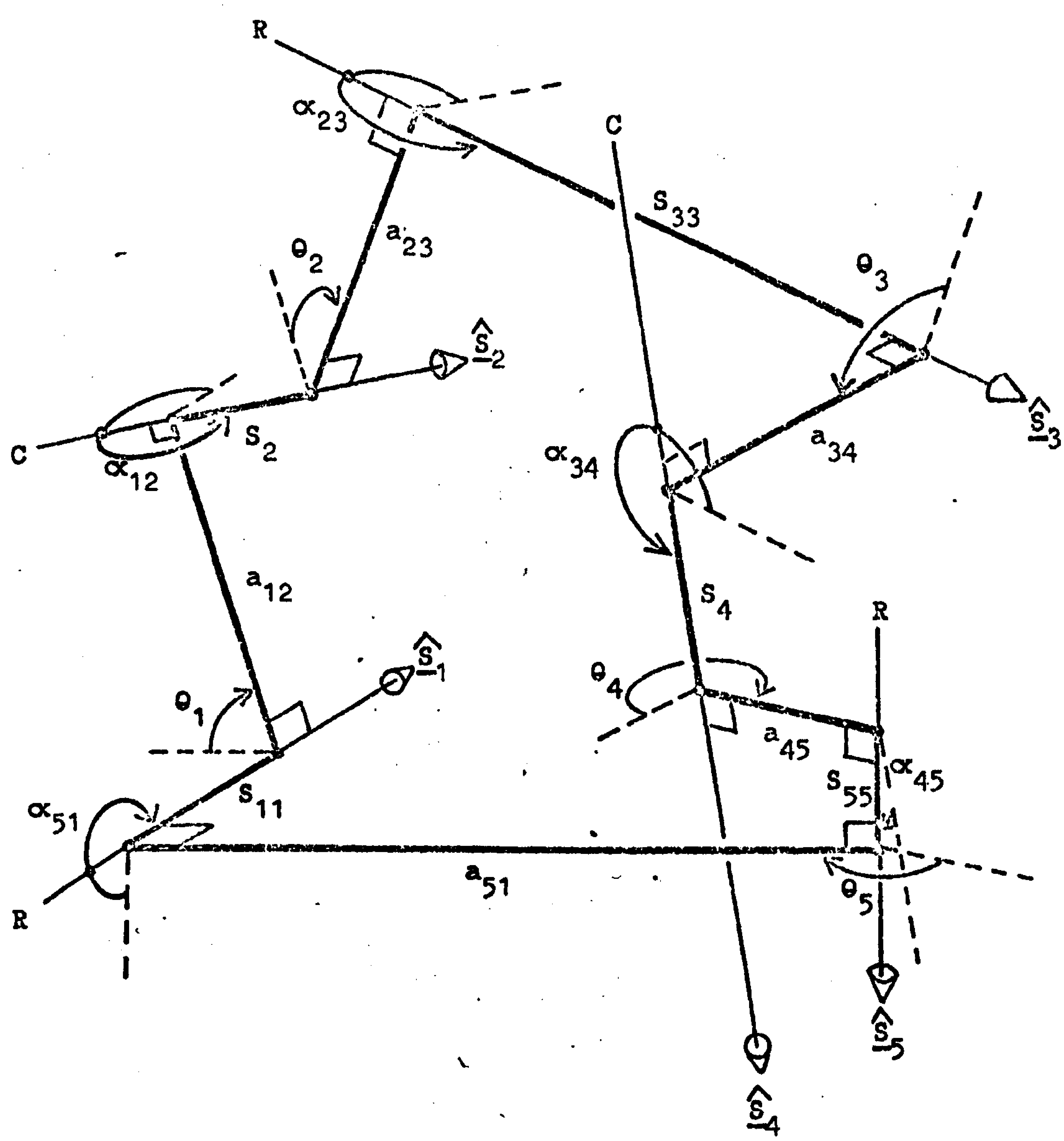


Figure 2.24 Representation of the Five-Link RCRCR, RRCRC and RCRRC Mechanisms.

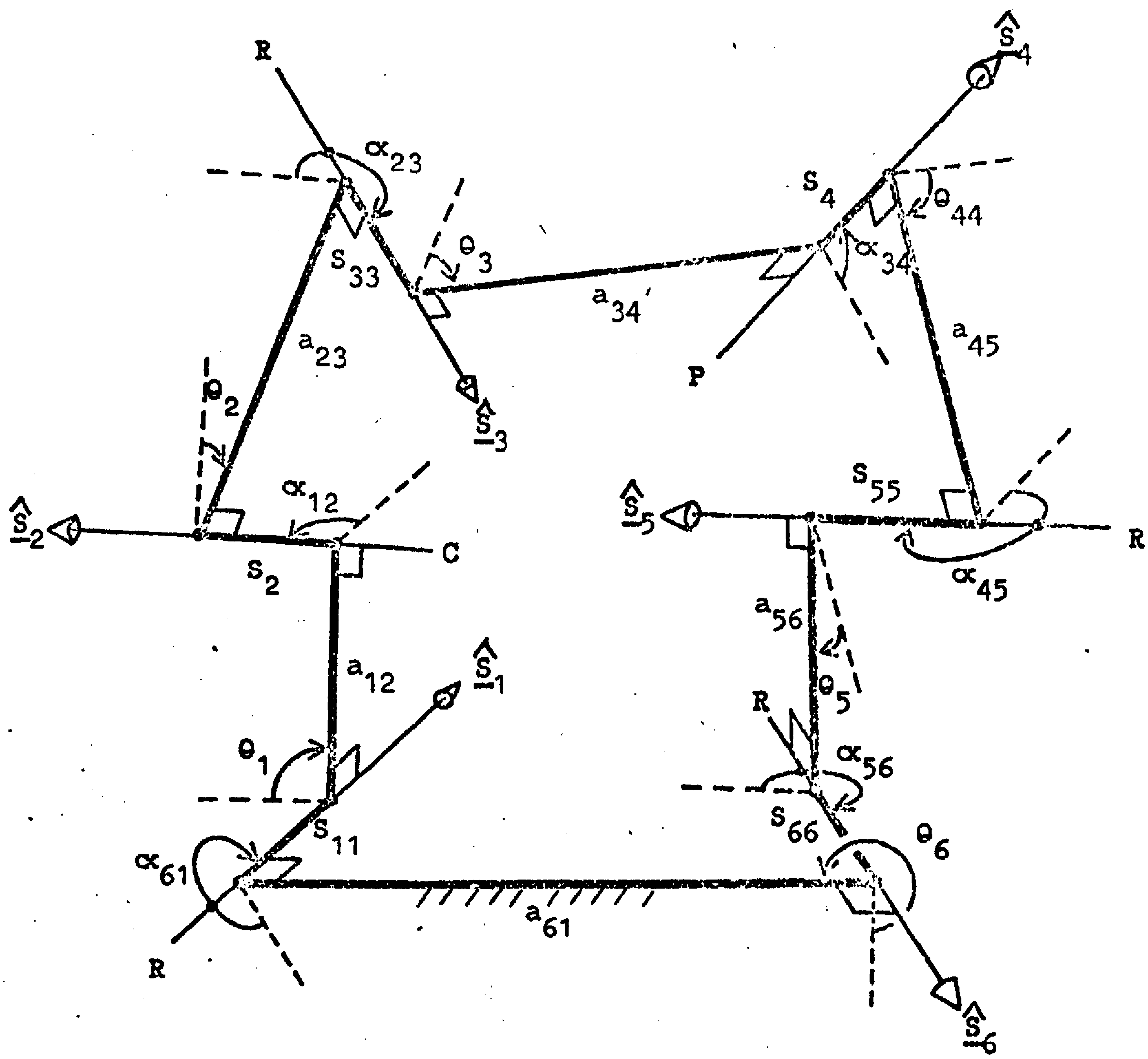


Figure 2.25 Representation of the Six-Link RCRPRR Mechanism.

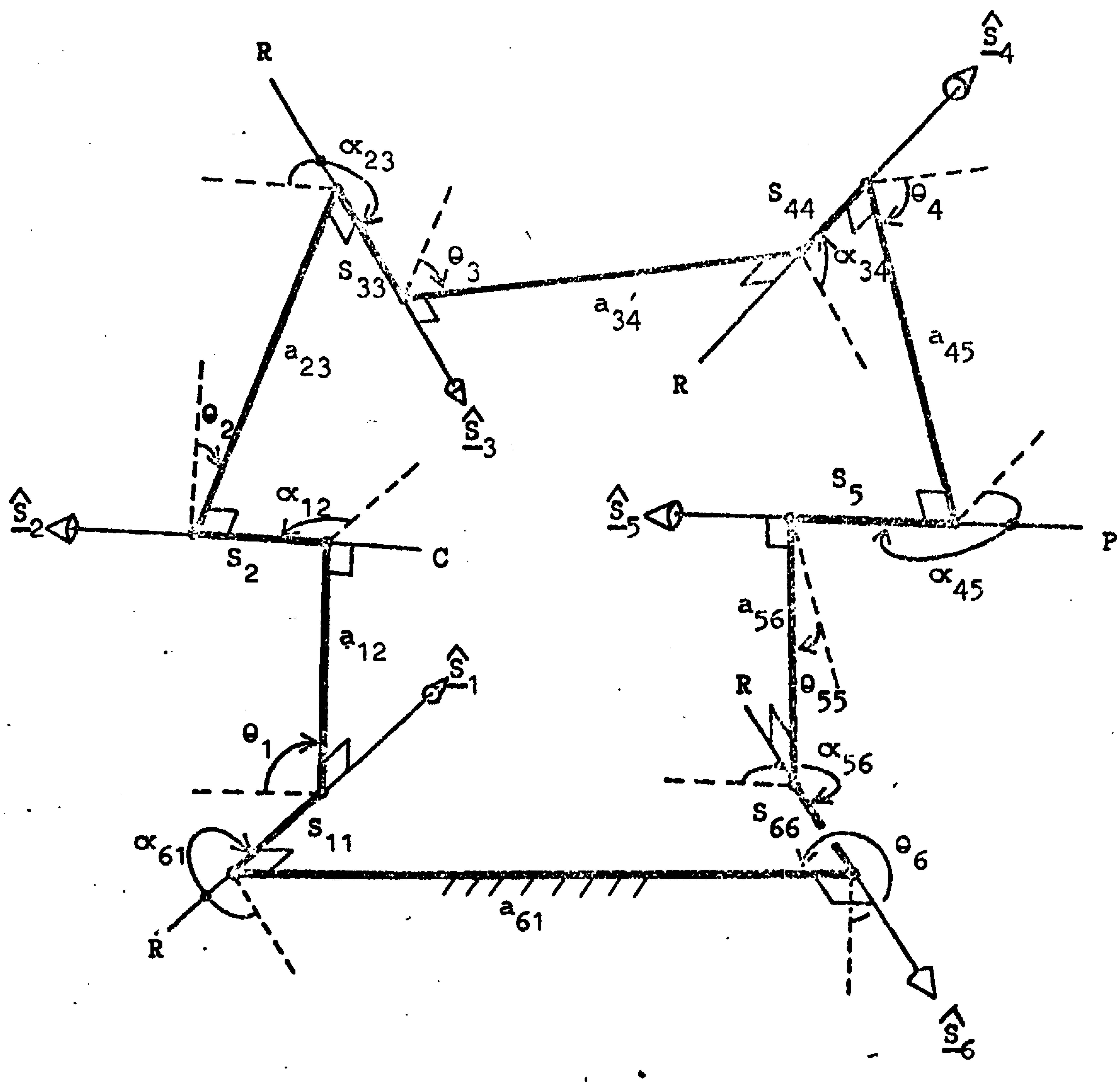


Figure 2.26 Representation of the Six-Link RCRRPR Mechanism.

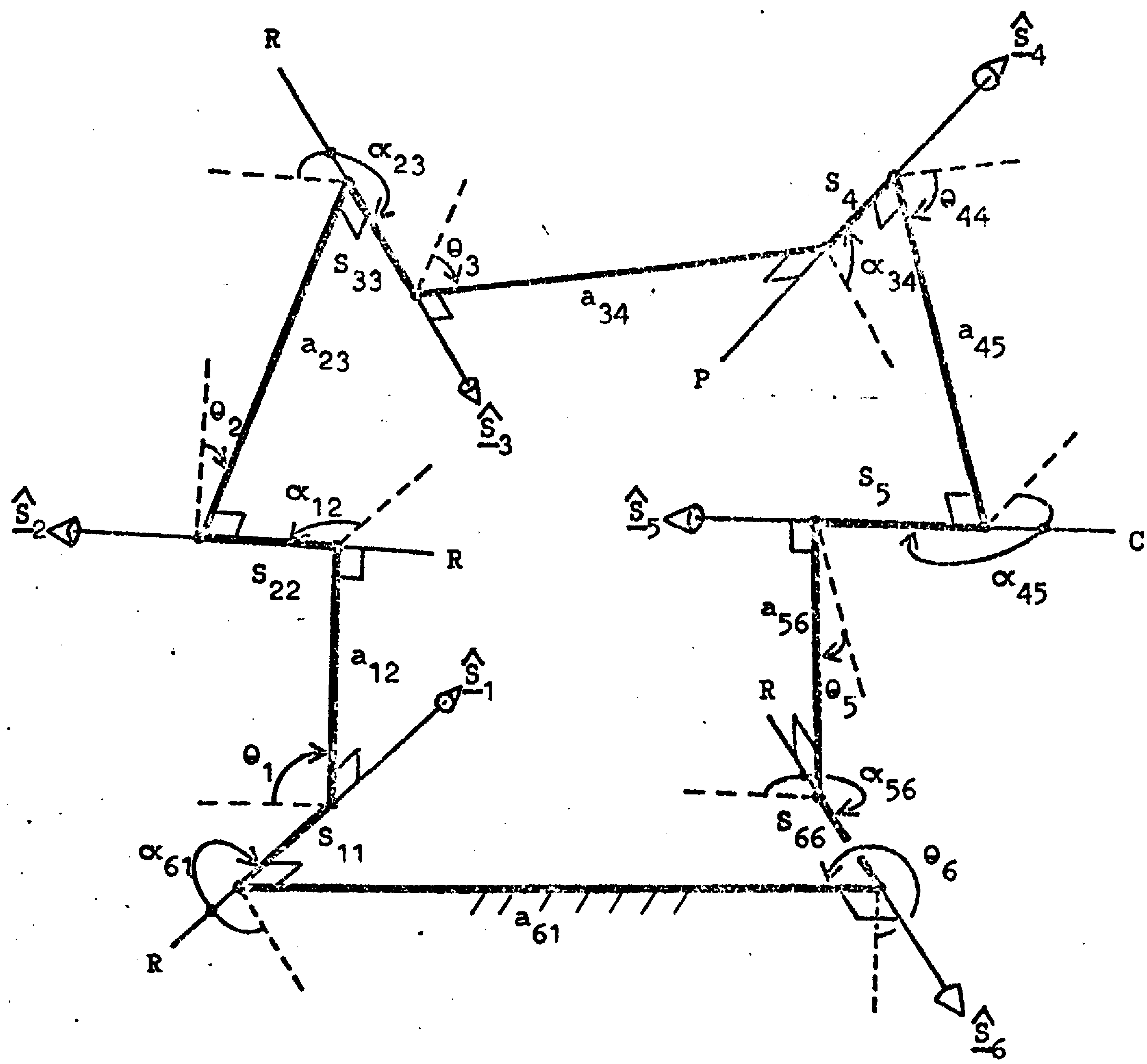


Figure 2.27 Representation of the Six-Link RRRPCR Mechanism.

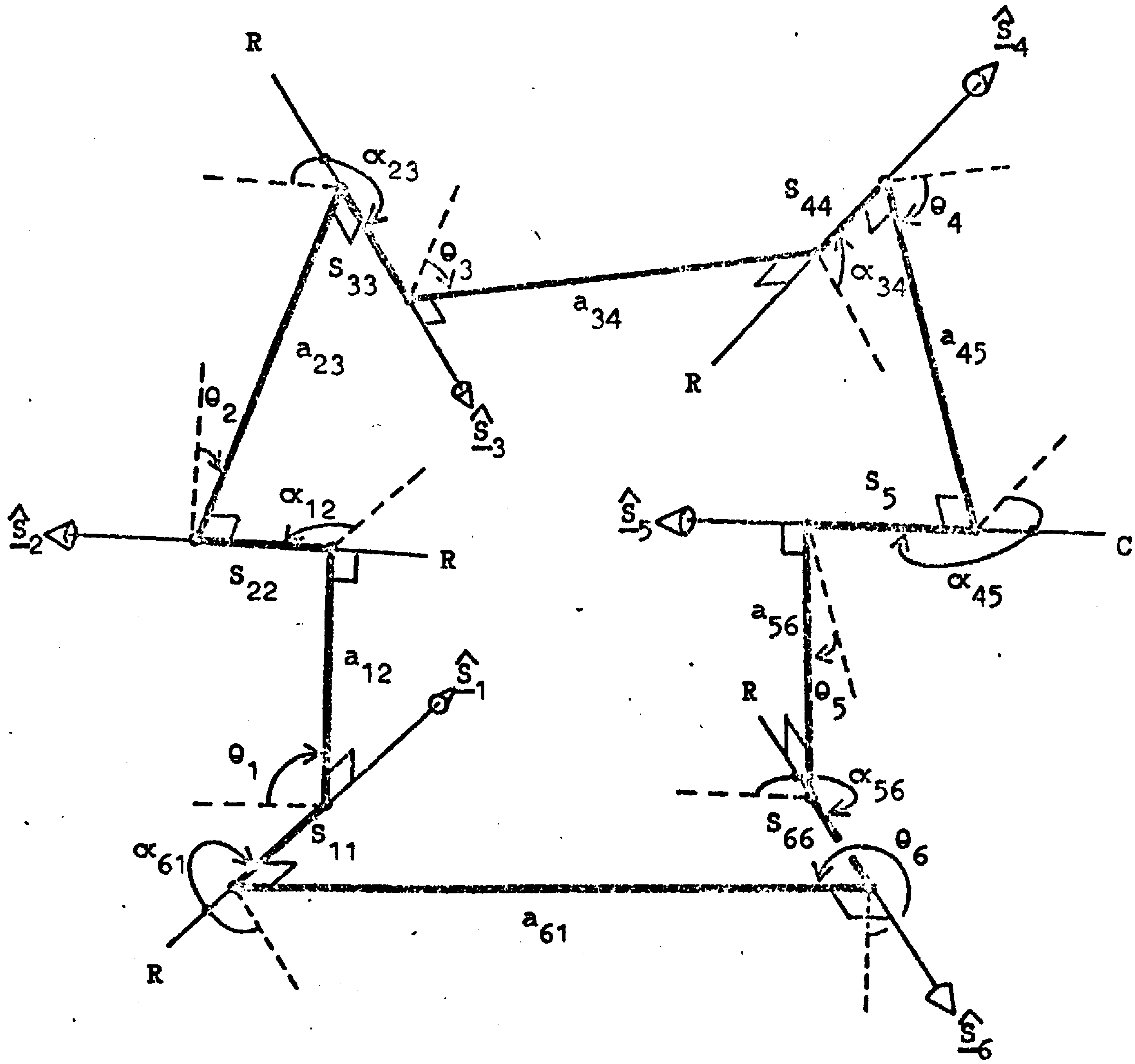


Figure 2.28 Representation of the Six-Link 5R-C Mechanisms.

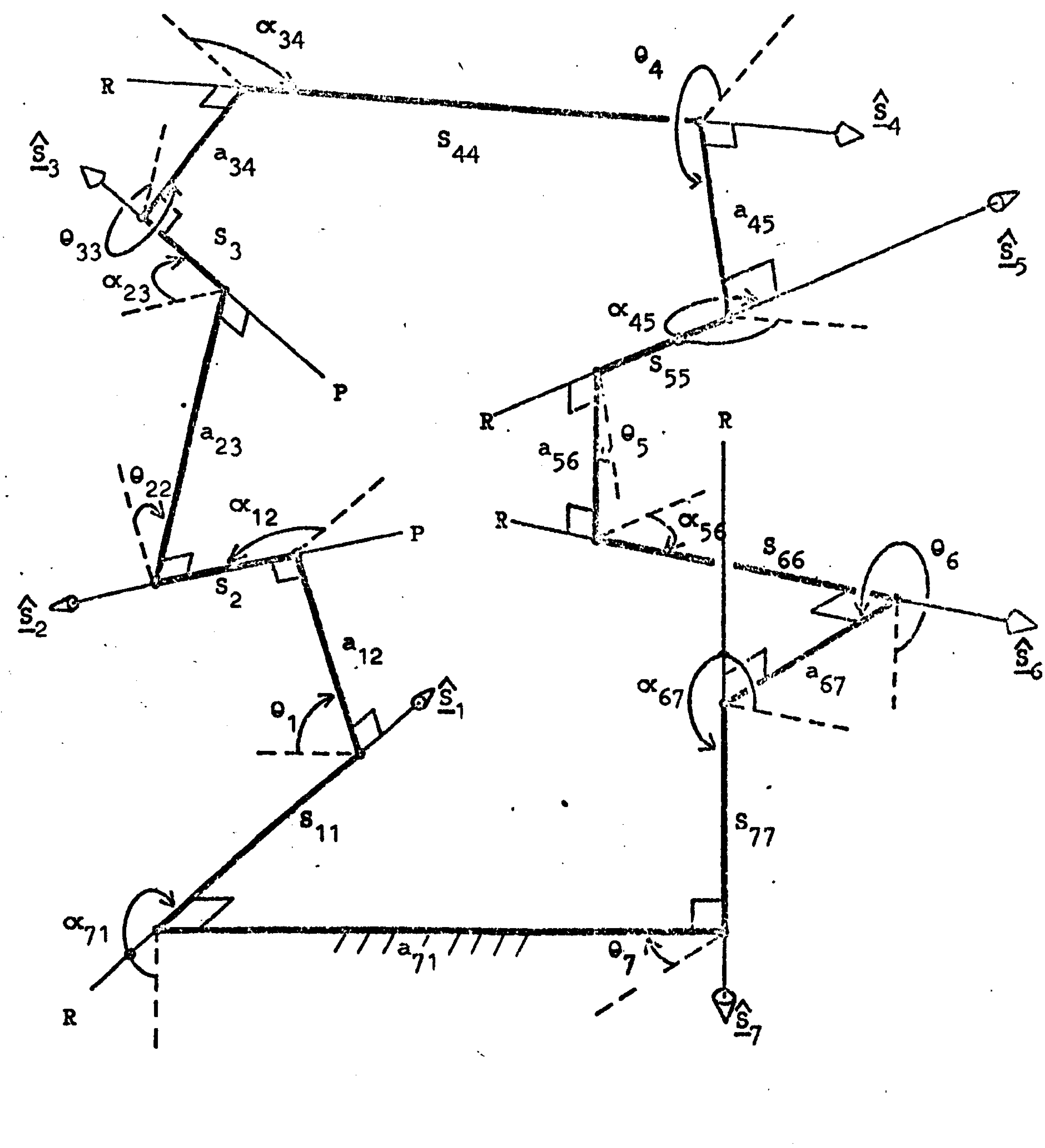


Figure 2.29 Representation of the Seven-Link RPPRRRR Mechanism.

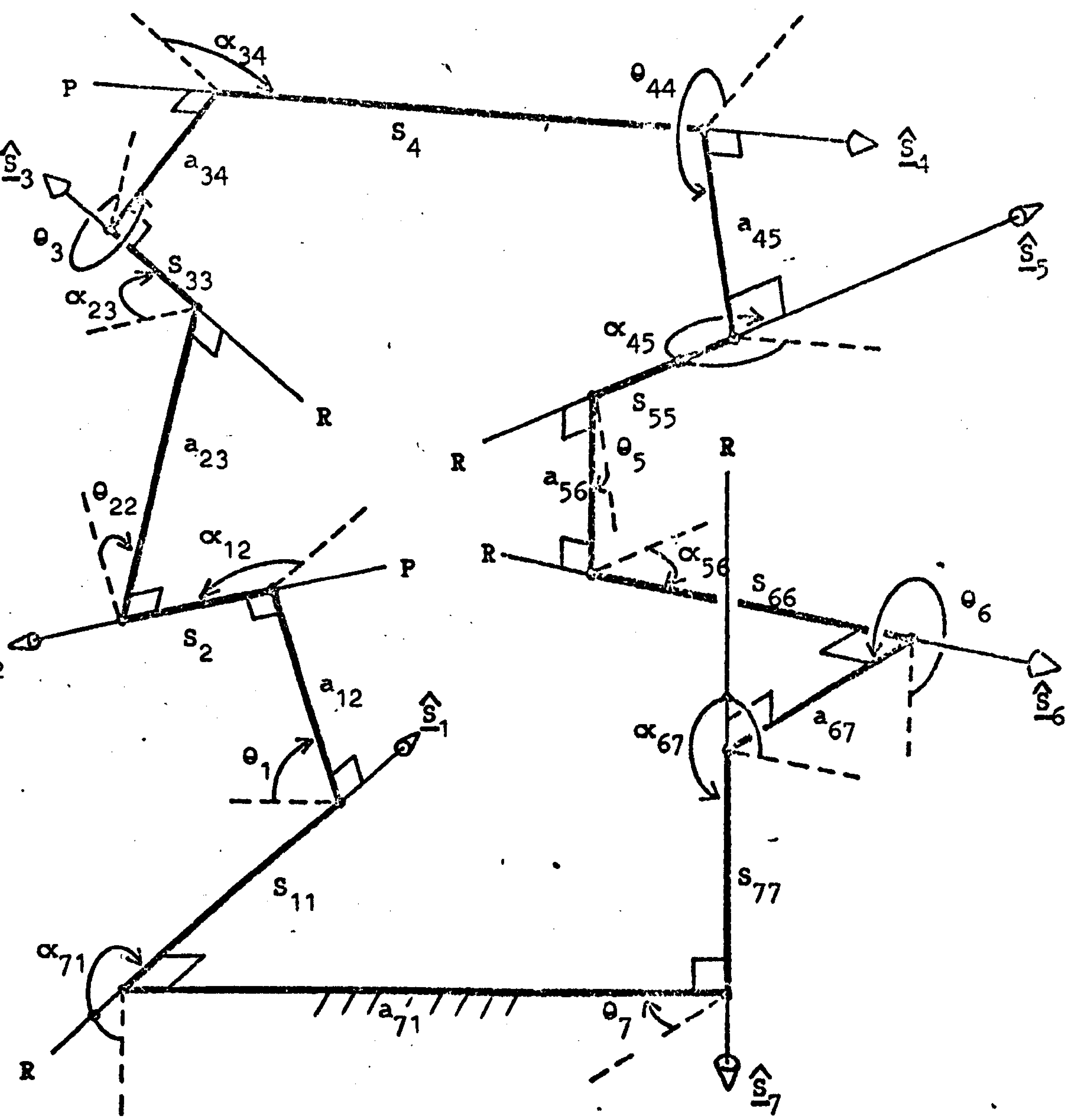


Figure 2.30 Representation of the Seven-Link RPRPRRR Mechanism.

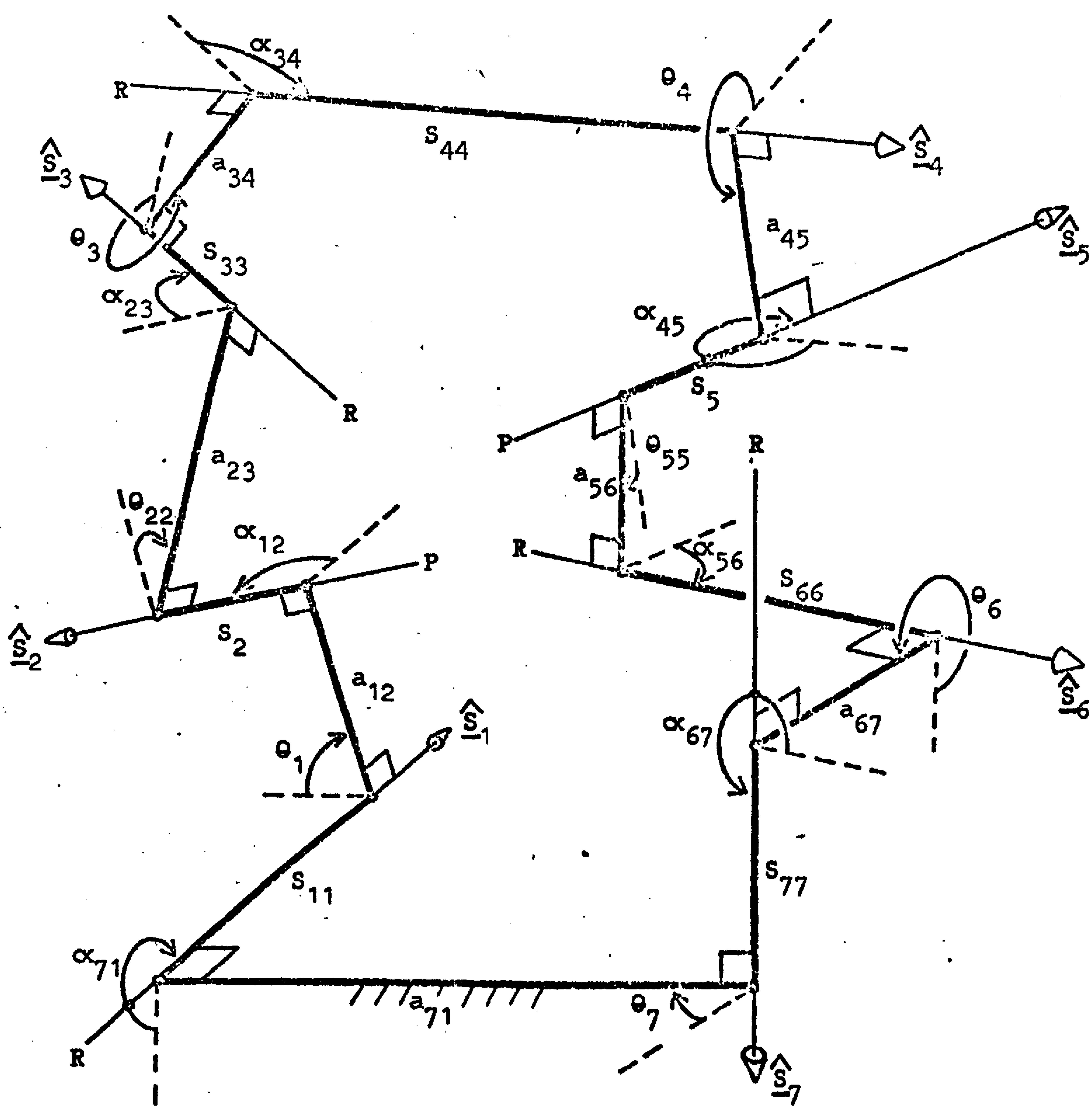


Figure 2.31 Representation of the Seven-Link RPRRPRR Mechanism.

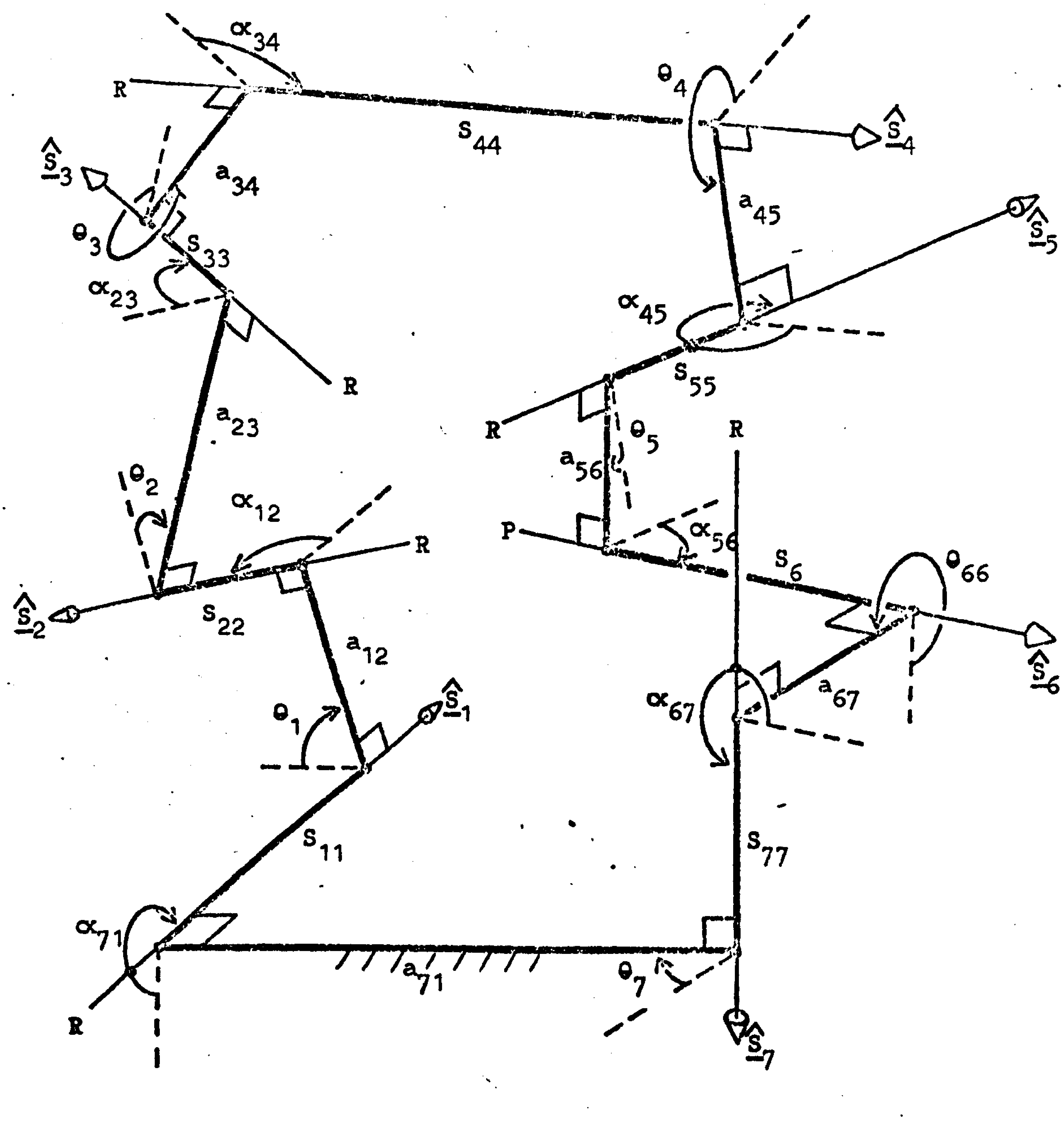


Figure 2.32 Representation of the Seven-Link RRRRRPR Mechanism.

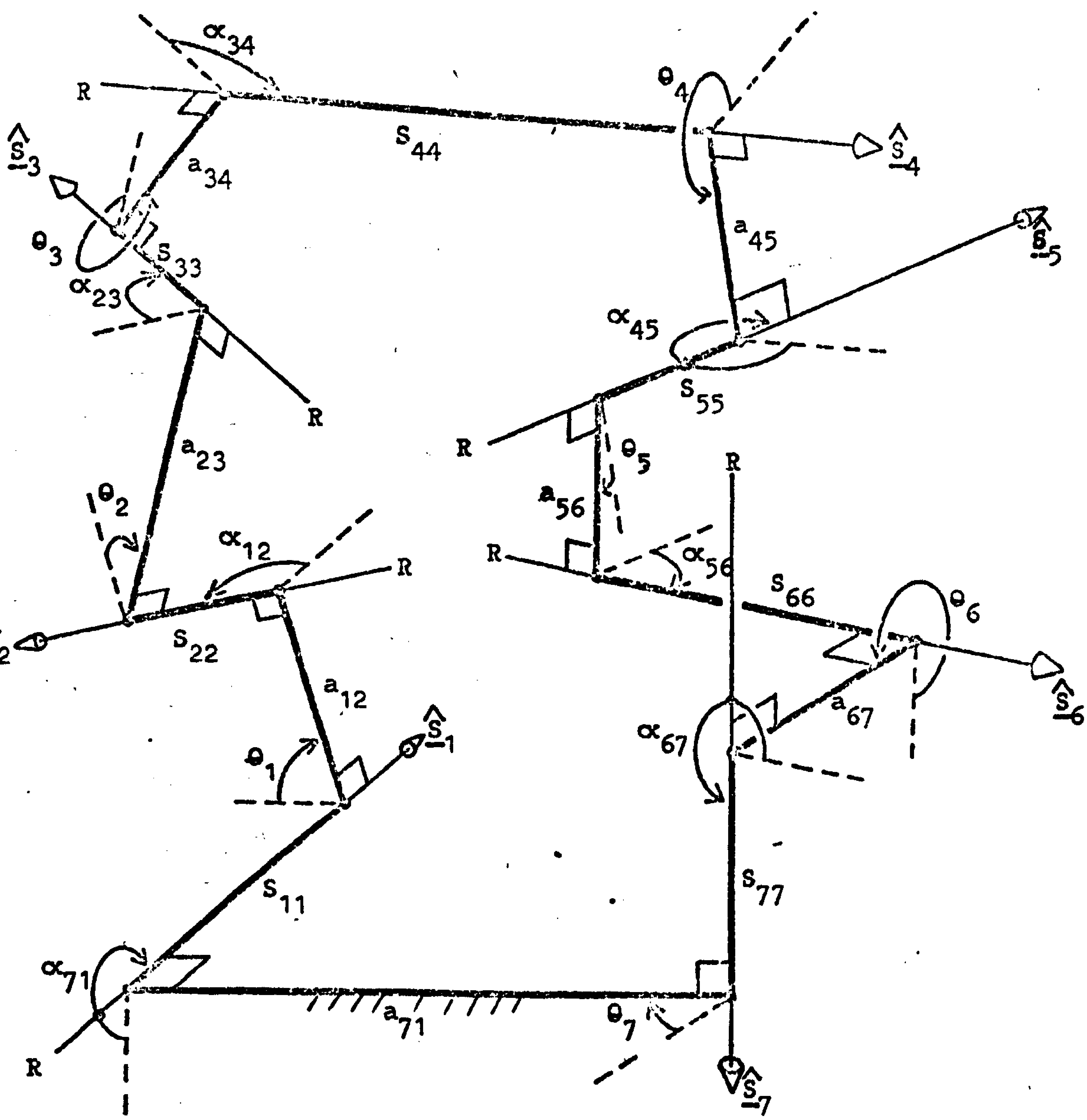


Figure 2.33 Representation of the Seven-Link RRRRRRR Mechanism.

CHAPTER 3

DUAL NUMBERS

AND

DUAL VECTORS

3.1 Introduction.

In the previous chapter the concepts of 'line' and 'line vector' were used in an intuitive and geometrical sense to represent the pair axes of spatial mechanisms and structures. However, in order to define such terms rigorously and to develop a proper algebraic representation for such quantities, it is necessary to investigate the nature of vectors in general. Thus the distinction between a line vector and a geometric (or free) vector must be appreciated, and, in addition, an understanding of the concepts of dual number, dual angle and dual vector is required.

Because a single comprehensive treatise on these subjects does not appear to be available in the literature, the author has correlated much relevant information on the matter and presented this, together with his own thoughts in the following sections. This work has led to a novel and concise proof of the Principle of Transference, amongst other interesting properties and representations of the dual plane.

3.2 Arrows.

A free vector is a quantity having both magnitude and direction but no specific location in space. In this respect it is not identified with any particular line segment but represents the collection of all line segments with the same direction and magnitude. Thus, following reference [29], one can define an arrow as a quantity having both magnitude (length) and direction, but also having a definite position. It can be represented by a fixed directed line segment with an arrow-head defining its direction. The length of the segment then represents the arrow's magnitude, whilst the 'blunt-end' defines its position. This is illustrated by Figure 3.1(a) which shows several arrows with various magnitudes, directions and positions. (They may represent the velocity at different points in a fluid, for example).

An arrow may be denoted by the symbol, $\bar{a}$, and two such arrows, $\bar{a}$ and $\bar{b}$, are considered to be equal if and only if they have the same length, direction and position. Thus any two of the arrows shown in Figure 3.1(b) are distinct, although they have the same length and direction.

It may be noted that there are ∞^6 possible arrows in three-dimensional Euclidean space, since there is a possible ∞^3 at each point, and an ∞^3 of points. Clearly, therefore, arrows are not the familiar quantities known as geometric vectors and can only be combined when at the same point.

3.3 Geometric Vectors.

The set of all arrows having the same magnitude and direction, but with different positions, is of considerable interest in describing, say, the displacement of a body in space (i.e. a translation), since all the points of that body will be displaced an equal distance in the same direction by the translation and their individual motions can be represented by arrows. This set is termed a geometric vector, and any member (arrow) of the set will determine the same geometric vector. In other words, a geometric vector has only direction and magnitude, and hence is also referred to as a free vector. It is conventional to select an arrow positioned at the origin to represent a free vector although this choice is not an essential one and it depends largely on the circumstances. Thus the geometric vector, $\underline{v}$, in Figure 3.1(b) (which is the set of all arrows $\bar{v}_1$, $\bar{v}_2$, ... etc.), is represented by the arrow, $\bar{v}$, positioned at the origin.

Since the collection of all possible geometric vectors is in one-one correspondence with the set of all possible arrows at the origin, it can be seen that there are ∞^3 free vectors in all, and hence they can be put into one-one correspondence with the points of three-dimensional space. This natural correspondence is an extremely useful one and, as a consequence, geometric (free) vectors are employed universally as position vectors.

3.4 Geometric Vectors as Equivalence Classes of Arrows.

A geometric vector can be described more concisely by means of the mathematical concept of equivalence class, as follows:- (See Appendix II).

Thus the relation, "has the same magnitude and direction as", is an equivalence relation on the set of all arrows, since it satisfies the three defining properties of reflexivity, symmetry and transitivity. As a consequence of this, it partitions the original set of all arrows into non-overlapping subsets, (i.e. equivalence classes) which are nothing more than geometric vectors. In other words, any two geometric vectors, $\underline{v}$ and $\underline{u}$ for example, are either identical or else they do not have any element (arrow) in common. (In Figure 3.1(b) the geometric vector, $\underline{v}$, is the equivalence class which contains the elements $\bar{v}$, $\bar{v}_1$, $\bar{v}_2$, ... etc.)

Henceforth, the word 'vector' will refer to the familiar 'geometric vector'.

3.5 Line Vectors.

In the study of a spatial mechanism the concept of a line vector is of paramount importance, since at any particular instant each pair axis of the mechanism defines a unique directed line in three-dimensional Euclidean space. Of the many ways of representing such lines algebraically, the most useful for present purposes is by means of unit line vectors.

Unlike a free vector, a line vector, as defined by Brand [4], is a quantity which is restricted to lie in a definite straight line, although it does possess a magnitude and direction. However, it can be freely located at any point on the defining line. In the terminology of the previous section, a line vector is an equivalence class of arrows which share a common magnitude and direction, and lie in the same straight line. As with any equivalence class, a line vector may be represented by any one of its constituent arrows although it is sometimes convenient (see also Yang [44]) to use that arrow whose position (i.e. 'blunt-end') is closest to the origin. Thus, in Figure 3.2, the two distinct line vectors, $\underline{\hat{v}}$ and $\underline{\hat{u}}$, are represented by the arrows, $\bar{v}$ and $\bar{u}$, rather than by $\bar{v}'$ and $\bar{u}'$. A unit line vector is a line vector whose arrows are of unit magnitude and clearly any line vector, $\underline{\hat{V}}$, of length, λ , may be written as, $\underline{\hat{V}} = \lambda \underline{\hat{v}}$, where $\underline{\hat{v}}$ is a unit line vector.

It is clear that each line vector is a collection of ∞^1 arrows (one for each point of the defining line) and since there are ∞^6 arrows in all, there must therefore be ∞^5 line vectors in three-dimensional Euclidean space. Now, since each unit line vector generates a single infinity of these (see above equation) there must be an ∞^4 distinct unit line vectors in space. Thus it is possible to put the totality of unit line vectors in one-one correspondence with the ∞^4 lines of space. In other words it is legitimate to represent a line (or pair axis), in three-dimensional space, by a unit line vector.

3.6 Dual Vectors.

In contrast with free vectors, which require three independent co-ordinates to specify them (two for a unit free vector), line vectors in general require five independent quantities and unit line vectors are specified by four such co-ordinates. In practice however, a line vector, $\hat{\underline{V}}$, is normally represented by its Plücker co-ordinates which are two vectors, $\underline{v}$ and $\underline{v}_0$, where $\underline{v}$ is the free vector with the same magnitude and direction as $\hat{\underline{V}}$, and $\underline{v}_0$ represents the moment of $\hat{\underline{V}}$ about the origin, O (see Brand [4] and Yang [44]). This is illustrated by Figure 3.3 from which it can be seen that:-

$$\underline{r} \times \underline{v} = \underline{v}_0 \tag{3.1a}$$

and

$$\underline{r} \cdot \underline{v} = d \tag{3.1b}$$

Equation (3.1a) is that of the line to which $\hat{\underline{V}}$ is bound, referred to the origin, O, and one may obtain $\underline{r}$ (the position vector of a point on the line) uniquely from equations (3.1) in the following form (see Brand [3]):-

$$\underline{r} = (\underline{v} \times \underline{v}_0 + d\,\underline{v})/(\underline{v} \cdot \underline{v}) \tag{3.2}$$

From (3.1) it is clear that $\underline{v}$ and $\underline{v}_0$ are perpendicular in general and if $\hat{\underline{V}}$ is a unit line vector then the following conditions apply:-

$$\underline{v} \cdot \underline{v} = 1, \qquad \underline{v} \cdot \underline{v}_0 = 0 \tag{3.3}$$

Equations (3.3) confirm that there are only four independent co-ordinates amongst the six components of the two Plücker co-ordinates ($\underline{v}$ and $\underline{v}_0$), in the case of a unit line vector. Following Brand [4], it is possible to amalgamate the free vectors, $\underline{v}$ and $\underline{v}_0$, into a dual vector:-

$$\hat{\underline{V}} = \underline{v} + \epsilon\, \underline{v}_0 \qquad (3.4)$$

where, by definition, $\epsilon^2 = 0$.

However, in general, dual vectors are not subject to the restrictions (3.3) and hence require six independent real scalars for their specification. Such quantities are then referred to as motors. Clearly there are an ∞^6 motors in all and therefore they stand in one-one correspondence with the totality of arrows in three-dimensional space.

3.7 Dual Numbers.

In the previous section a dual vector was defined in a rather arbitrary and imprecise manner by equation (3.4). It is the author's intention to define the concept rigorously in order to more easily understand and manipulate dual vector representations. For this purpose it is necessary to investigate the properties of dual numbers as introduced by Clifford [5].

Basically a dual number is a rather similar quantity to a complex number and the two have much in common. In fact, it is possible to construct three distinct types of 'complex' number from the points of the Cartesian plane, by defining three different multiplication rules. These three 'complex number' constructions are characterised by their respective 'imaginary' units (i, ϵ and j) which have properties:-

$$\begin{aligned} i^2 &= -1 \\ \epsilon^2 &= 0 \\ j^2 &= +1 \end{aligned} \qquad (3.5)$$

The first type is simply the familiar complex number, which has wide application throughout many disciplines. The second type was introduced by Clifford [5] and referred to as a dual number. (Here, the word 'dual' has no connection with the concept of 'dual space' as used in the treatment of vector spaces). Its use has so far been restricted to applications in mechanics and recently [10, 44, 45, 46 etc.] it has been used to considerable advantage in the study of spatial linkages.

The third type of complex number has not, in the author's knowledge, been used to any great extent in applied mathematics as yet.

For the purposes of this dissertation a good understanding of the second type (the dual number) is essential. Thus, the set $\mathbb{D}$, of dual numbers may be considered to be the set, $\mathbb{R}^2$, of ordered pairs, (a, a_0), of real numbers, on which are defined two binary operations:-'addition' and 'multiplication'. These operations are represented by the usual symbols (+ and x), although they combine two dual numbers rather than two reals. Thus addition is defined component-wise, as is usual, and subtraction is defined as the inverse operation to addition. Hence for two dual numbers, (a, a_0) and (b, b_0) one has:-

$$(a, a_0) + (b, b_0) = (a + b, a_0 + b_0)$$
$$\text{and} \quad (a, a_0) - (b, b_0) = (a - b, a_0 - b_0) \qquad (3.6)$$

Here the addition and subtraction symbols on the L.H.S. of the equations refer to these operations in the dual plane, whereas those on the R.H.S. refer to the normal addition and subtraction on the real line. Two dual numbers, (a, a_0) and (b, b_0), are considered to be equal if and only if $a = b$ and $a_0 = b_0$.

Multiplication on the dual plane is defined as follows:-

$$(a, a_0) \times (b, b_0) = (a \times b, a \times b_0 + a_0 \times b)$$
$$= (a.b, a.b_0 + a_0.b) \qquad (3.7)$$

Finally, scalar multiplication can be defined as for all ordered n-tuples of real numbers. Thus:-

$$\alpha.(a, a_0) = (\alpha a, \alpha a_0) \qquad (3.8)$$

where α is any real number.

It is clear from these definitions that, as with the complex numbers, there is an isomorphism (i.e. a one-one correspondence) between dual numbers of the form, $(a, 0)$, and the real numbers. Hence a dual number may be represented as follows:-

$$(a,\ a_0) = (a,\ 0) + (0,\ a_0)$$
$$= (a,\ 0).(1,\ 0) + (a_0,\ 0).(0,\ 1)$$
$$= a.(1,\ 0) + a_0.(0,\ 1) \qquad (3.9)$$

If one now defines $\varepsilon = (0,\ 1)$ (3.10)

and identifies (1, 0) with the real unit, 1, then from (3.9), a dual number may be written in the following "Gaussian" form:-

$$(a,\ a_0) = a + \varepsilon\ a_0 \qquad (3.11)$$

where 'a' is termed the real or primary part, 'a_0' is called the secondary (or, sometimes, dual) part and ε is referred to as the dual unit.

Equation (3.11) is the usual representation of a dual number and it can be seen from definition (3.7) that $\varepsilon^2 = \varepsilon^3 = \varepsilon^4 = \ldots = 0$, since

$$\varepsilon^2 = (0,\ 1).(0,\ 1) = (0,\ 0) \qquad (3.12)$$

The dual conjugate or conjugate of a dual number may also be defined. Thus the conjugate of $(a,\ a_0) = a + \varepsilon\ a_0$ is denoted and defined by:-

$$\overline{(a,\ a_0)} = a - \varepsilon\ a_0 \qquad (3.13)$$

from which it is clear that, $(a,\ a_0).\overline{(a,\ a_0)} = a^2$, and that $(a,\ a_0)$ and $\overline{(a,\ a_0)}$ are mirror image points in the real axis.

It is now possible to define an operation of division (inverse to multiplication) in the following way, with the proviso that division by a pure dual number (of the form $\varepsilon\ a_0$) is excluded.

Thus:-

$$\frac{(a,\ a_0)}{(b,\ b_0)} = \frac{(a + \varepsilon\ a_0)}{(b + \varepsilon\ b_0)}$$
$$= \frac{(a + \varepsilon\ a_0).(b - \varepsilon\ b_0)}{(b + \varepsilon\ b_0).(b - \varepsilon\ b_0)}$$
$$= \frac{a}{b} + \varepsilon\,\frac{(a_0.b - a.b_0)}{b^2} \qquad (3.14)$$

This operation is not a true multiplicative inverse, since it cannot be defined for all non-zero dual numbers. However, it is possible to obviate these difficulties somewhat, as will be seen later.

For manipulative purposes it is more convenient to use the Gaussian notation, $(a + \epsilon a_0)$, rather than, (a, a_0), to represent a dual number since the usual commutative and distributive laws of real algebra then apply, together with the property, $\epsilon^2 = 0$. Thus multiplication may be carried out, for example, as follows:-

$$\begin{aligned}(a + \epsilon a_0).(b + \epsilon b_0) &= a.b + \epsilon a_0.b + \epsilon a.b_0 + \epsilon^2 a_0.b_0 \\ &= a.b + \epsilon(a_0.b + a.b_0)\end{aligned} \qquad (3.15)$$

It has been customary in the past to represent the unique dual number, $a + \epsilon a_0$, by the symbol, $\hat{a}$, where:-

$$\hat{a} = a + \epsilon a_0 \qquad (3.16a)$$

(the circumflex over a letter is termed the dual symbol). However, this representation is not an entirely satisfactory one since it does not determine the secondary part of the dual number. Thus, from definition (3.16a), the dual number $\widehat{\pi/2}$, for example, is taken to be:-

$$\widehat{\pi/2} = \pi/2 + \epsilon a_0 \qquad (3.16b)$$

where a_0 is arbitary, and hence does not strictly represent a <u>single</u> dual number, but the set of such numbers with real part, $\pi/2$. Nevertheless, the dual symbol is widely used in the literature and fortunately no confusion occurs. This apparent ambiguity in the notation will be discussed later in connection with the Principle of Transference.

3.8 Argand Representation of Dual Numbers.

Since a dual number is an ordered pair of real numbers, it is possible to represent the set of duals geometrically by the points of an infinite Euclidean plane. This is in complete analogy with the familiar representation of complex numbers on an Argand diagram and Figure 3.4 (which is a comparison between the complex plane, **C**, and the dual plane **D**) illustrates this correspondence. As with complex numbers, there are three possible algebraic representations for dual numbers, which may be outlined in the following sections.

3.8.1 Gaussian or Cartesian Representation.

If in Figure 3.4 the points of the dual plane, $\mathbb{D}$, are represented by Cartesian co-ordinates, x and y, then one has the Gaussian representation already described above, and the notation used there applies. The operation of addition can then be thought of in terms of the parallelogram rule for the addition of two-dimensional vectors. Thus the dual number, $\hat{a} = a + \epsilon\, a_0$, behaves in exactly the same way as the complex number, $z = x + iy$, for the purposes of addition and subtraction.

3.8.2 'Polar' Representation.

It proves of no advantage to represent dual numbers in terms of the usual polar co-ordinates, r and θ, as is done for complex numbers, since such a representation does not yield a simpler multiplication rule (i.e. for complex numbers, if $z_1 = r_1.(\cos\theta_1 + i \sin\theta_1)$ and $z_2 = r_2.(\cos\theta_2 + i \sin\theta_2)$ then:- $z_1.z_2 = r_1.r_2.[\cos(\theta_1 + \theta_2) + i \sin(\theta_1 + \theta_2)]$). However, it is possible to represent a dual number in a 'pseudo-polar' form which does lead to a simpler product rule. The co-ordinates used are ρ and t where:-

$$\rho = a \text{ (the real part of } \hat{a})$$

$$\text{and} \qquad t = a_0/a = \tan\theta \qquad (3.17)$$

referred to as 'pseudo-polar' co-ordinates.

Figure 3.5(a) illustrates the co-ordinate lines for various constant values of ρ (straight lines parallel to the dual axis) and t (straight lines passing through the origin, with slope 't').

From equations (3.17) it is clear that:-

$$a = \rho$$

$$\text{and} \qquad a_0 = \rho\, t \qquad (3.18)$$

is the inverse co-ordinate transformation and hence:-

$$\hat{a} = a + \epsilon\, a_0 = \rho + \epsilon \rho\, t$$

$$= \rho(1 + \epsilon\, t) \qquad (3.19)$$

Equation (3.19) is the 'pseudo-polar' form of a dual-number and is analogous to the polar representation of a complex number.

The product of two dual numbers can now be expressed in a very simple form. Thus if:-

$$\hat{a} = a + \varepsilon a_0 = \rho_1(1 + \varepsilon t_1)$$
$$\text{and} \quad \hat{b} = b + \varepsilon b_0 = \rho_2(1 + \varepsilon t_2) \qquad (3.20)$$

$$\text{then} \quad \hat{a}.\hat{b} = \rho_1(1 + \varepsilon t_1).\rho_2(1 + \varepsilon t_2)$$
$$= \rho_1.\rho_2[1 + \varepsilon(t_1 + t_2)] \qquad (3.21)$$

In other words the product can be written down immediately. (CF: $r_1.r_2[\cos(\theta_1 + \theta_2) + i \sin(\theta_1 + \theta_2)]$ for complex numbers). By induction it is clear from equation (3.21) that:-

$$(\hat{a})^n = [\rho(1 + \varepsilon t)]^n$$
$$= \rho^n(1 + \varepsilon nt) \qquad (3.21)$$

for positive integral 'n'. Equation (3.21) is an equivalent expression to De Moivre's Theorem, for complex numbers, and holds true for rational values of n provided that $\hat{a}$ is not a pure dual number (i.e. of the form εa_0), since division and extraction of roots may then be defined.

Figure 3.5(b) illustrates how the product of two dual numbers, $\hat{a}$ and $\hat{b}$, may be obtained graphically using the 'pseudo-polar' representation.

3.8.3 Exponential Representation.

Assuming that the power series expansion for e^x is valid for dual number arguments, it is possible to derive, for the duals, expressions similar to the identities obtained by Euler for the complex numbers (i.e. $e^{i\theta} \equiv \cos\theta + i \sin\theta$, and $e^{-i\theta} \equiv \cos\theta - i \sin\theta$). Thus one has:-

$$e^{\varepsilon t} = 1 + (\varepsilon t) + \frac{(\varepsilon t)^2}{2!} + \frac{(\varepsilon t)^3}{3!} + \ldots\ldots\ldots$$

$$= 1 + \varepsilon t \qquad (3.22)$$

since $\varepsilon^2 = \varepsilon^3 = \varepsilon^4 = \ldots\ldots = 0$.

If, in addition, the rules for indices are assumed to be valid here, then:-

$$e^{\hat{a}} = e^{(a + \varepsilon a_0)}$$
$$= e^a.e^{\varepsilon a_0}$$
$$= e^a.(1 + \varepsilon a_0) \qquad (3.23)$$

$$\rho(1+\epsilon t) = e^{\ln\rho}.(1+\epsilon t)$$
$$= e^{\ln\rho}.e^{\epsilon t}$$
$$= e^{(\ln\rho + \epsilon t)}$$
$$= \rho e^{\epsilon t} \quad (3.24)$$

Hence, every dual number has an exponential representation.

3.9 The Dual Numbers as an Algebraic Structure.

Since dual numbers may be represented by geometric (position) vectors in the dual plane, they form a vector space over the real numbers under the operation of addition. This can be verified by the fact that the vector space axioms presented in Herstein [20] are satisfied for the set, $\mathbb{D}$, of dual numbers (see Appendix II.).

In addition to its vector space structure, $\mathbb{D}$ also forms an associative ring under the two binary operations of addition and multiplication defined in the previous sections. This follows immediately since $\mathbb{D}$ is an abelian group under addition, is closed under the associative operation of multiplication and its elements satisfy the following two distributive laws:-

$$\hat{a}.(\hat{b}+\hat{c}) = \hat{a}.\hat{b} + \hat{a}.\hat{c}$$
and
$$(\hat{b}+\hat{c}).\hat{a} = \hat{b}.\hat{a} + \hat{c}.\hat{a} \quad (3.25)$$

(In fact, $\mathbb{D}$ is a commutative ring with unit element, although it is not a field since it has zero divisors, and hence differs, in this respect, from the complex numbers).

Finally, the set of dual numbers, $\mathbb{D}$, is an algebra over the field of real numbers, since it is both a vector space over the reals and an associative ring, and in addition its elements satisfy the following axiom:-

$$\alpha.(\hat{a}.\hat{b}) = (\alpha.\hat{a}).\hat{b} = \hat{a}.(\alpha.\hat{b}) \quad (3.26)$$

where $\hat{a}$ and $\hat{b}$ are dual numbers and α is real.
(see Herstein [20] and Appendix II.).

Of the three main algebraic structures that $\mathbb{D}$ possesses (i.e. vector space, associative ring and algebra), the most important, for the purposes of this dissertation, is the ring structure, since the properties of the latter will lead to a novel and concise proof of the Principle of Transference, which forms a basis for the analyses presented here.

3.10 The Ring of Dual Numbers.

The set $\mathbb{D}$ of dual numbers forms a ring under the operations of addition and multiplication defined above and, by reference to Herstein [20] or Appendix II., it is clear that the dual axis (or subset of pure duals), denoted by H, is an ideal of this ring. Thus one may construct the cosets of this ideal, H, and these are of the form; $\hat{a} + H$, $\hat{b} + H$, $2\hat{a} + H$, etc., where $\hat{a}$ and $\hat{b}$ are arbitrary dual numbers. It can be seen from Figure 3.6 that the cosets of H, when represented on an Argand diagram, are straight lines parallel to the dual axis (H), and passing through the points $\hat{a}$, $\hat{b}$, $2\hat{a}$ etc. Although each coset, like H, is a single infinity of dual numbers, it is characterised by a unique real number (the point at which it cuts the real axis). Hence the cosets of H may be put into one-one correspondence with the real numbers in a natural way.

This set of cosets of H is denoted by, $\mathbb{D}/H$, and by defining appropriate operations of 'addition' and 'multiplication' on $\mathbb{D}/H$, the latter will have the structure of a ring. This ring is called the quotient ring of H in $\mathbb{D}$ (see Herstein [20]). Addition and multiplication are defined by:

$$(\hat{a} + H) + (\hat{b} + H) = (\hat{a} + \hat{b}) + H$$

and $$(a + H).(b + H) = (a.b) + H \qquad (3.27)$$

and with these two operations, $\mathbb{D}/H$ satisfies the ring axioms (see Appendix II.).

However, since it is the only non-trivial ideal of $\mathbb{D}$, H must be a maximal ideal and this means that $\mathbb{D}/H$ is a field (see Herstein [20]). Consequently one may define an operation of division in $\mathbb{D}/H$, although this cannot be done for $\mathbb{D}$ itself. Furthermore, from Figure 3.6, the cosets of H are clearly in one-one correspondence with the real numbers ($\mathbb{R}$), and so $\mathbb{D}/H$ and $\mathbb{R}$ are isomorphic as fields. In other words, denoting a typical coset $(\hat{a} + H)$ by $(a + H)$ where 'a' is the point at which it cuts the real axis (see Figure 3.6), one may define the operations, +, −, × and ÷ on D/H, in terms of these same operations on the set of reals, as follows:-

$$(a + H) + (b + H) = (a + b) + H$$
$$(a + H) - (b + H) = (a - b) + H$$
$$(a + H) \times (b + H) = (a \times b) + H$$
$$(a + H) \div (b + H) = (a \div b) + H \qquad (3.28)$$

Alternatively, since a coset may be thought of as an equivalence class (see Appendix II.) of dual numbers related under the equivalence relation, "has the same real part as", one may represent a typical coset, a + H, in a very brief form as, [a]. This is in accordance with accepted conventions and [a] is taken to be the equivalence class of all elements which are related to 'a'. (In this case it is the set of all those dual numbers with real part, 'a'). Using this equivalence class notation one may restate (3.28) very briefly as follows:-

$$[a] + [b] = [a + b]$$
$$[a] - [b] = [a - b]$$
$$[a] \times [b] = [a \times b]$$
$$[a] \div [b] = [a \div b] \qquad (3.29)$$

Clearly this definition of division in terms of cosets or equivalence classes obviates the difficulties mentioned earlier in defining such an operation on the set of duals, **D**, since it automatically excludes division by a pure dual number, whose coset would be (0 + H) or just H, and whose equivalence class is [0]. Neither (3.28) nor (3.29) yields a meaningful result for such a quotient, since a ÷ 0 is not defined. As a final point it must be noted that the advantage in using cosets or equivalence classes is that any single element of a particular class may be selected as a representative of the whole class. This property is of fundamental importance in the proof of the Principle to Transference, to be dealt with later.

3.11 Functions of a Dual Variable.

Equations (3.28) and (3.29) define the four algebraic operations of addition, subtraction, multiplication and division on the set of cosets or equivalence classes of dual numbers by means of the natural isomorphism of $\mathbb{D}/H$ with $\mathbb{R}$, the reals. (see Figure 3.6). For consistency one must define a function of a coset in the same manner, as follows:-

$$f((a + H)) = (f(a) + H) \tag{3.30}$$

or equivalently:-

$$f([a]) = [f(a)] \tag{3.31}$$

In practice, however, one requires a definition for a function of a particular dual number. Thus, in analogy with the definitions given by Spiegel [34] for functions of a complex variable, one can define the value of a function with a dual number argument by means of a Taylor series expansion about the real part of the dual argument. Thus if $\hat{a} = a + \varepsilon a_0$, then:-

$$f(\hat{a}) = f(a) + f'(a)(\hat{a} - a) + \frac{f''(a)}{2!}(\hat{a} - a)^2 + \ldots\ldots\ldots$$

$$= f(a) + \varepsilon a_0 . f'(a) \tag{3.32}$$

since $\varepsilon^2 = \varepsilon^3 = \ldots = 0$.

In other words, if $f(\hat{a})$ is identified with $\widehat{f(a)}$ and written:-

$$\begin{aligned} f(\hat{a}) &= f(a + \varepsilon a_0) \\ &= f(a) + \varepsilon f_0(a) \\ &= \widehat{f(a)} \end{aligned} \tag{3.33a}$$

then $$f_0(a) = a_0 f'(a) \tag{3.33b}$$

In particular, one has from (3.32):-

$$\begin{aligned} \sin(\hat{a}) &= \sin(a + \varepsilon a_0) \\ &= \sin a + \varepsilon a_0 . \cos a \end{aligned} \tag{3.34a}$$

$$\begin{aligned} \cos(\hat{a}) &= \cos(a + \varepsilon a_0) \\ &= \cos a - \varepsilon a_0 . \sin a \end{aligned} \tag{3.34b}$$

and $$\begin{aligned} \tan(\hat{a}) &= \tan(a + \varepsilon a_0) \\ &= \tan a + \varepsilon a_0 . \sec^2 a \end{aligned} \tag{3.34c}$$

Furthermore, all the usual trigonometrical identities and relationships are valid for dual number arguments. Thus, for example:-

$$\sin^2(\hat{a}) + \cos^2(\hat{a}) \equiv 1 \qquad (3.35)$$

as can be seen from equations (3.34a) and 3.34b).

3.12 Dual Angles.

The relative position and orientation of two skew lines (i.e. unit line vectors) in space can be represented by a dual number, $\hat{\alpha}$ for example, where the real part of $\hat{\alpha}$ is the real angle, α, between the two lines and the dual part,(usually denoted by 'a' rather than α_0) is the common perpendicular distance between the lines. This dual number, $\hat{\alpha}$, is then termed the dual angle between the skew lines. Thus the dual angle between the adjacent unit line vectors $\hat{\underline{S}}_i$ and $\hat{\underline{S}}_j$ in Figure 1.1 is denoted by $\hat{\alpha}_{ij}$, where:-

$$\hat{\alpha}_{ij} = \alpha_{ij} + \varepsilon\, a_{ij} \qquad (3.36)$$

whilst the dual angle between the adjacent common perpendiculars, $\hat{\underline{a}}_{ij}$ and $\hat{\underline{a}}_{jk}$ is denoted by $\hat{\theta}_j$, where:-

$$\hat{\theta}_j = \theta_j + \varepsilon\, S_j \qquad (3.37)$$

(again it is conventional to use the symbol, S, rather than θ_0 for the secondary part of $\hat{\theta}$, when dealing with dual angles).

This concept of a dual angle is due to Study [35] and has proved of considerable importance in the analysis of spatial mechanisms.

3.13 Dual Vectors in Terms of Dual Angles and Dual Numbers.

Having examined the properties of dual numbers in some detail, it is now possible to define a dual vector more rigorously as an ordered triplet of such dual numbers, in analogy with the definition of a real vector as an ordered triplet of real numbers. Thus, if $\underline{v}$ is a real vector, where:-

$$\underline{v} = \begin{bmatrix} a_1 \\ a_2 \\ a_3 \end{bmatrix} \qquad (3.38)$$

and a_1, a_2, a_3 are real numbers, one may define a corresponding dual vector, $\hat{\underline{v}}$,

where:-

$$\hat{\underline{v}} = \begin{bmatrix}(a_1, a_{01})\\(a_2, a_{02})\\(a_3, a_{03})\end{bmatrix} = \begin{bmatrix}a_1 + \varepsilon a_{01}\\a_2 + \varepsilon a_{02}\\a_3 + \varepsilon a_{03}\end{bmatrix} \quad (3.39)$$

and $a_1 + \varepsilon a_{01}$, $a_2 + \varepsilon a_{02}$, $a_3 + \varepsilon a_{03}$ are three dual numbers.

Clearly, from (3.38), (3.39) and the rules for the addition of vectors, one may write:-

$$\hat{\underline{v}} = \begin{bmatrix}a_1\\a_2\\a_3\end{bmatrix} + \varepsilon \begin{bmatrix}a_{01}\\a_{02}\\a_{03}\end{bmatrix} = \underline{v} + \varepsilon \underline{v}_0 \quad (3.40)$$

where:-

$$\underline{v}_0 = \begin{bmatrix}a_{01}\\a_{02}\\a_{03}\end{bmatrix} \quad (3.41)$$

is a real vector. (Compare (3.4) and (3.40)).

In general a dual vector, $\hat{\underline{v}}$, represents a <u>motor</u> (see Brand [4]) and Figure 3.7 illustrates the relationship between $\hat{\underline{v}}$, $\underline{v}$ and $\underline{v}_0$, (Note: a motor may be thought of as the sum of a <u>line vector</u> and a <u>couple</u>.) in this case. However, if in equation (3.40) one has the condition:-

$$\underline{v} \cdot \underline{v}_0 = 0 \quad (3.42)$$

then $\hat{\underline{v}}$ represents a line vector, and if in addition:-

$$\underline{v} \cdot \underline{v} = 1 \quad (3.43)$$

then $\hat{\underline{v}}$ represents a unit line vector, and may be used to describe a pair axis in space as previously discussed (see also equation (3.4)). In this case $\underline{v}$ and $\underline{v}_0$ are the Plücker co-ordinates of the line.

A line vector $\hat{\underline{v}}$ which passes through the origin has zero moment about the latter (i.e. $\underline{v}_0 = \underline{0}$) and hence may be identified with the free vector, $\underline{v}$, for that particular choice of origin.

Now any unit free vector $\underline{v}$ can be specified by the three real angles, α_1, α_2, α_3, at which it intersects the co-ordinate axes $\underline{i}$, $\underline{j}$ and $\underline{k}$ (considered to be three intersecting unit line vectors). Alternatively and more usually, one may specify it by its three direction cosines (D. C.'s), $\cos\alpha_1$, $\cos\alpha_2$, and $\cos\alpha_3$, which are also the real number components of $\underline{v}$ and where:-

$$\cos\alpha_1 = \underline{i} \cdot \underline{v}$$
$$\cos\alpha_2 = \underline{j} \cdot \underline{v}$$
$$\cos\alpha_3 = \underline{k} \cdot \underline{v} \qquad (3.44)$$

and $$\cos^2\alpha_1 + \cos^2\alpha_2 + \cos^2\alpha_3 = 1 \qquad (3.45)$$

This situation is illustrated by Figure 3.8(a).

In an analogous manner one may specify the unit line vector, $\hat{\underline{v}} = \underline{v} + \epsilon\, \underline{v}_0$, by the three dual angles, $\hat{\alpha}_1$, $\hat{\alpha}_2$ and $\hat{\alpha}_3$, defining its position and orientation relative to $\underline{i}$, $\underline{j}$ and $\underline{k}$, respectively, where:-

$$\hat{\alpha}_1 = \alpha_1 + \epsilon\, a_1$$
$$\hat{\alpha}_2 = \alpha_2 + \epsilon\, a_2$$
$$\hat{\alpha}_3 = \alpha_3 + \epsilon\, a_3 \qquad (3.46)$$

(α_1, α_2, α_3 are the real angles at which $\underline{v}$ intersects $\underline{i}$, $\underline{j}$ and $\underline{k}$, and a_1, a_2, a_3 are the common perpendiculars between $\underline{v}$ and the latter, respectively). This is illustrated by Figure 3.8(b).

Now from Figure 3.9 it is clear that:-

$$\underline{i} \cdot \underline{v}_0 = -\, a_1\sin\alpha_1$$

and similarly
$$\underline{j} \cdot \underline{v}_0 = -\, a_2\sin\alpha_2$$
$$\underline{k} \cdot \underline{v}_0 = -\, a_3\sin\alpha_3 \qquad (3.47)$$

since $\underline{v}_0$ is the moment of $\underline{v}$ about the origin. Thus, from (3.44) and (3.47) the components of $\underline{v}$ and $\underline{v}_0$ are given by:-

$$\underline{v} = \begin{bmatrix} \cos\alpha_1 \\ \cos\alpha_2 \\ \cos\alpha_3 \end{bmatrix}, \quad \underline{v}_0 = \begin{bmatrix} -\, a_1\sin\alpha_1 \\ -\, a_2\sin\alpha_2 \\ -\, a_3\sin\alpha_3 \end{bmatrix} \qquad (3.48)$$

and hence $\underline{\hat{v}}$ may be written:-

$$\underline{\hat{v}} = \begin{bmatrix} \cos\alpha_1 \\ \cos\alpha_2 \\ \cos\alpha_3 \end{bmatrix} + \epsilon \begin{bmatrix} -a_1\sin\alpha_1 \\ -a_2\sin\alpha_2 \\ -a_3\sin\alpha_3 \end{bmatrix} = \begin{bmatrix} \cos\alpha_1 - \epsilon\, a_1\sin\alpha_1 \\ \cos\alpha_2 - \epsilon\, a_2\sin\alpha_2 \\ \cos\alpha_3 - \epsilon\, a_3\sin\alpha_3 \end{bmatrix} \tag{3.49}$$

However, from (3.34b),

$$\begin{aligned} \cos\hat{\alpha}_1 &= \cos\alpha_1 - \epsilon\, a_1\sin\alpha_1 \\ \cos\hat{\alpha}_2 &= \cos\alpha_2 - \epsilon\, a_2\sin\alpha_2 \\ \cos\hat{\alpha}_3 &= \cos\alpha_3 - \epsilon\, a_3\sin\alpha_3 \end{aligned} \tag{3.50}$$

and consequently the three dual number components of the unit line vector, $\underline{\hat{v}}$, are $\cos\hat{\alpha}_1$, $\cos\hat{\alpha}_2$, and $\cos\hat{\alpha}_3$, (see equation (3.39)). Furthermore from (3.48) and (3.42) one has:-

$$a_1\sin\alpha_1\cos\alpha_1 + a_2\sin\alpha_2\cos\alpha_2 + a_3\sin\alpha_3\cos\alpha_3 = 0 \tag{3.51}$$

and, after squaring and adding equations (3.50) one obtains, from (3.45) and (3.51), the following identity:-

$$\cos^2\hat{\alpha}_1 + \cos^2\hat{\alpha}_2 + \cos^2\hat{\alpha}_3 \equiv 1 \tag{3.52}$$

It is clear, therefore, that $\cos\hat{\alpha}_1$, $\cos\hat{\alpha}_2$, and $\cos\hat{\alpha}_3$ may be considered to be the <u>dual direction cosines</u> of $\underline{\hat{v}}$ and they uniquely specify the latter.

As a final point it must be noted that only two real direction cosines are independent (from (3.45)) since a line through the origin is determined by two real co-ordinates. Similarly only two dual direction cosines can be independent (from (3.52)) and since these involve four real numbers, this is in agreement with the number of variables required to specify a line in space. (see also Chapter 2 and Appendix I.).

3.14 Spherical and Spatial Geometry and the Principle of Transference.

It is now possible to compare the behaviour of unit line vectors through the origin (i.e. free vectors) with that of general unit line vectors in space. From the previous section it is clear that a general unit line vector, $\underline{\hat{v}} = \underline{v} + \epsilon\, \underline{v}_0$, may be related to a <u>unique</u> free vector, $\underline{v}$, such that if $\underline{\hat{v}}$ is orientated with respect to $\underline{i}$, $\underline{j}$ and $\underline{k}$, by the three dual angles $\hat{\alpha}_1$, $\hat{\alpha}_2$ and $\hat{\alpha}_3$ (given by (3.46)), then $\underline{v}$ intersects $\underline{i}$, $\underline{j}$ and $\underline{k}$ at the real angles α_1, α_2 and α_3, respectively.

Furthermore, since a set of intersecting unit line vectors (or free vectors) effectively represents a spherical polygon (see Brand [4] and Todhunter and Leatham [36]) and a set of general unit line vectors represents a spatial polygon (see Yang [44]), it is clear that the correspondence between $\hat{\underline{v}}$ and $\underline{v}$ is one between spatial and spherical geometry. For each spatial triangle, for example, defined by the three unit line vectors, $\hat{\underline{S}}_1$, $\hat{\underline{S}}_2$ and $\hat{\underline{S}}_3$, orientated with respect to one another by the relative dual angles $\hat{\alpha}_{12}$, $\hat{\alpha}_{23}$, $\hat{\alpha}_{31}$ (see Figure 3.10(a)), there is a unique spherical triangle defined by $\underline{S}_1$, $\underline{S}_2$ and $\underline{S}_3$ which intersect each other at the real angles α_{12}, α_{23}, and α_{31} (see Figure 3.10(b)).

The correspondence between equivalent spatial and spherical configurations is attributed to Dobrovolski [8]. However, it is not a one-one correspondence since a spherical polygon is related to an infinite number of spatial polygons.

Nevertheless, it is possible to transfer laws relating to spherical polygons into equivalent spatial laws using the <u>Principle of Transference</u>, formulated originally by Kotelnikov [26]. The principle may be formally stated as follows:-

> All valid laws and formulae relating to a system of intersecting unit line vectors (and hence involving real variables) are equally valid when applied to an equivalent system of skew unit line vectors, if each real variable, α, in the formulae is replaced by the corresponding dual variable, $\hat{\alpha} = \alpha + \epsilon a$, and each constant, k, is replaced by the dual constant, $\hat{k} = k + \epsilon 0$ (i.e. $\hat{k}$ has zero secondary part). Here 'k' is not a parameter but a definite fixed number.

As an example of the Principle, one may apply it to equation (3.45) and obtain equation (3.52) directly. (Note: if the constant, 1, on the R.H.S. of (3.45) were not replaced by $1 + \epsilon 0$ then (3.51) would not be correct and $\hat{\underline{v}}$ would be a motor). The proof of the Principle of Transference is now fairly straightforward and depends on the properties of dual numbers outlined in this chapter.

Thus, suppose one has some law relating two sides and the included angle of a spherical triangle, for example (see Figure 3.10(b)). This may be expressed as:-

$$f(\alpha_{12}, \alpha_{23}, \theta_2) = 0 \qquad (3.53)$$

But since the real numbers are isomorphic to the cosets of dual numbers, discussed previously, one may write, from (3.53):-

$$(f(\alpha_{12}, \alpha_{23}, \theta_2) + H) = (0 + H) \qquad (3.54)$$

or in equivalence class notation:-

$$[f(\alpha_{12}, \alpha_{23}, \theta_2)] = [0] \qquad (3.55)$$

Now from (3.30) and (3.31) one may rewrite (3.54) and (3.55) in the form:-

$$f((\alpha_{12} + H), (\alpha_{23} + H), (\theta_2 + H)) = (0 + H) \qquad (3.56)$$

and

$$f([\alpha_{12}], [\alpha_{23}], [\theta_2]) = [0] \qquad (3.57)$$

Finally by making use of the properties of equivalence classes (see Appendix II.) one may select a particular element from each of the latter as a representative of the class and obtain from (3.57):-

$$f((\alpha_{12} + \epsilon\, a_{12}), (\alpha_{23} + \epsilon\, a_{23}), (\theta_2 + \epsilon\, S_2)) = 0 + \epsilon\, 0 \qquad (3.58)$$

Expanding (3.58) by means of the rules for a function of dual variables (see previous sections), one obtains two real equations (after equating primary and secondary parts), one of which is equation (3.53), the original spherical law. The dual number $0 + \epsilon\, 0$ on the R.H.S. of (3.58) <u>must</u> be chosen as the representative of the equivalence class, [0], in (3.57) in order that (3.58) will reduce to (3.53) when $a_{12} = a_{23} = S_2 = 0$.

Equation (3.58) is the required equation applicable to the spatial triangle corresponding to the original spherical triangle (see Figure 3.10(a)). This completes the proof of the Principle of Transference, and clearly the procedure is valid for all polygons and relevant formulae. The Principle is of fundamental importance since it enables one to derive dual number loop equations describing spatial mechanisms (represented by spatial polygons), directly from the much simpler real number loop equations describing equivalent spherical mechanisms (represented by spherical polygons).

Thus, suppose one has some law relating two sides and the included angle of a spherical triangle, for example (see Figure 3.10(b)). This may be expressed as:-

$$f(\alpha_{12}, \alpha_{23}, \theta_2) = 0 \qquad (3.53)$$

But since the real numbers are isomorphic to the cosets of dual numbers, discussed previously, one may write, from (3.53):-

$$(f(\alpha_{12}, \alpha_{23}, \theta_2) + H) = (0 + H) \qquad (3.54)$$

or in equivalence class notation:-

$$[f(\alpha_{12}, \alpha_{23}, \theta_2)] = [0] \qquad (3.55)$$

Now from (3.30) and (3.31) one may rewrite (3.54) and (3.55) in the form:-

$$f((\alpha_{12} + H), (\alpha_{23} + H), (\theta_2 + H)) = (0 + H) \qquad (3.56)$$

and $$f([\alpha_{12}], [\alpha_{23}], [\theta_2]) = [0] \qquad (3.57)$$

Finally by making use of the properties of equivalence classes (see Appendix II.) one may select a particular element from each of the latter as a representative of the class and obtain from (3.57):-

$$f((\alpha_{12} + \epsilon\, a_{12}), (\alpha_{23} + \epsilon\, a_{23}), (\theta_2 + \epsilon\, S_2)) = 0 + \epsilon\, 0 \qquad (3.58)$$

Expanding (3.58) by means of the rules for a function of dual variables (see previous sections), one obtains two real equations (after equating primary and secondary parts), one of which is equation (3.53), the original spherical law. The dual number $0 + \epsilon\, 0$ on the R.H.S. of (3.58) must be chosen as the representative of the equivalence class, [0], in (3.57) in order that (3.58) will reduce to (3.53) when $a_{12} = a_{23} = S_2 = 0$.

Equation (3.58) is the required equation applicable to the spatial triangle corresponding to the original spherical triangle (see Figure 3.10(a)). This completes the proof of the Principle of Transference, and clearly the procedure is valid for all polygons and relevant formulae. The Principle is of fundamental importance since it enables one to derive dual number loop equations describing spatial mechanisms (represented by spatial polygons), directly from the much simpler real number loop equations describing equivalent spherical mechanisms (represented by spherical polygons).

In practice one obtains equation (3.58) immediately from (3.53) by the device of " introducing the dual symbol ^ ". In this way the dual of (3.53) is:-

$$f(\hat{\alpha}_{12}, \hat{\alpha}_{23}, \hat{\theta}_2) = \hat{0} \tag{3.59}$$

where, by convention:-

$$\hat{\alpha}_{12} = \alpha_{12} + \epsilon\, a_{12}$$
$$\hat{\alpha}_{23} = \alpha_{23} + \epsilon\, a_{23}$$
$$\hat{\theta}_2 = \theta_2 + \epsilon\, S_2$$

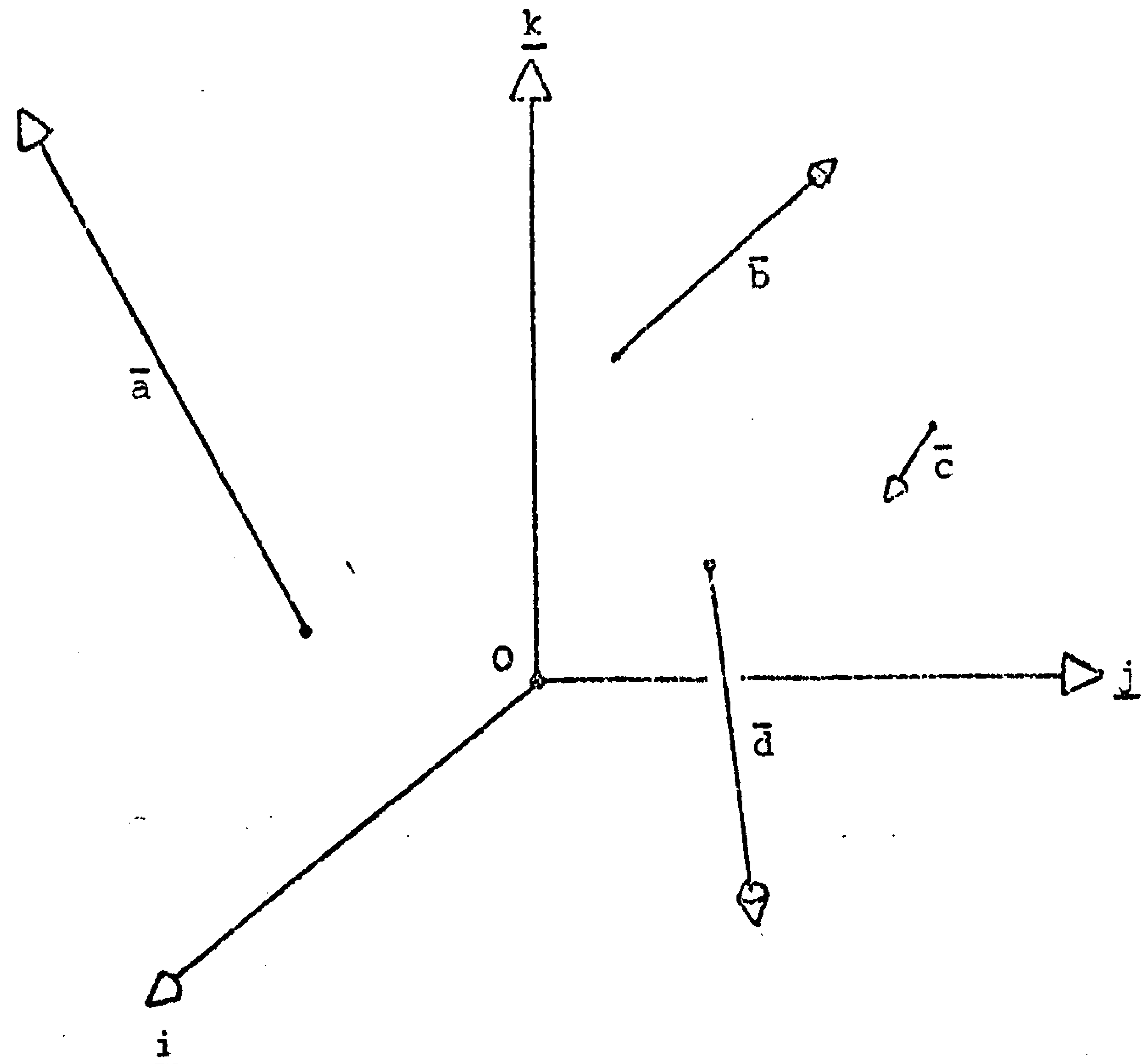
and $$\hat{0} = 0 + \epsilon\, 0 \tag{3.60}$$

Equation (3.59) is then expanded into a primary equation (identical to (3.53)) and a secondary equation using the algebra of dual numbers outlined previously.

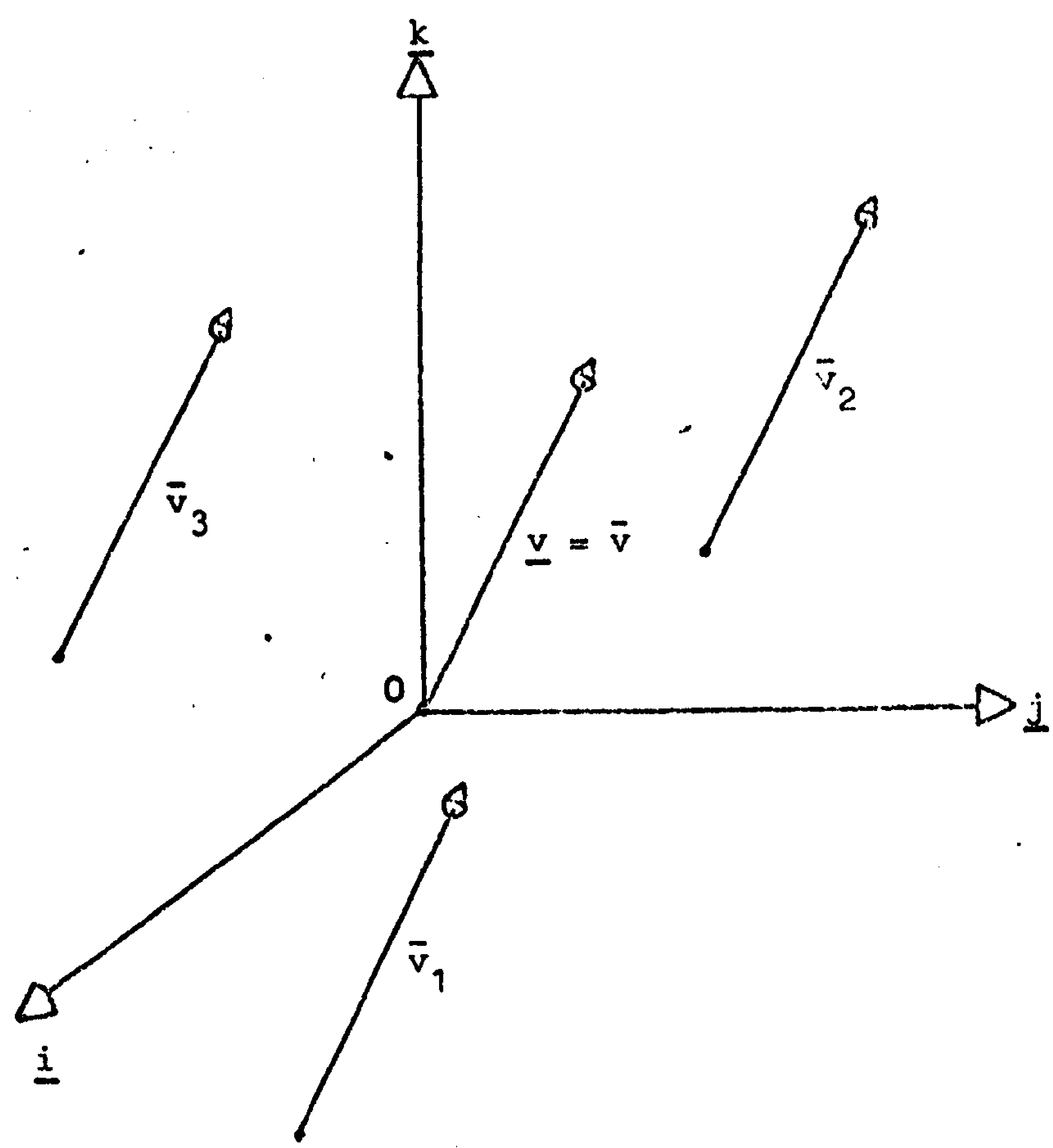
The use of the dual symbol should not now lead to any ambiguities since, for example, the dual of any constant such as $\pi/2$, is taken to be (see also equation (3.16(b)):-

$$\pi/2 = \pi/2 + \epsilon\, 0 \tag{3.61}$$

In Chapter 4, a complete system of loop equations for spherical polygons is derived and classified, and, using the Principle of Transference, these then yield corresponding dual number loop equations for spatial polygons. The subsequent analyses of spatial mechanisms, presented in later chapters is based on these dual equations.



(a) Several Distinct Arrows.



(b) The Arrows Defining the Geometric Vector, $\underline{v}$.

Figure 3.1 Representation of Arrows and Geometric Vectors.

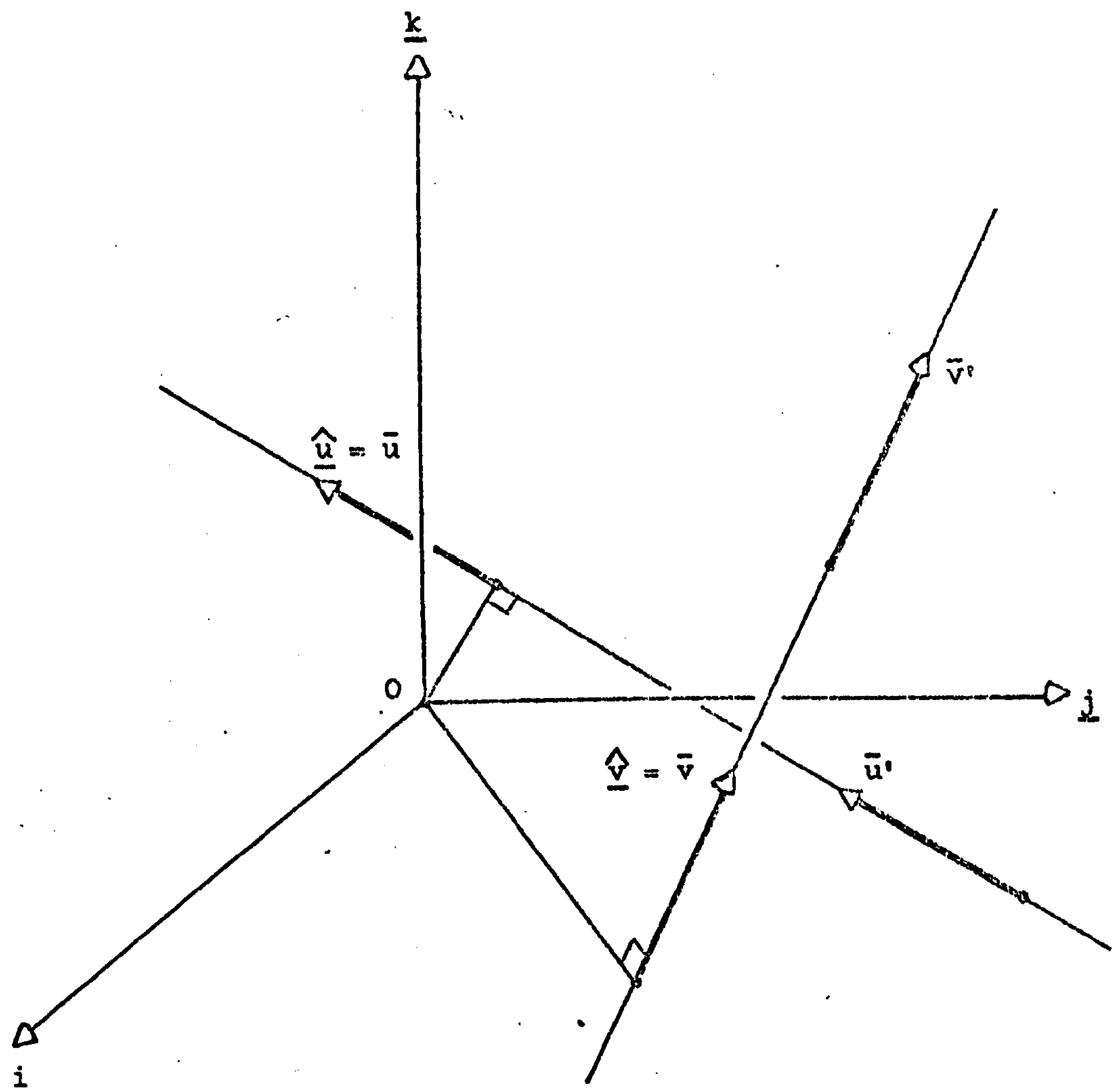


Figure 3.2 Representation of Line Vectors by the Arrows Closest to the Origin.

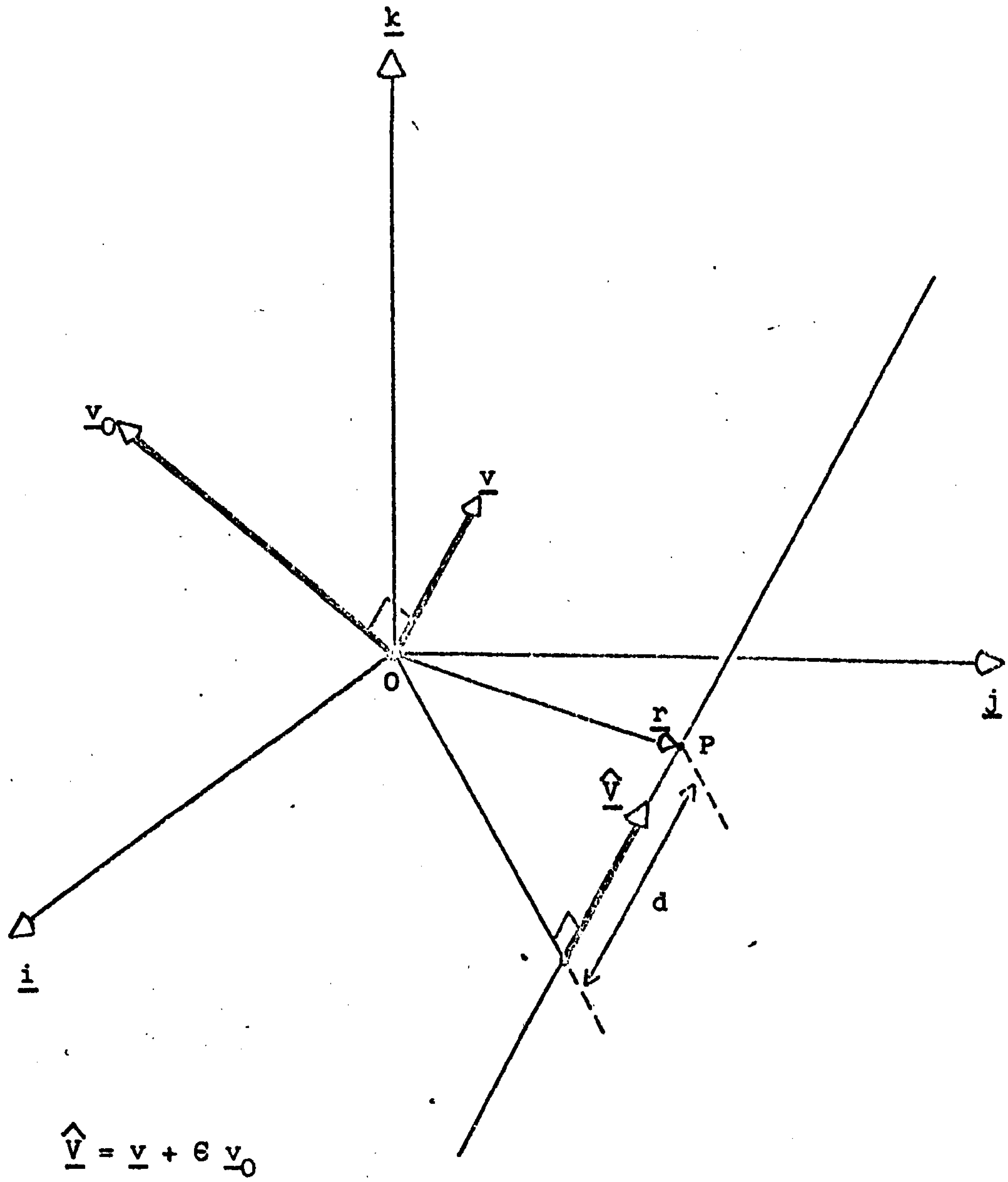
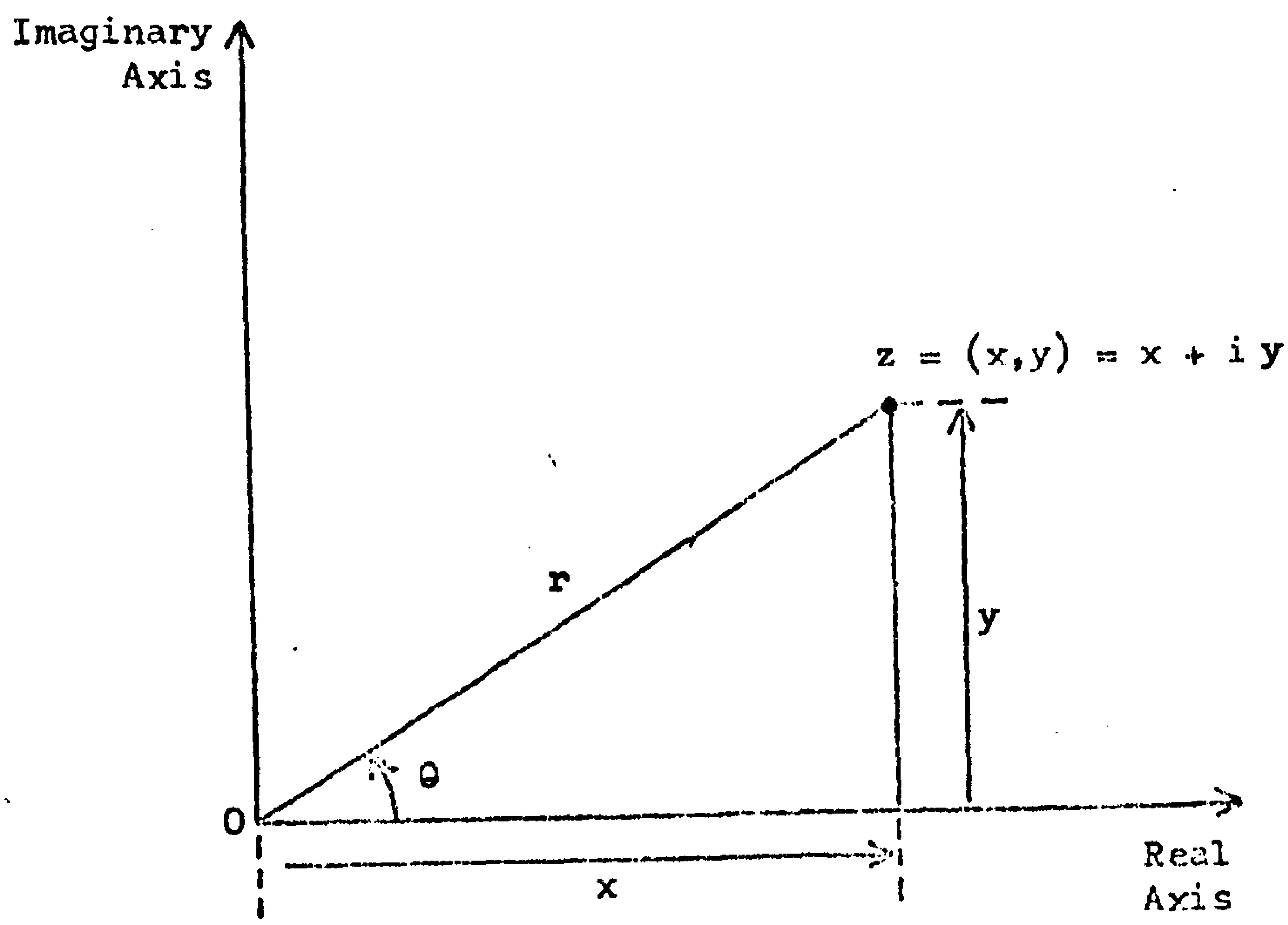
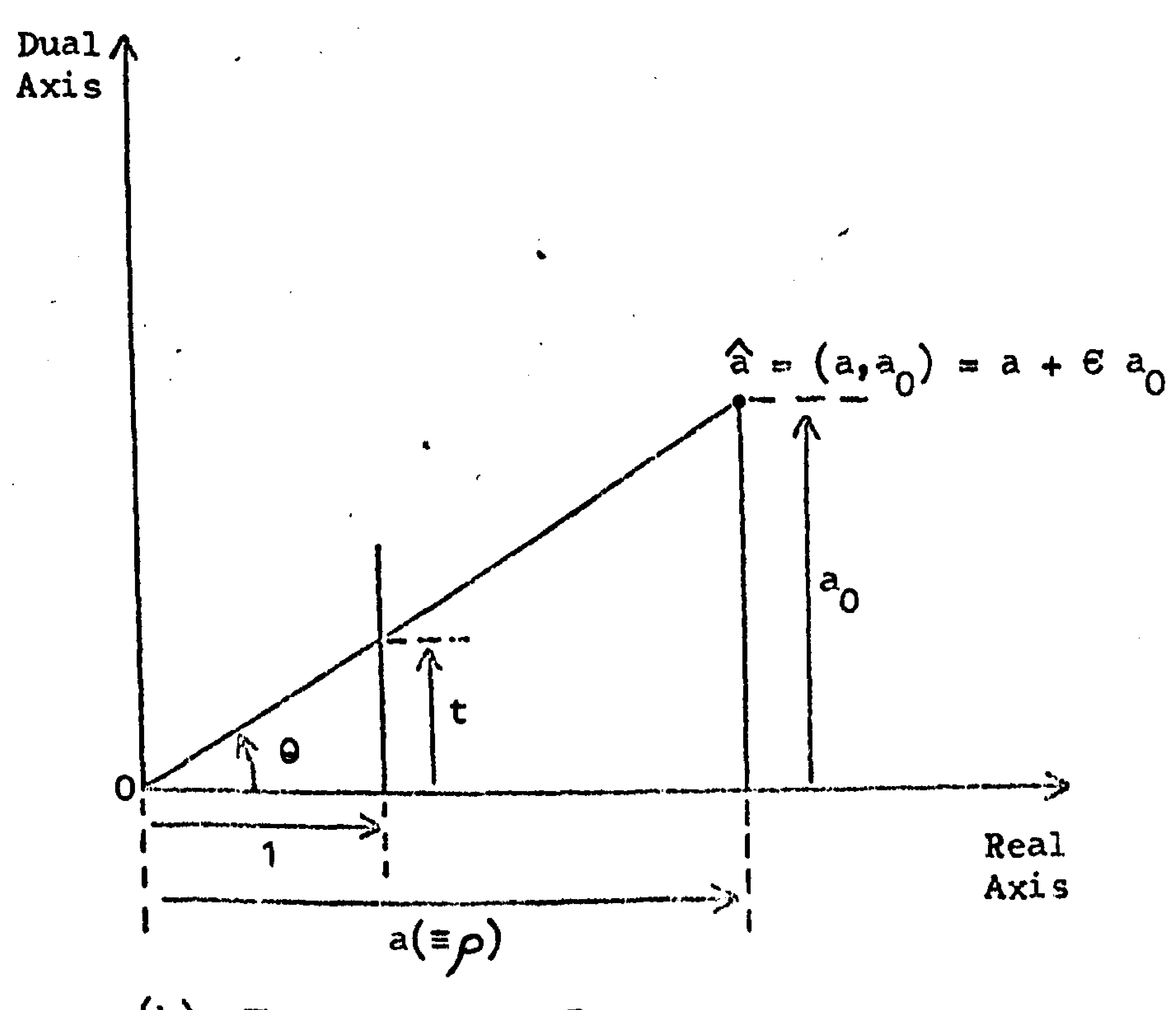


Figure 3.3 The Significance of the Plücker Co-ordinates, $\underline{v}$ and $\underline{v}_0$, of a Line Vector, $\underline{\hat{V}}$.

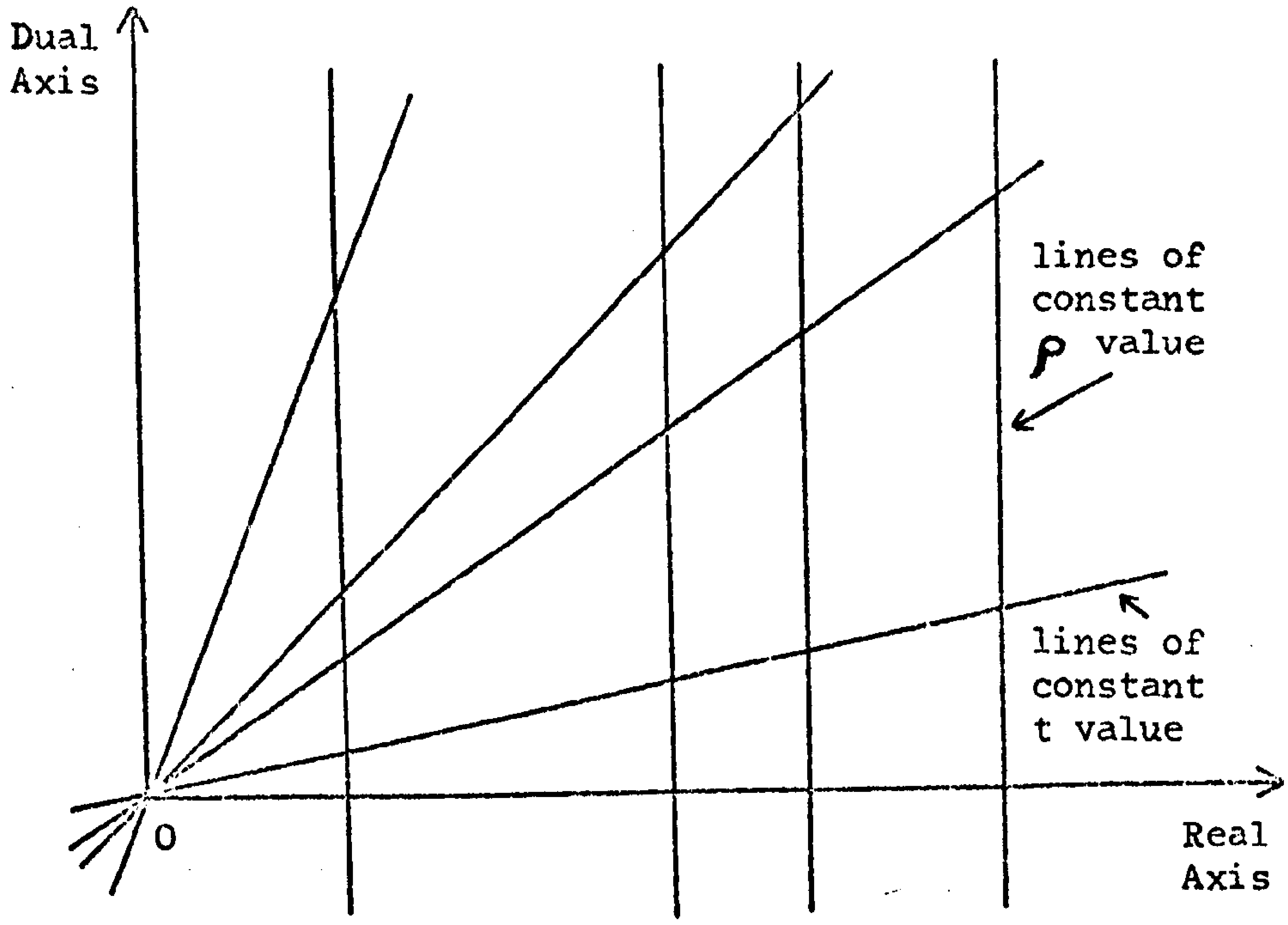


(a) The Complex Plane, $\mathbb{C}$.

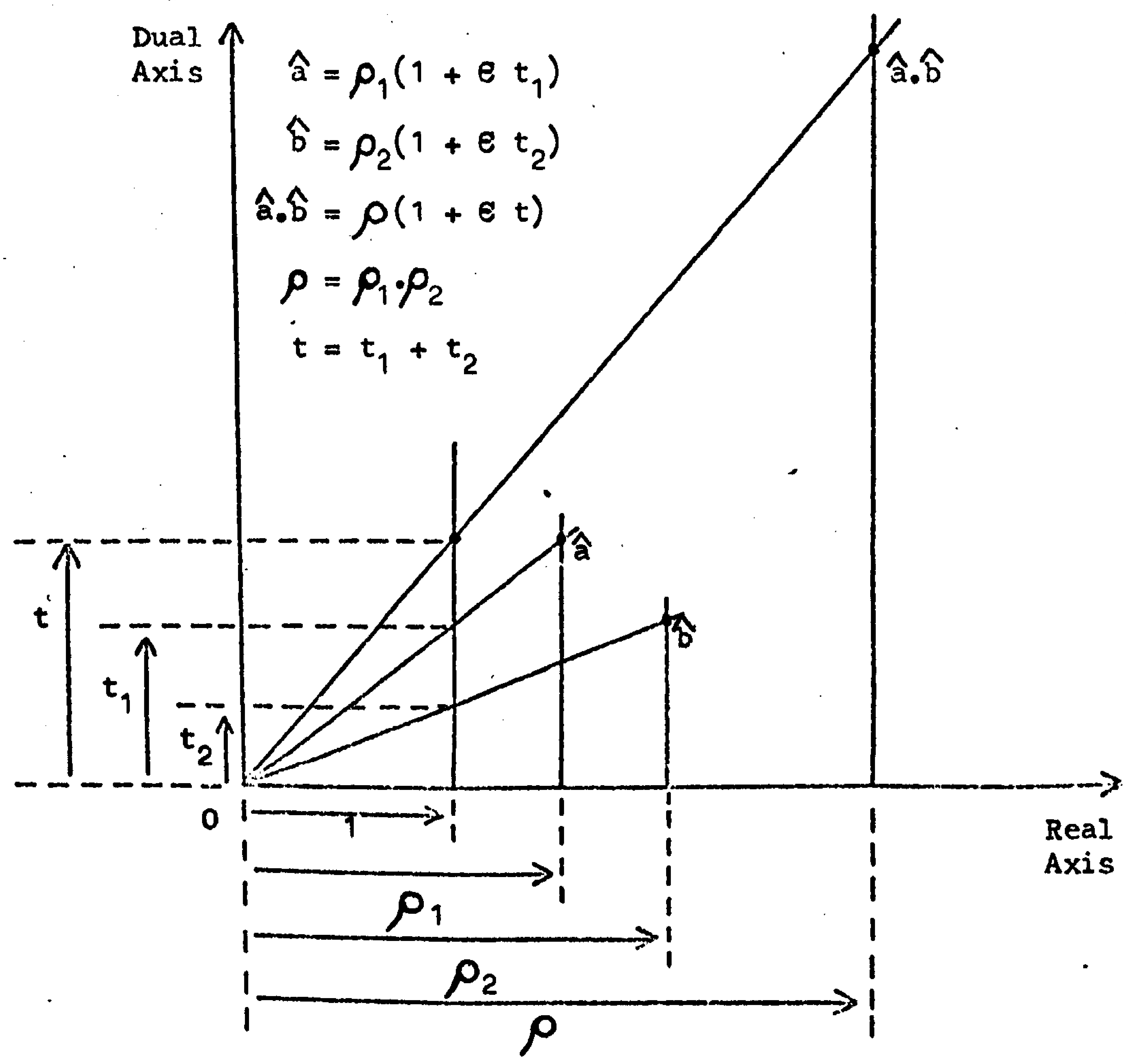


(b) The Dual Plane, $\mathbb{D}$.

Figure 3.4 Comparison between the Complex and the Dual Planes.



(a) Lines of Constant t and ρ Values.



(b) Obtaining the Product of two Dual Numbers Graphically.

Figure 3.5 'Pseudo-Polar' Representation of Dual Numbers.

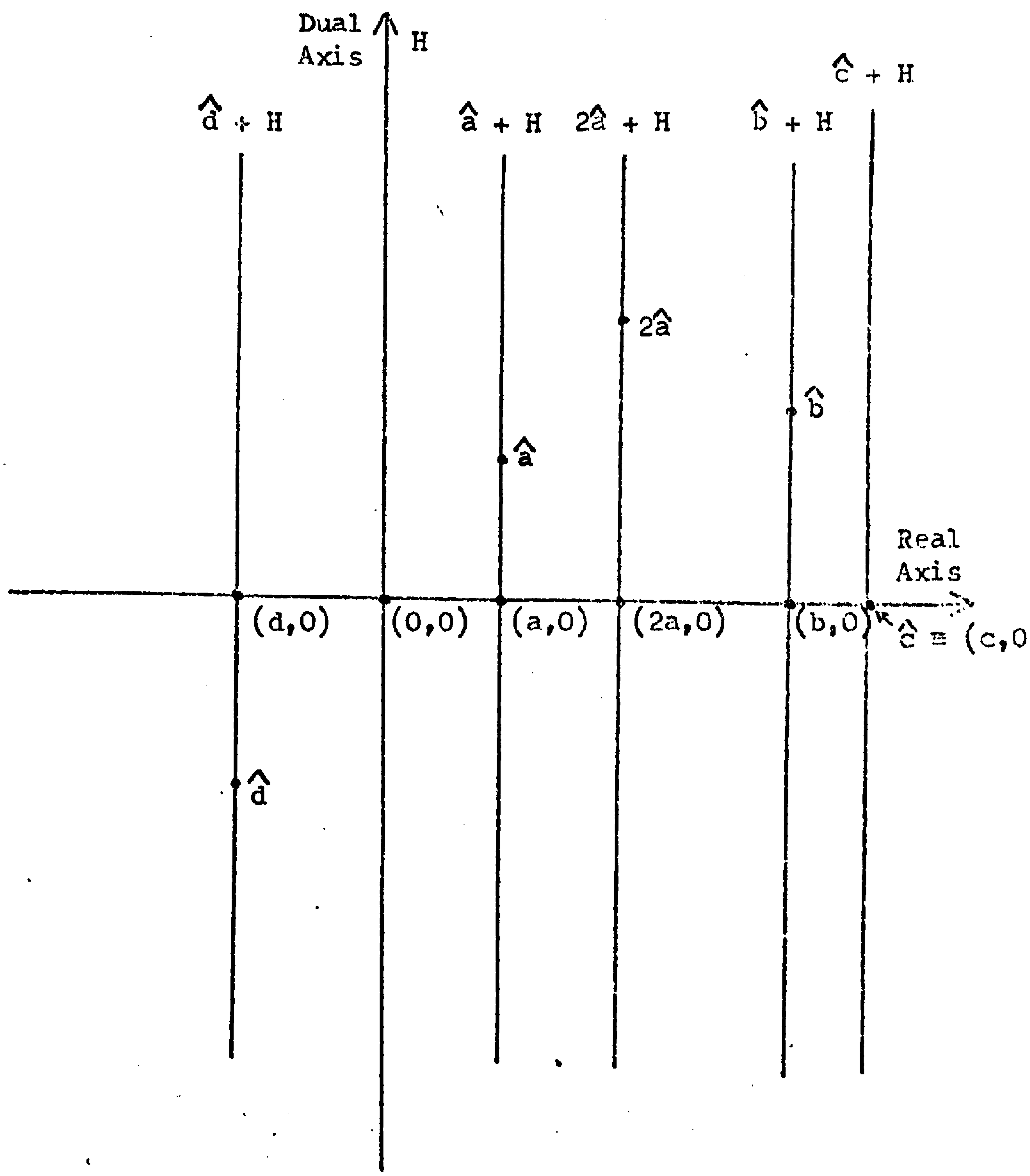
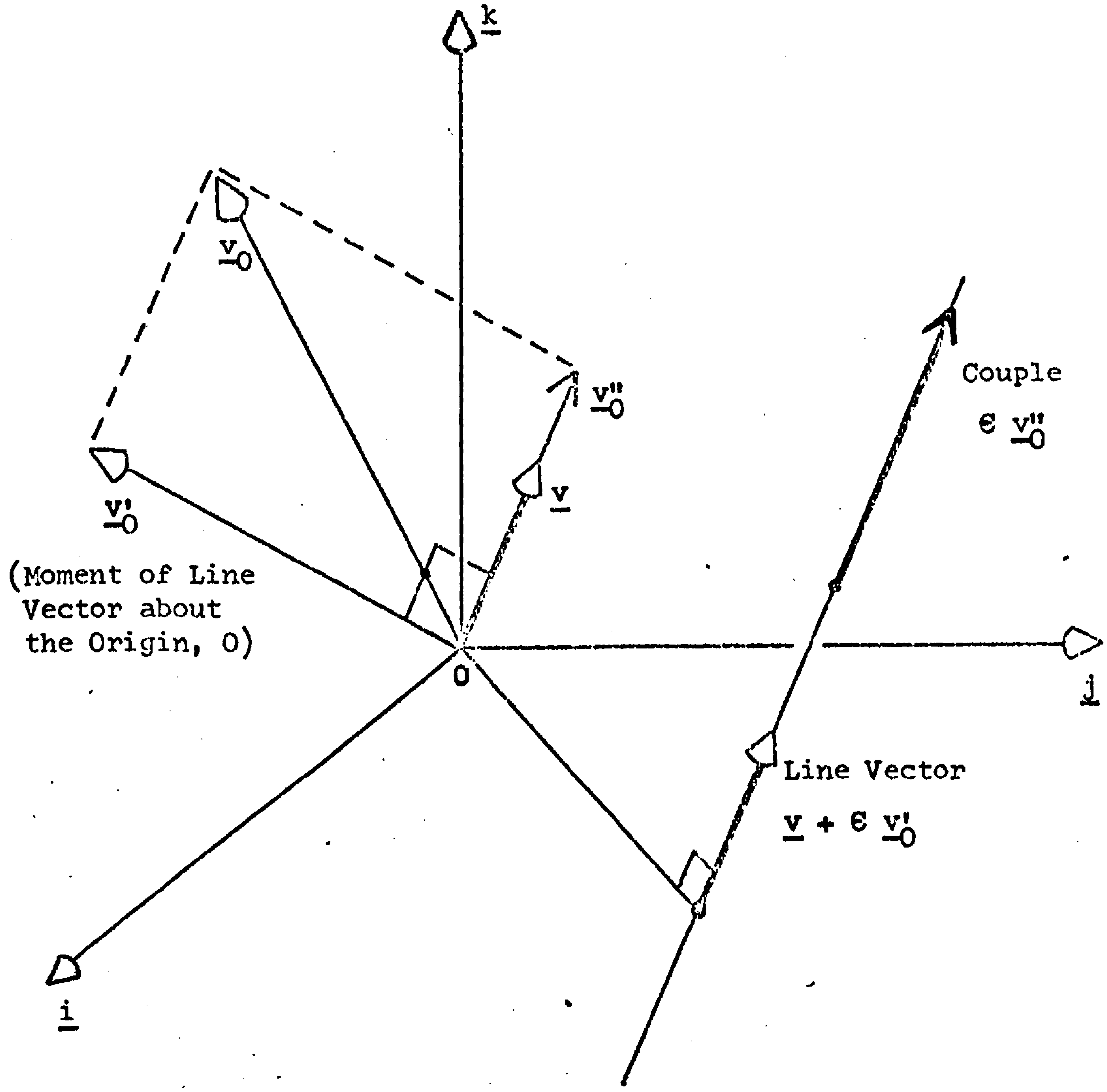


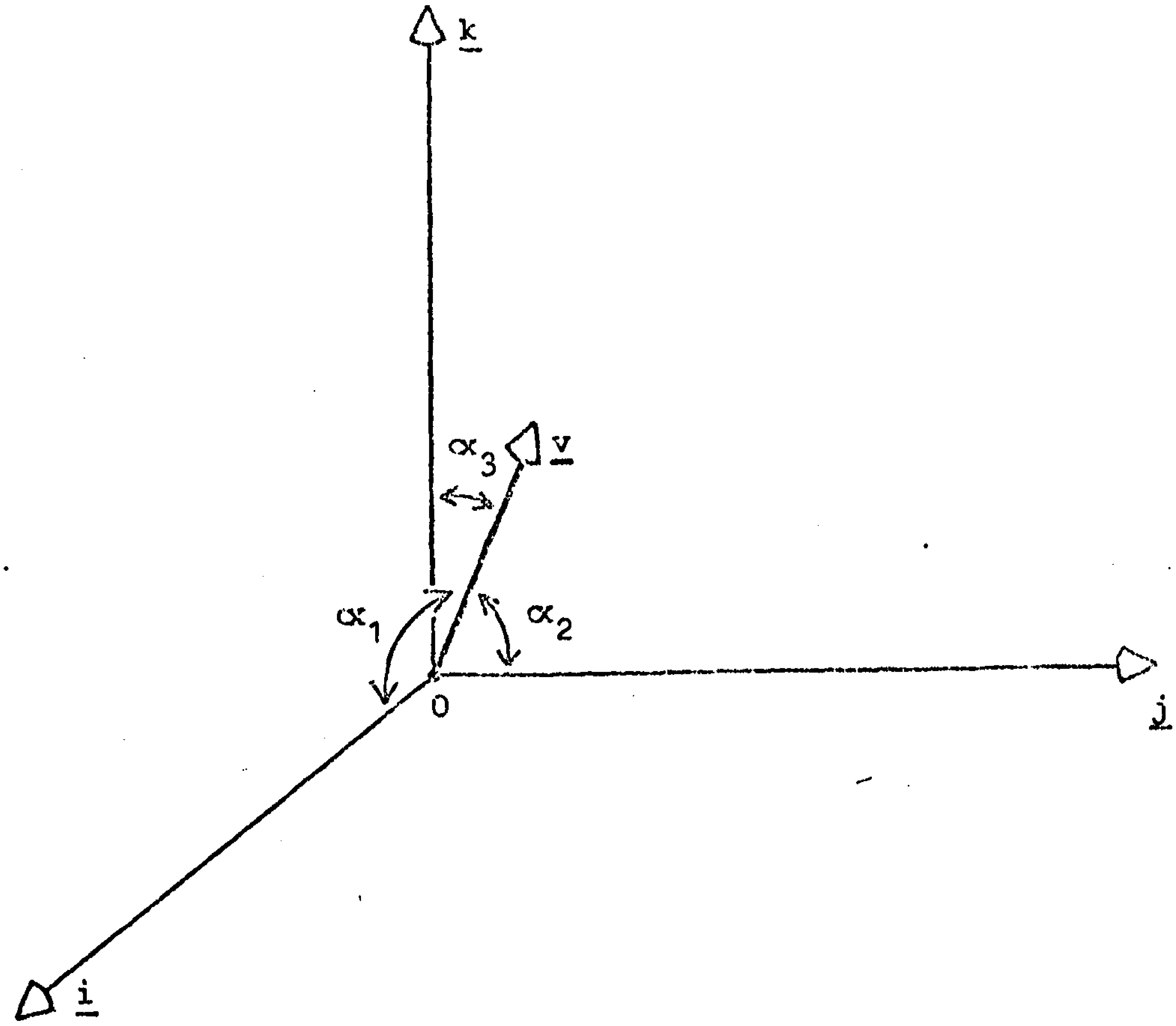
Figure 3.6 The Cosets of H (the Dual Axis).



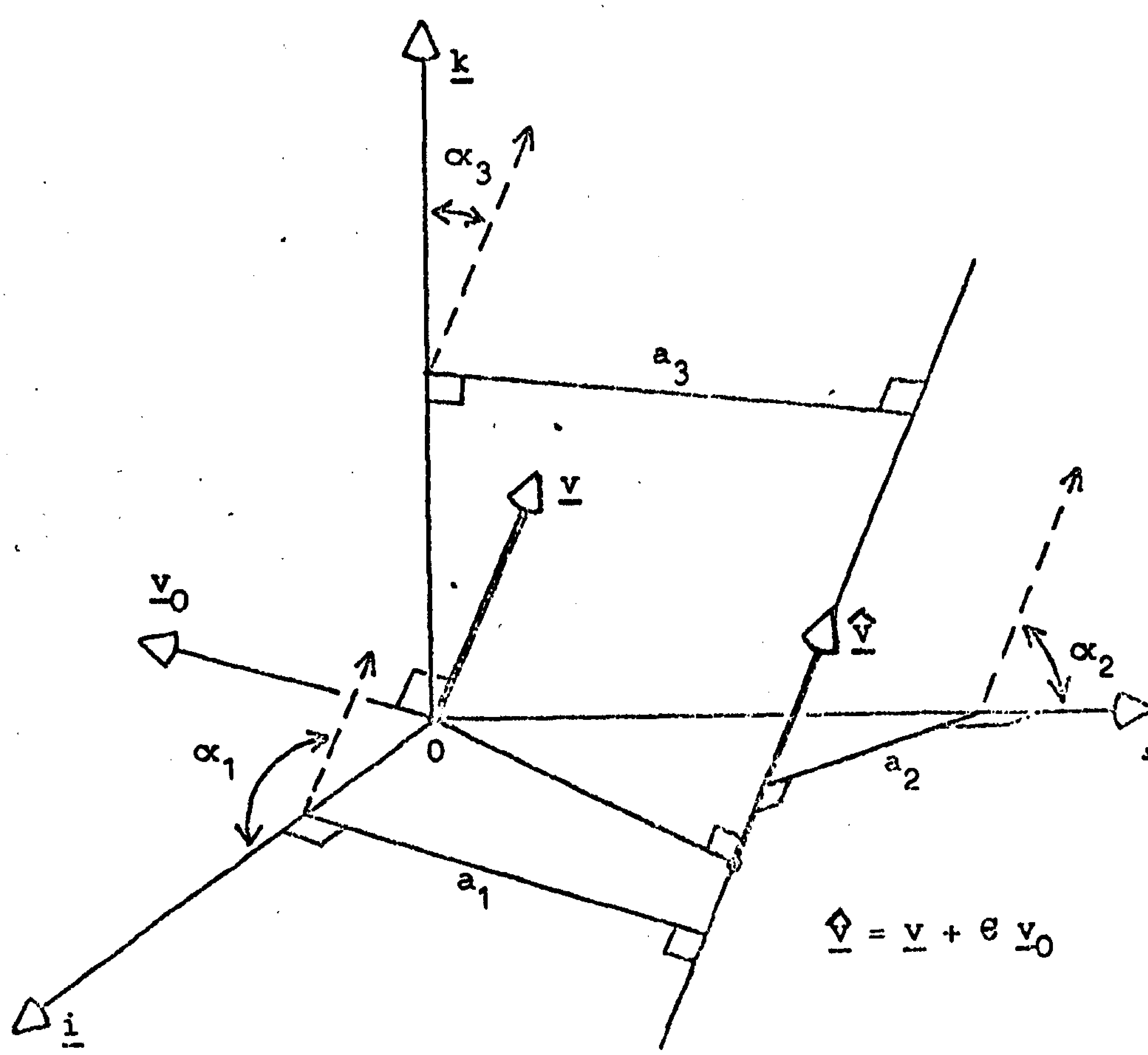
Motor, $\hat{v} = v + \epsilon\, v_0$

where $\quad v_0 = v_0' + v_0''$

Figure 3.7 The Significance of the Components of a General Dual Vector, $\hat{v}$, in Representing a Motor.

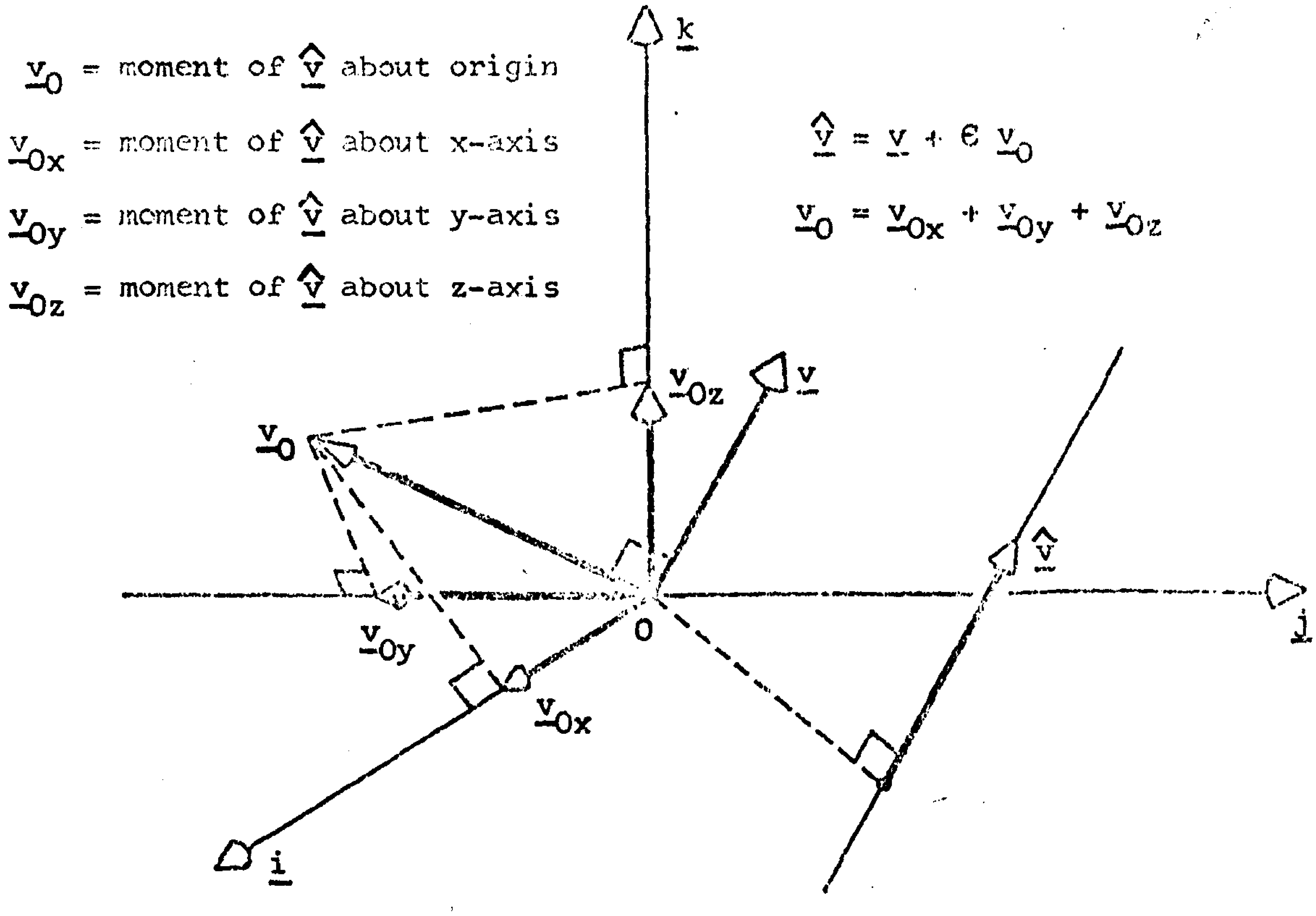


(a) Orientation of a Unit Free Vector, v .

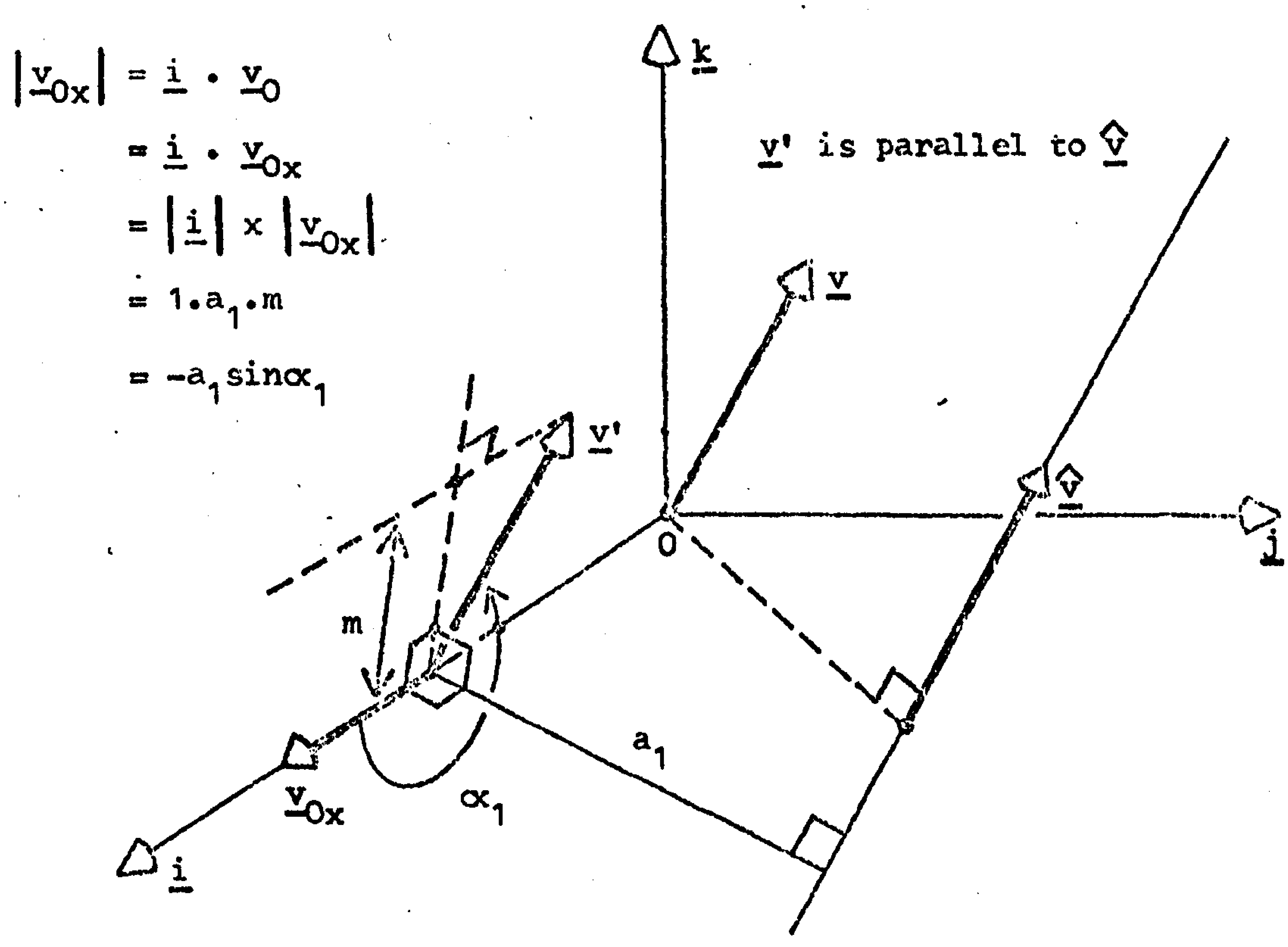


(b) Orientation of a Unit Line Vector, $\hat{v}$.

Figure 3.8 Comparison between a Unit Free Vector and a Unit Line Vector.

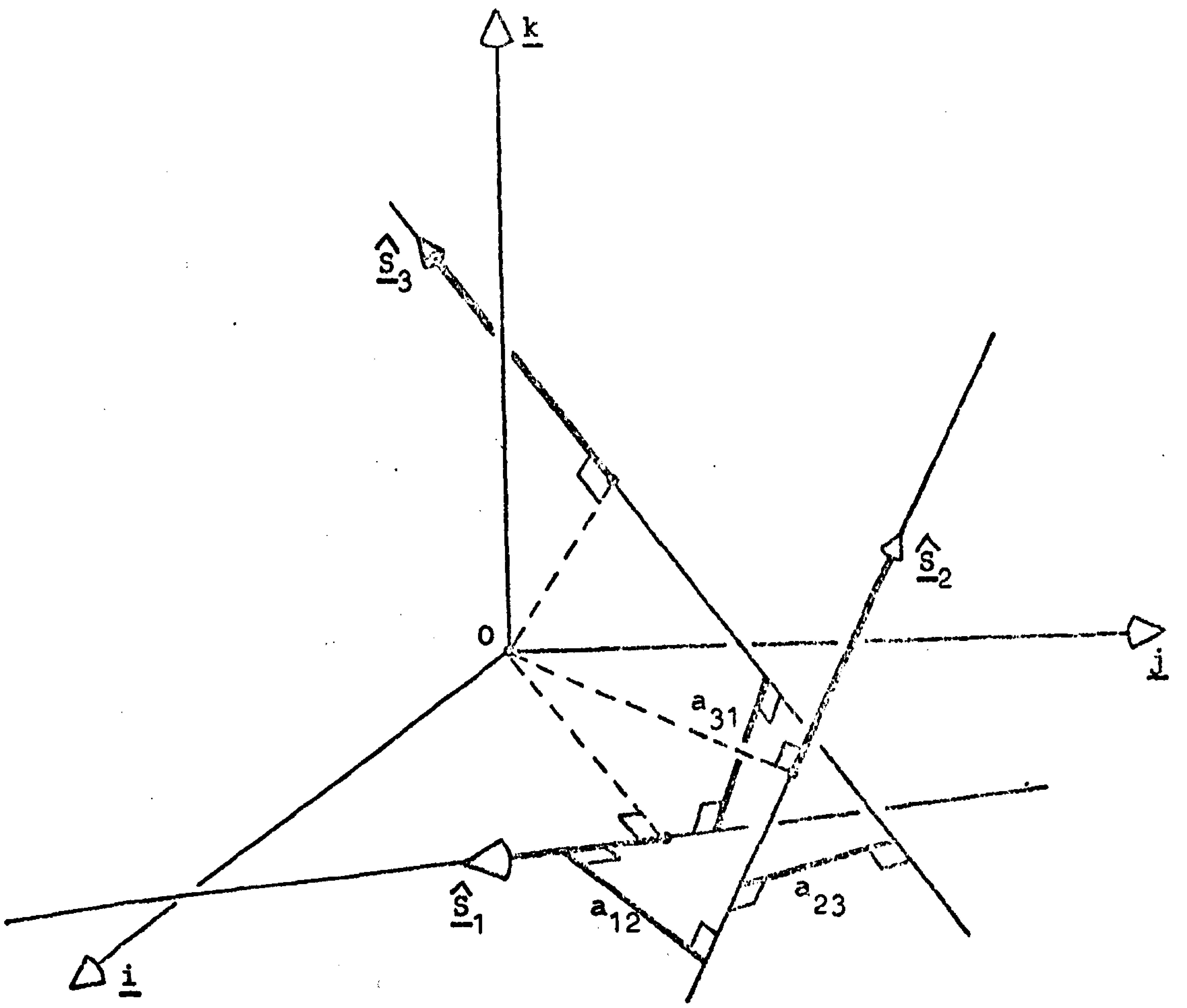


(a) The Components of $\underline{v}_0$.

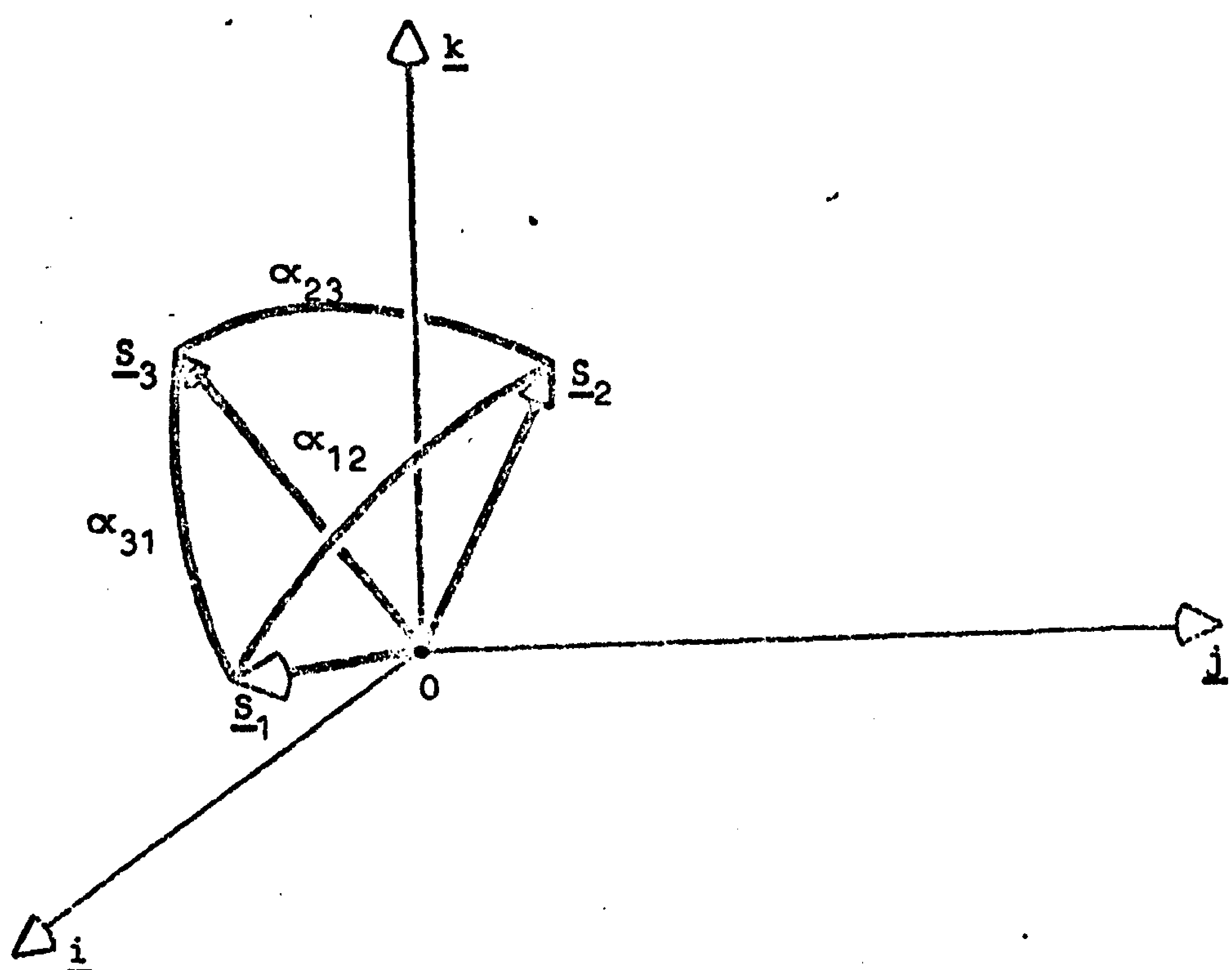


(b) The Magnitude of $\underline{v}_{0x}$.

Figure 3.9 The Significance of the Components of $\underline{v}_0$.



(a) The Three Unit Line Vectors Defining a Spatial Triangle.



(b) The Three Unit Line Vectors Defining a Spherical Triangle.

Figure 3.10 Comparison between Spherical and Spatial Geometry.

CHAPTER 4

DERIVATION OF LOOP EQUATIONS FOR SPHERICAL AND SPATIAL POLYGONS

4.1 Introduction.

In this chapter, loop equations are derived for spherical and spatial linkages using spherical trigonometry. Now it has been demonstrated in Chapter 3 (as a consequence of the Principle of Transference) that loop equations for spherical polygons can be extended to corresponding equations for spatial polygons simply by introducing the dual symbol. Consequently it is only necessary to derive spherical loop equations since the angular relationships for a spatial polygon are clearly identical to those of an equivalent spherical polygon. This concept of equivalent spherical and spatial configurations is due to Dobrovolski [8] and the importance of the dual angle in this context is now apparent.

The approach used here will be to derive a system of spherical loop equations from the fundamental trigonometrical laws for a spherical triangle. Then, by presenting a relatively simple and concise notation, these derived equations for spherical polygons can be categorised into a natural scheme of sine, sine-cosine and cosine laws, in analogy with the laws for a spherical triangle.

Earlier an attempt was made by Duffy [9] to introduce a unified notation for five, six and seven-link spherical mechanisms by extending the notation for the four-link spherical mechanism adopted by Yang and Freudenstein [46]. This was accomplished by adding triangles and using the sine and cosine laws [36] for the spherical triangle. However, although some equations were in a compact form, others were not, and much new notation had to be introduced.

In the past, what was not immediately clear was, firstly, the nature of the dependence of one loop equation on another, secondly, which equations could be considered to be basic or fundamental and finally how they should be classified.

In this chapter loop equations are derived for four, five, six and seven-link spherical mechanisms using the sine, cosine, and in addition the sine-cosine laws [36] for spherical triangles. The incorporation of the sine-cosine law for the spherical triangle, which has only recently been utilized in the analysis, has led to a significant change in the perspective of loop equations for spherical polygons. It is now clear which are fundamental formulae and what their origin is. Furthermore, from an algebraic point of view, the derivation of the various laws is now considerably more straightforward, whereas in [9], without the sine-cosine law, it was necessary to describe the derivations using lengthy Appendices.

The basis of these loop equations is hence a knowledge of spherical trigonometry and, in particular, an understanding of the basic laws relating the sides and angles of a spherical triangle.

4.2 The Spherical Triangle.

The spherical triangle (vertices 1, 2 and 3), defined by the three intersecting unit line vectors $\underline{S}_1$, $\underline{S}_2$ and $\underline{S}_3$ (see Chapter 3) is illustrated by Figure 4.1. The sides of the triangle are arcs of great circles of a sphere with unit radius and centre at O, the point of intersection of the line vectors. They are designated α_{12}, α_{23}, and α_{31}, as shown, whilst the exterior angles of the triangle are denoted by θ_1, θ_2 and θ_3. (The angles are considered to be positive when measured in an anti-clockwise sense as viewed from outside the sphere).

There are three fundamental laws for a spherical triangle and it is demonstrated in Todhunter and Leatham [36] that these formulae are valid for any spherical triangle (i.e. they are not restricted to triangles for which the angles and sides lie within the first quadrant). The three laws are the sine, sine-cosine and cosine laws. Thus the sine law is normally written as:-

$$\frac{\sin\alpha_{12}}{\sin\theta_3} = \frac{\sin\alpha_{23}}{\sin\theta_1} = \frac{\sin\alpha_{31}}{\sin\theta_2} \qquad (4.1)$$

which is, effectively, three equations; the three cyclic permutations of the cosine law are written as:-

$$\cos\alpha_{12} = \cos\alpha_{23}\cos\alpha_{31} - \sin\alpha_{23}\sin\alpha_{31}\cos\theta_3 \tag{4.2a}$$

$$\cos\alpha_{23} = \cos\alpha_{31}\cos\alpha_{12} - \sin\alpha_{31}\sin\alpha_{12}\cos\theta_1 \tag{4.2b}$$

$$\cos\alpha_{31} = \cos\alpha_{12}\cos\alpha_{23} - \sin\alpha_{12}\sin\alpha_{23}\cos\theta_2 \tag{4.2c}$$

and the six cyclic permutations of the sine-cosine law are written as:-

$$\sin\alpha_{12}\cos\theta_1 = -(\cos\alpha_{23}\sin\alpha_{31} + \sin\alpha_{23}\cos\alpha_{31}\cos\theta_3) \tag{4.3a}$$

$$\sin\alpha_{23}\cos\theta_2 = -(\cos\alpha_{31}\sin\alpha_{12} + \sin\alpha_{31}\cos\alpha_{12}\cos\theta_1) \tag{4.3b}$$

$$\sin\alpha_{31}\cos\theta_3 = -(\cos\alpha_{12}\sin\alpha_{23} + \sin\alpha_{12}\cos\alpha_{23}\cos\theta_2) \tag{4.3c}$$

$$\sin\alpha_{31}\cos\theta_1 = -(\cos\alpha_{23}\sin\alpha_{12} + \sin\alpha_{23}\cos\alpha_{12}\cos\theta_2) \tag{4.3d}$$

$$\sin\alpha_{12}\cos\theta_2 = -(\cos\alpha_{31}\sin\alpha_{23} + \sin\alpha_{31}\cos\alpha_{23}\cos\theta_3) \tag{4.3e}$$

$$\sin\alpha_{23}\cos\theta_3 = -(\cos\alpha_{12}\sin\alpha_{31} + \sin\alpha_{12}\cos\alpha_{31}\cos\theta_1) \tag{4.3f}$$

The three basic laws may be derived in a number of different ways, and the derivations are presented in [36] and in Brand [4]. However, the solution of any spherical triangle may be obtained using only the sine law, (4.1), and the cosine law, (4.2a-c), and hence the sine-cosine law is rarely used. Nevertheless, as will be seen, the latter is of considerable importance in the derivation of formulae for spherical polygons with more than three sides.

It must be noted that, of the three basic laws, only two are independent (apart from cyclic permutations) and each may be derived from a combination of the other two. Thus the sine-cosine law (4.3f), for example, may be considered to be a combination of the two cosine laws, (4.2a) and (4.2b), by observing that (4.2a) + $\cos\alpha_{31}$ x (4.2b) gives:-

$$\begin{matrix}\cos\alpha_{12} \\ +\cos\alpha_{31}\cos\alpha_{23}\end{matrix} = \begin{matrix}(\cos\alpha_{23}\cos\alpha_{31} - \sin\alpha_{23}\sin\alpha_{31}\cos\theta_3) \\ +\cos\alpha_{31}.(\cos\alpha_{31}\cos\alpha_{12} - \sin\alpha_{31}\sin\alpha_{12}\cos\theta_1)\end{matrix} \tag{4.4a}$$

which, upon simplification, becomes:-

$$\cos\alpha_{12}\sin^2\alpha_{31} = -\sin\alpha_{31}.(\sin\alpha_{23}\cos\theta_3 + \cos\alpha_{31}\sin\alpha_{12}\cos\theta_1) \tag{4.4b}$$

Rearranging equation (4.4b) one obtains:-

$$\sin\alpha_{23}\cos\theta_3 = -(\cos\alpha_{12}\sin\alpha_{31} + \sin\alpha_{12}\cos\alpha_{31}\cos\theta_1) \tag{4.4c}$$

which is identical to equation (4.3f), the sine-cosine law.

Finally, if the angles α_{12}, α_{23} and α_{31} of the spherical triangle are small the triangle should not differ significantly from a planar triangle, and hence the three laws should reduce to the corresponding planar laws. This is indeed found to be the case, if one makes the usual approximations for the sines and cosines of a small angle. Thus equations (4.1), (4.2a) and (4.3a) become respectively:-

$$\frac{\alpha_{12}}{\sin\theta_3} = \frac{\alpha_{23}}{\sin\theta_1} = \frac{\alpha_{31}}{\sin\theta_2} \qquad (4.5a)$$

(the sine law for a plane triangle).

$$\alpha_{12}^2 = \alpha_{23}^2 + \alpha_{31}^2 + 2.\alpha_{23}\alpha_{31}\cos\theta_3 \qquad (4.5b)$$

(the cosine law for a plane triangle).

and $$\alpha_{12}\cos\theta_1 = -(\alpha_{31} + \alpha_{23}\cos\theta_3) \qquad (4.5c)$$

(the 'sine-cosine' law for a plane triangle).

4.3 Notation for a Spherical Dyad.

Throughout the course of the work presented here, an extremely useful short-hand notation was developed for describing lengthy trigonometrical expressions, which led to the realisation that the three triangle laws could be extended to other spherical polygons. Thus, it was found convenient to define six basic trigonometric forms (involving two adjacent sides and the included angle) for a spherical triangle and these forms are clearly applicable to any spherical dyad. With reference to Figure 4.2(a) one may define the six basic forms as follows:-

$$X_j = \sin\alpha_{ij}\sin\theta_j \qquad (4.6a)$$

$$Y_j = -(\cos\alpha_{ij}\sin\alpha_{jk} + \sin\alpha_{ij}\cos\alpha_{jk}\cos\theta_j) \qquad (4.6b)$$

$$Z_j = (\cos\alpha_{ij}\cos\alpha_{jk} - \sin\alpha_{ij}\sin\alpha_{jk}\cos\theta_j) \qquad (4.6c)$$

$$\bar{X}_j = \sin\alpha_{jk}\sin\theta_j \qquad (4.7a)$$

$$\bar{Y}_j = -(\cos\alpha_{jk}\sin\alpha_{ij} + \sin\alpha_{jk}\cos\alpha_{ij}\cos\theta_j) \qquad (4.7b)$$

$$\bar{Z}_j = (\cos\alpha_{jk}\cos\alpha_{ij} - \sin\alpha_{jk}\sin\alpha_{ij}\cos\theta_j) \qquad (4.7c)$$

where each of the symbols X_j, Y_j, Z_j, $\bar{X}_j$, $\bar{Y}_j$ and $\bar{Z}_j$ is to be considered as a particular function of Θ_j. Thus, for example, $X_j \equiv X(\Theta_j)$, $Y_j \equiv Y(\Theta_j)$, $Z_j \equiv Z(\Theta_j)$ etc. Now from (4.6c) and (4.7c) it is apparent that Z_j is identically equal to $\bar{Z}_j$ and hence there exist only five distinct forms. However, it will be appreciated later that it is convenient to make a distinction between Z_j and $\bar{Z}_j$ for completeness, consistency and greater ease in manipulating the notation when considering spherical polygons with more than three sides.

As an aid to memory, it may be noted that the expressions for, say, X_j, Y_j, and Z_j are defined by approaching the angle Θ_j in an anti-clockwise sense along the preceeding side α_{ij}, whilst those for $\bar{X}_j$, $\bar{Y}_j$, and $\bar{Z}_j$ are defined by approaching Θ_j in a clockwise sense along the succeeding side, α_{jk}.

This scheme of barred and unbarred symbols was considered to be more suitable than one based on double-suffices, in order to resolve the ambiguities inherent in, say, the symbol X_j.

4.4 Identities for a Spherical Dyad.

From a consideration of equations (4.6) and (4.7) the following identities may easily be derived for a spherical dyad:-

$$Z_j \equiv \bar{Z}_j \quad (4.8)$$

$$X_j^2 + Y_j^2 + Z_j^2 \equiv 1 \quad (4.9a)$$

$$\bar{X}_j^2 + \bar{Y}_j^2 + \bar{Z}_j^2 \equiv 1 \quad (4.9b)$$

$$(\sin\alpha_{jk} Z_j + \cos\alpha_{jk} Y_j) \equiv -\sin\alpha_{ij} \cos\Theta_j \quad (4.10a)$$

$$(\cos\alpha_{jk} Z_j - \sin\alpha_{jk} Y_j) \equiv \cos\alpha_{ij} \quad (4.10b)$$

$$(\sin\alpha_{ij} \bar{Z}_j + \cos\alpha_{ij} \bar{Y}_j) \equiv -\sin\alpha_{jk} \cos\Theta_j \quad (4.11a)$$

$$(\cos\alpha_{ij} \bar{Z}_j - \sin\alpha_{ij} \bar{Y}_j) \equiv \cos\alpha_{jk} \quad (4.11b)$$

These identities have proved to be extremely useful in the derivation of loop equations and half-tangent laws (see Chapter 5.), and are clearly valid for all spherical polygons.

4.5 The Laws for a Spherical Triangle.

Using the notation introduced above, it is now possible to rewrite the three basic laws for a spherical triangle in a very concise form. Thus equations (4.1), (4.3b) and (4.2b) become respectively:-

Sine Law	$X_1 = \sin\alpha_{23}\sin\theta_2$	(4.12a)
Sine-Cosine Law	$Y_1 = \sin\alpha_{23}\cos\theta_2$	(4.12b)
Cosine Law	$Z_1 = \cos\alpha_{23}$	(4.12c)

whilst equations (4.1), (4.3f) and (4.2b) involve barred symbols and become:-

Sine Law	$\bar{X}_1 = \sin\alpha_{23}\sin\theta_3$	(4.13a)
Sine-Cosine Law	$\bar{Y}_1 = \sin\alpha_{23}\cos\theta_3$	(4.13b)
Cosine Law	$\bar{Z}_1 = \cos\alpha_{23}$	(4.13c)

All cyclic permutations can now easily be obtained from equations (4.12) and (4.13) by considering Figure 4.2(b), and an exhaustive list is given in Appendix III.

At this point it must be emphasised that, whereas equations (4.6) and (4.7) are identities defining the symbols X_j, Y_j, etc., equations (4.12) and (4.13) represent relationships that X_1, Y_1, etc., must satisfy.

The advantages of this particular notation for the spherical triangle are fairly minimal, however, and the real significance of the symbology will not become apparent until it is applied to the other spherical polygons.

4.6 Polar Spherical Triangles.

The spherical triangle with vertices 1,2 and 3 is illustrated in Figure 4.3. The sides of this triangle, being arcs of great circles, each define an axis perpendicular to the planes in which they lie and passing through the centre of the sphere. Thus the side α_{12} defines the axis represented by the unit line vector $\underline{a}_{12}$, chosen such that $\underline{S}_1$, $\underline{S}_2$ and $\underline{a}_{12}$ form a dextral set. The points 3', 1' and 2' at which $\underline{a}_{12}$, $\underline{a}_{23}$ and $\underline{a}_{31}$ intersect the sphere are termed the poles of the respective sides α_{12}, α_{23} and α_{31}, and these poles define a second spherical triangle with sides θ_1, θ_2 and θ_3, and exterior angles α_{12}, α_{23} and α_{31} as shown in Figure 4.3. The points 1, 2 and 3 at which $\underline{S}_1$, $\underline{S}_2$ and $\underline{S}_3$ intersect the sphere (i.e. the vertices of the original triangle) are themselves the poles of the sides θ_1, θ_2 and θ_3 of the second triangle.

The two triangles are thus said to be polar with respect to each other since there is a complete reciprocity between the vector sets $(\underline{S}_1, \underline{S}_2, \underline{S}_3)$ and $(\underline{a}_{12}, \underline{a}_{23}, \underline{a}_{31})$, and hence between the spherical triangles they determine (see also Brand [4]). As a consequence of this direct correspondence, it is possible to interchange the roles of the angles and sides of a spherical triangle and hence obtain equivalent sine, sine-cosine and cosine laws in terms of two exterior angles and an included side. Hence, making the following definitions in analogy with equations (4.6) and (4.7):-

$$U_{ij} = \sin\theta_i \sin\alpha_{ij} \tag{4.14a}$$

$$V_{ij} = -(\cos\theta_i \sin\theta_j + \sin\theta_i \cos\theta_j \cos\alpha_{ij}) \tag{4.14b}$$

$$W_{ij} = (\cos\theta_i \cos\theta_j - \sin\theta_i \sin\theta_j \cos\alpha_{ij}) \tag{4.14c}$$

one obtains a scheme of laws completely analogous to (4.12) and (4.13). Thus:-

Sine Law $$U_{12} = \sin\theta_3 \sin\alpha_{23} \tag{4.15a}$$

Sine-Cosine Law $$V_{12} = \sin\theta_3 \cos\alpha_{23} \tag{4.15b}$$

Cosine Law $$W_{12} = \cos\theta_3 \tag{4.15c}$$

Sine Law $$U_{21} = \sin\theta_3 \sin\alpha_{31} \tag{4.16a}$$

Sine-Cosine Law $$V_{21} = \sin\theta_3 \cos\alpha_{31} \tag{4.16b}$$

Cosine Law $$W_{21} = \cos\theta_3 \tag{4.16c}$$

Notice that there is no necessity to introduce barred symbols (such as $\bar{U}_{ij}$ etc.) since there is a double suffix involved in the U, V and W expressions and hence U_{ji} ($\neq U_{ij}$) takes the place of $\bar{U}_{ij}$.

Finally, it is possible to derive identities analogous to equations (4.8), (4.9), (4.10) and (4.11) for these polar expressions. Thus:-

$$W_{ij} \equiv W_{ji} \tag{4.17}$$

$$U_{ij}^2 + V_{ij}^2 + W_{ij}^2 \equiv 1 \tag{4.18a}$$

$$U_{ji}^2 + V_{ji}^2 + W_{ji}^2 \equiv 1 \tag{4.18b}$$

$$(\sin\theta_j W_{ij} + \cos\theta_j V_{ij}) \equiv -\sin\theta_i \cos\alpha_{ij} \tag{4.19a}$$

$$(\cos\theta_j W_{ij} - \sin\theta_j V_{ij}) \equiv \cos\theta_i \tag{4.19b}$$

$$(\sin\theta_i W_{ji} + \cos\theta_i V_{ji}) \equiv -\sin\theta_j \cos\alpha_{ij} \tag{4.20a}$$

$$(\cos\theta_i W_{ji} - \sin\theta_i V_{ji}) \equiv \cos\theta_j \tag{4.20b}$$

However, it was felt that it would be pointless and also superfluous to include exhaustive lists of these polar-triangle laws in an appendix since throughout the course of the work presented here, their relevance and usefulness has proved minimal.

4.7 Combinations of the Basic Laws.

It is possible to derive certain additional formulae from various combinations of the three basic laws for a spherical triangle. Thus, restating (4.12a), one obtains:-

$$\sin\alpha_{31} \sin\theta_1 = \bar{X}_2 \tag{4.21a}$$

Multiplying (4.12c) by $\sin\alpha_{12}$, (4.12b) by $\cos\alpha_{12}$ and adding, one obtains, after making use of identity (4.10a) (with i = 3, j = 1 and k = 2):-

$$\sin\alpha_{31} \cos\theta_1 = \bar{Y}_2 \tag{4.21b}$$

Finally, multiplying (4.12c) by $\cos\alpha_{12}$, (4.12b) by $\sin\alpha_{12}$ and subtracting, one obtains, after making use of identity (4.10b) (with $i = 3$, $j = 1$ and $k = 2$):-

$$\cos\alpha_{31} = \bar{Z}_2 \tag{4.21c}$$

However, in carrying out these manipulations, the additional formulae that have been obtained (i.e. (4.21a), (4.21b) and (4.21c)) are simply cyclic permutations of the three basic laws (equations (4.13)) and hence nothing new has been achieved. This is not the case when considering the other spherical polygons, however, and certain distinct subsidiary formulae can be derived in the above manner. Nevertheless, in the case of the triangle, what has been shown is that the three laws involving barred symbols and the three not involving barred symbols, are completely interdependent, and hence there are clearly only three basic or fundamental laws (i.e. equations (4.12a), (4.12b) and (4.12c)) termed the sine, sine-cosine and cosine laws, from which all other relationships can be derived.

This dependence of the many varieties of formula on the three basic laws is of fundamental importance when considering the spherical polygons with more than three sides, and it will be discussed in greater detail later in this chapter.

4.8 The Spherical Quadrilateral.

The spherical quadrilateral (vertices 1, 2, 3 and 4) defined by the four intersecting unit line vectors $\underline{S}_1$, $\underline{S}_2$, $\underline{S}_3$ and $\underline{S}_4$ is illustrated by Figure 4.4(a). The sides of the quadrilateral are designated α_{12}, α_{23}, α_{34}, α_{41} and the exterior angles are denoted by θ_1, θ_2, θ_3 and θ_4 as shown.

Consider the spherical quadrilateral to be formed from the two spherical triangles 123 and 134 with common side α_{31}. The loop equations for the quadrilateral can then be derived using the sine, sine-cosine and cosine laws for these two triangles, which are respectively:-

$$\sin\alpha_{31}\sin\theta_1' = \sin\alpha_{23}\sin\theta_2 \tag{4.22a}$$

$$-(\cos\alpha_{31}\sin\alpha_{12} + \sin\alpha_{31}\cos\alpha_{12}\cos\theta_1') = \sin\alpha_{23}\cos\theta_2 \tag{4.22b}$$

$$(\cos\alpha_{31}\cos\alpha_{12} - \sin\alpha_{31}\sin\alpha_{12}\cos\theta_1') = \cos\alpha_{23} \tag{4.22c}$$

and:-

$$\sin\alpha_{31}\sin\theta_1'' = X_4 \tag{4.23a}$$

$$\sin\alpha_{31}\cos\theta_1'' = Y_4 \tag{4.23b}$$

$$\cos\alpha_{31} = Z_4 \tag{4.23c}$$

In order to inter-relate the two sets of equations, a relationship between θ_1', θ_1'' and the desired θ_1, is required. From Figure 4.4(a) it is clear that this relationship is:-

$$\pi - \theta_1' = \theta_1'' - \theta_1 \tag{4.24}$$

and hence:-

$$\sin\theta_1' = (\sin\theta_1''\cos\theta_1 - \cos\theta_1''\sin\theta_1) \tag{4.25a}$$

and

$$\cos\theta_1' = -(\cos\theta_1''\cos\theta_1 + \sin\theta_1''\sin\theta_1) \tag{4.25b}$$

It is now possible to deduce three basic laws for the spherical quadrilateral, which are termed the <u>sine</u>, <u>sine-cosine</u> and <u>cosine</u> laws (because of their fundamental similarity to the corresponding triangle laws) as follows:-

(i) <u>Sine Law</u>

Substituting for $\sin\theta_1'$ from (4.25a) into the L.H.S. of equation (4.22a) and re-grouping terms, one obtains:-

$$(\sin\alpha_{31}\sin\theta_1'')\cos\theta_1 - (\sin\alpha_{31}\cos\theta_1'')\sin\theta_1 = \sin\alpha_{23}\sin\theta_2 \tag{4.26}$$

By means of (4.23a) and (4.23b), equation (4.26) becomes:-

$$X_4\cos\theta_1 - Y_4\sin\theta_1 = \sin\alpha_{23}\sin\theta_2 \tag{4.27}$$

Now, by defining:-

$$X_{41} = X_4\cos\theta_1 - Y_4\sin\theta_1 \tag{4.28}$$

equation (4.27) may be written as:-

$$X_{41} = \sin\alpha_{23}\sin\theta_2 \tag{4.29}$$

which is termed the <u>sine</u> law for a spherical quadrilateral, since it is analogous in form to equation (4.12a) for the spherical triangle. Again the symbol, X, has the meaning of a function, whilst it suffices denote that it has two arguments (θ_4 and θ_1 in this case).

(ii) <u>Sine-Cosine Law.</u>

Substituting for $\cos\theta_1'$ from (4.25b) into the L.H.S. of equation (4.22b) and re-grouping terms, one obtains:-

$$-\sin\alpha_{12}(\cos\alpha_{31}) + \cos\alpha_{12}(\sin\alpha_{31}\cos\theta_1'')\cos\theta_1 + \cos\alpha_{12}(\sin\alpha_{31}\sin\theta_1'')\sin\theta_1 = \sin\alpha_{23}\cos\theta_2 \tag{4.30}$$

By means of (4.23a), (4.23b) and (4.23c) this becomes:-

$$\cos\alpha_{12}(X_4\sin\theta_1 + Y_4\cos\theta_1) - \sin\alpha_{12}Z_4 = \sin\alpha_{23}\cos\theta_2 \tag{4.31}$$

Now, by defining:-

$$Y_{41} = \cos\alpha_{12}(X_4\sin\theta_1 + Y_4\cos\theta_1) - \sin\alpha_{12}Z_4 \tag{4.32}$$

equation (4.31) may be written as:-

$$Y_{41} = \sin\alpha_{23}\cos\theta_2 \tag{4.33}$$

which is termed the <u>sine-cosine</u> law for a spherical quadrilateral, since it is analogous in form to equation (4.12b), for the spherical triangle.

(iii) <u>Cosine Law</u>

Substituting for $\cos\theta_1'$ from (4.25b) into the L.H.S. of equation (4.22c) and re-grouping terms, one obtains:-

$$\cos\alpha_{12}(\cos\alpha_{31}) + \sin\alpha_{12}(\sin\alpha_{31}\cos\theta_1'')\cos\theta_1 + \sin\alpha_{12}(\sin\alpha_{31}\sin\theta_1'')\sin\theta_1 = \cos\alpha_{23} \tag{4.34}$$

By means of (4.23a), (4.23b) and (4.23c) this becomes:-

$$\sin\alpha_{12}(X_4\sin\theta_1 + Y_4\cos\theta_1) + \cos\alpha_{12}Z_4 = \cos\alpha_{23} \tag{4.35}$$

Now, by defining:-

$$Z_{41} = \sin\alpha_{12}(X_4\sin\theta_1 + Y_4\cos\theta_1) + \cos\alpha_{12}Z_4 \tag{4.36}$$

equation (4.35) may be written as:-

$$Z_{41} = \cos\alpha_{23} \tag{4.37}$$

which is termed the cosine law for a spherical quadrilateral, since it is analogous in form to equation (4.12c) for the spherical triangle. Notice that from (4.36) and definitions (4.6) (with i = 3, j = 4, k = 1) it can easily be shown that $Z_{41} \equiv Z_{14}$ (i.e. the 'Z' expression is symmetric with respect to its suffices).

Now, equations (4.29), (4.33) and (4.37) (the sine, sine-cosine and cosine laws for a spherical quadrilateral) are considered to be fundamental formulae and are sufficient to determine completely any spherical quadrilateral provided any five of its eight elements (i.e. four sides and four exterior angles) are specified. The definitions (4.28), (4.32) and (4.36) can clearly be written in general form (applicable to any three adjacent spherical links) as:-

$$X_{ij} = X_i\cos\theta_j - Y_i\sin\theta_j \tag{4.38a}$$

$$Y_{ij} = \cos\alpha_{jk}(X_i\sin\theta_j + Y_i\cos\theta_j) - \sin\alpha_{jk}Z_i \tag{4.38b}$$

$$Z_{ij} = \sin\alpha_{jk}(X_i\sin\theta_j + Y_i\cos\theta_j) + \cos\alpha_{jk}Z_i \tag{4.38c}$$

$$X_{kj} = \bar{X}_k\cos\theta_j - \bar{Y}_k\sin\theta_j \tag{4.39a}$$

$$Y_{kj} = \cos\alpha_{ij}(\bar{X}_k\sin\theta_j + \bar{Y}_k\cos\theta_j) - \sin\alpha_{ij}\bar{Z}_k \tag{4.39b}$$

$$Z_{kj} = \sin\alpha_{ij}(\bar{X}_k\sin\theta_j + \bar{Y}_k\cos\theta_j) + \cos\alpha_{ij}\bar{Z}_k \tag{4.39c}$$

where i, j and k are in ascending consecutive cyclic order, and with the aid of (4.38), (4.39) and Figure 4.4(b) all cyclic permutations of the three basic laws may be easily obtained. Appendix III. contains a complete list of the latter.

Finally, it must be noted that a set of laws may be derived for the polar quadrilateral, in complete analogy with equations (4.15) and (4.16).

A representative of each law may be written as follows:-

$$U_{123} = \sin\theta_4 \sin\alpha_{34} \quad (4.40a)$$

$$V_{123} = \sin\theta_4 \cos\alpha_{34} \quad (4.40b)$$

$$W_{123} = \cos\theta_4 \quad (4.40c)$$

where the U, V and W expressions are defined by:-

$$U_{123} = U_{12}\cos\alpha_{23} - V_{12}\sin\alpha_{23} \quad (4.41a)$$

$$V_{123} = \cos\theta_3(U_{12}\sin\alpha_{23} + V_{12}\cos\alpha_{23}) - \sin\theta_3 W_{12} \quad (4.41b)$$

$$W_{123} = \sin\theta_3(U_{12}\sin\alpha_{23} + V_{12}\cos\alpha_{23}) + \cos\theta_3 W_{12} \quad (4.41c)$$

and U_{12}, V_{12} and W_{12} may be obtained from definitions (4.14).

However, the author is of the opinion that, since the polar laws (for all spherical polygons) closely resemble the basic laws, in form, it is unnecessary and would be largely irrelevant to include exhaustive lists of these polar laws, in view of the ease with which they may be written down using the above notation.

4.9 Identities for a Spherical Quadrilateral.

In addition to identities (4.8), (4.9), (4.10) and (4.11), which are valid for any spherical dyad, it is possible to derive the following identities from equations (4.38) and (4.39), using (4.6) and (4.7), for a spherical quadrilateral:-

$$Z_{ij} \equiv Z_{ji} \quad (4.42)$$

$$X_{ij}^2 + Y_{ij}^2 + Z_{ij}^2 \equiv 1 \quad (4.43a)$$

$$X_{kj}^2 + Y_{kj}^2 + Z_{kj}^2 \equiv 1 \quad (4.43b)$$

$$(\sin\alpha_{jk} Z_{ij} + \cos\alpha_{jk} Y_{ij}) \equiv (X_i \sin\theta_j + Y_i \cos\theta_j) \quad (4.44a)$$

$$(\cos\alpha_{jk} Z_{ij} - \sin\alpha_{jk} Y_{ij}) \equiv Z_i \quad (4.44b)$$

$$(\sin\alpha_{ij} Z_{kj} + \cos\alpha_{ij} Y_{kj}) \equiv (\bar{X}_k \sin\theta_j + \bar{Y}_k \cos\theta_j) \quad (4.45a)$$

$$(\cos\alpha_{ij} Z_{kj} - \sin\alpha_{ij} Y_{kj}) \equiv \bar{Z}_k \quad (4.45b)$$

Finally, the following two identities, which are self-evident, may be stated, as they are useful in some of the derivations:-

$$\cos\theta_j(X_i\cos\theta_j - Y_i\sin\theta_j) + \sin\theta_j(X_i\sin\theta_j + Y_i\cos\theta_j) \equiv X_i \tag{4.46a}$$

$$\sin\theta_j(X_i\cos\theta_j - Y_i\sin\theta_j) - \cos\theta_j(X_i\sin\theta_j + Y_i\cos\theta_j) \equiv -Y_i \tag{4.46b}$$

4.10 Subsidiary Formulae for a Spherical Quadrilateral.

In analogy with the triangle laws it is possible to derive certain subsidiary formulae from the basic quadrilateral sine, sine-cosine and cosine laws (i.e. equations (4.29), (4.33) and (4.37)). Thus, restating (4.29) one obtains:-

$$(X_4\cos\theta_1 - Y_4\sin\theta_1) = \bar{X}_2 \tag{4.47a}$$

Multiplying (4.37) by $\sin\alpha_{12}$, (4.33) by $\cos\alpha_{12}$ and adding, one obtains, after making use of identity (4.44a) (with i = 4, j = 1, k = 2):-

$$(X_4\sin\theta_1 + Y_4\cos\theta_1) = -\bar{Y}_2 \tag{4.47b}$$

Finally, multiplying (4.37) by $\cos\alpha_{12}$, (4.33) by $\sin\alpha_{12}$ and subtracting, one obtains, after making use of identity (4.44b) (with i = 4, j = 1, k = 2):-

$$Z_4 = \bar{Z}_2 \tag{4.47c}$$

Equations (4.47a), (4.47b) and (4.47c) are termed <u>subsidiary</u> sine, sine-cosine and cosine laws respectively and are extremely useful for the purposes of displacement analyses.

However, some further laws may be obtained from equations (4.47) by using identities (4.46a) and (4.46b), and these are:-

$$X_4 = (\bar{X}_2\cos\theta_1 - \bar{Y}_2\sin\theta_1) \tag{4.48a}$$

$$-Y_4 = (\bar{X}_2\sin\theta_1 + \bar{Y}_2\cos\theta_1) \tag{4.48b}$$

$$Z_4 = \bar{Z}_2 \tag{4.48c}$$

These laws are merely cyclic permutations of (4.47a,b,c), but their complete functional dependence on the latter, and hence on the initial laws (4.29), (4.33) and (4.37) has clearly been demonstrated.

One final set of 'subsidiary' laws may be derived, with the aid of identities (4.10a) and (4.10b), which are:-

$$\sin\alpha_{34}\sin\theta_4 = X_{21} \quad (4.49a)$$

$$\sin\alpha_{34}\cos\theta_4 = Y_{21} \quad (4.49b)$$

$$\cos\alpha_{34} = Z_{21} \quad (4.49c)$$

and these are just cyclic permutations of the basic sine, sine-cosine and cosine laws (4.29), (4.33) and (4.37). However, what has been shown is that all the laws involving three out of the four angles of a spherical quadrilateral can be derived in a systematic way from various combinations of the three fundamental laws in those angles. Furthermore, in the course of the derivations, certain extra subsidiary relationships or formulae arise, such as equations (4.47b) and (4.47c), which were not apparent in the case of the triangle. All fundamental and subsidiary formulae for a spherical quadrilateral are listed in Appendix III.

4.11 The Spherical Pentagon.

The spherical pentagon (vertices 1, 2, 3, 4 and 5) defined by the five intersecting unit line vectors $\underline{S}_1$, $\underline{S}_2$, $\underline{S}_3$, $\underline{S}_4$ and $\underline{S}_5$ is illustrated by Figure 4.5(a). The sides of the pentagon are designated α_{12}, α_{23}, α_{34}, α_{45}, α_{51}, and the exterior angles are denoted by θ_1, θ_2, θ_3, θ_4 and θ_5 as shown.

Consider the spherical pentagon to be formed from a spherical triangle, 234, and a spherical quadrilateral, 1245, with common side, α_{42}. The loop equations for the pentagon can then be derived using the sine, sine-cosine and cosine laws for this triangle and quadrilateral, which are, respectively:-

$$\sin\alpha_{42}\sin\theta_2' = \sin\alpha_{34}\sin\theta_3 \quad (4.50a)$$

$$-(\cos\alpha_{42}\sin\alpha_{23} + \sin\alpha_{42}\cos\alpha_{23}\cos\theta_2') = \sin\alpha_{34}\cos\theta_3 \quad (4.50b)$$

$$(\cos\alpha_{42}\cos\alpha_{23} - \sin\alpha_{42}\sin\alpha_{23}\cos\theta_2') = \cos\alpha_{34} \quad (4.50c)$$

and:-

$$\sin\alpha_{42}\sin\theta''_2 = X_{51} \qquad (4.51a)$$

$$\sin\alpha_{42}\cos\theta''_2 = Y_{51} \qquad (4.51b)$$

$$\cos\alpha_{42} = Z_{51} \qquad (4.51c)$$

In addition, the following relationship holds (see Figure 4.5(a)):-

$$\pi - \theta'_2 = \theta''_2 - \theta_2 \qquad (4.52)$$

and hence one may derive the three laws for a pentagon as:-

Sine Law $\quad X_{512} = \sin\alpha_{34}\sin\theta_3 \qquad (4.53a)$

Sine-Cosine Law $\quad Y_{512} = \sin\alpha_{34}\cos\theta_3 \qquad (4.53b)$

Cosine Law $\quad Z_{512} = \cos\alpha_{34} \qquad (4.53c)$

where the terms X_{512}, Y_{512} and Z_{512} are defined by:-

$$X_{512} = X_{51}\cos\theta_2 - Y_{51}\sin\theta_2 \qquad (4.54a)$$

$$Y_{512} = \cos\alpha_{23}(X_{51}\sin\theta_2 + Y_{51}\cos\theta_2) - \sin\alpha_{23}Z_{51} \qquad (4.54b)$$

$$Z_{512} = \sin\alpha_{23}(X_{51}\sin\theta_2 + Y_{51}\cos\theta_2) + \cos\alpha_{23}Z_{51} \qquad (4.54c)$$

and X_{51}, Y_{51} and Z_{51} are obtained from equations (4.38) (with $i = 5$, $j = 1$).

Equations (4.53a), (4.53b) and (4.53c) (the sine, sine-cosine and cosine laws for a spherical pentagon) are fundamental formulae and are sufficient to determine completely any spherical pentagon provided any seven of its ten elements are specified. Furthermore, the definitions (4.54) can clearly be generalised for any three adjacent angles of a spherical polygon and from this it can be shown that $Z_{512} \equiv Z_{215}$.

A complete list of all cyclic permutations of the three basic laws for a spherical pentagon may be written with the aid of Figure 4.5(b) and is given in Appendix III.

4.12 Identities for a Spherical Pentagon.

Some further identities may be derived from the spherical pentagon laws, in addition to those which are valid for any spherical dyad, and the quadrilateral identities. These may be listed as follows and are valid for any three adjacent angles of a spherical polygon:-

$$Z_{ijk} \equiv Z_{kji} \tag{4.55}$$

$$X_{ijk}^2 + Y_{ijk}^2 + Z_{ijk}^2 \equiv 1 \tag{4.56}$$

$$(\sin\alpha_{kl} Z_{ijk} + \cos\alpha_{kl} Y_{ijk}) \equiv (X_{ij}\sin\theta_k + Y_{ij}\cos\theta_k) \tag{4.57a}$$

$$(\cos\alpha_{kl} Z_{ijk} - \sin\alpha_{kl} Y_{ijk}) \equiv Z_{ij} \tag{4.57b}$$

Finally, two useful self-evident identities may be listed as:-

$$\cos\theta_k(X_{ij}\cos\theta_k - Y_{ij}\sin\theta_k) + \sin\theta_k(X_{ij}\sin\theta_k + Y_{ij}\cos\theta_k) \equiv X_{ij} \tag{4.58a}$$

$$\sin\theta_k(X_{ij}\cos\theta_k - Y_{ij}\sin\theta_k) - \cos\theta_k(X_{ij}\sin\theta_k + Y_{ij}\cos\theta_k) \equiv -Y_{ij} \tag{4.58b}$$

4.13 Subsidiary Formulae for a Spherical Pentagon.

By adopting the procedures outlined for the quadrilateral it is possible to obtain two distinct groups of subsidiary laws for the spherical pentagon, where only one such group occurred for the quadrilateral. The subsidiary sine, sine-cosine and cosine laws in the first group may be written respectively as:-

$$(X_{51}\cos\theta_2 - Y_{51}\sin\theta_2) = \bar{X}_3 \tag{4.59a}$$

$$(X_{51}\sin\theta_2 + Y_{51}\cos\theta_2) = -\bar{Y}_3 \tag{4.59b}$$

$$Z_{51} = \bar{Z}_3 \tag{4.59c}$$

where (4.59a) is a restatement of (4.53a); equation (4.59b) is derived by adding $\cos\alpha_{23}$ x (4.53b) to $\sin\alpha_{23}$ x (4.53c) and using (4.57a); and finally equation (4.59c) is derived by subtracting $\sin\alpha_{23}$ x (4.53b) from $\cos\alpha_{23}$ x (4.53c) and using (4.57b).

In a similar manner the three subsidiary laws in the second group may be written:-

$$X_{51} = (\bar{X}_3\cos\theta_2 - \bar{Y}_3\sin\theta_2) \quad (4.60a)$$

$$-Y_{51} = (\bar{X}_3\sin\theta_2 + \bar{Y}_3\cos\theta_2) \quad (4.60b)$$

$$Z_{51} = \bar{Z}_3 \quad (4.60c)$$

and these are derived from equations (4.59) and identities (4.58). Notice that although (4.59c) and (4.60c) are identical the equation is included in each of the two groups since it possesses the characteristics of both.

Appendix III. contains an exhaustive list of all cyclic permutations of the subsidiary sine, sine-cosine and cosine laws for a spherical pentagon.

4.14 The Spherical Hexagon.

The spherical hexagon (vertices 1, 2, 3, 4, 5 and 6) defined by the six intersecting unit line vectors $\underline{S}_1$, $\underline{S}_2$, $\underline{S}_3$, $\underline{S}_4$, $\underline{S}_5$ and $\underline{S}_6$ is illustrated by Figure 4.6(a). The sides of the hexagon are designated α_{12}, α_{23}, α_{34}, α_{45}, α_{56}, α_{61} and the exterior angles are denoted by θ_1, θ_2, θ_3, θ_4 θ_5 and θ_6 as shown.

Considering the spherical hexagon to be formed by adding a spherical triangle, 345, to a spherical pentagon, 12356, with common side α_{53}, one may write the three laws for the triangle as:-

$$\sin\alpha_{53}\sin\theta'_3 = \sin\alpha_{45}\sin\theta_4 \quad (4.61a)$$

$$-(\cos\alpha_{53}\sin\alpha_{34} + \sin\alpha_{53}\cos\alpha_{34}\cos\theta'_3) = \sin\alpha_{45}\cos\theta_4 \quad (4.61b)$$

$$(\cos\alpha_{53}\cos\alpha_{34} - \sin\alpha_{53}\sin\alpha_{34}\cos\theta'_3) = \cos\alpha_{45} \quad (4.61c)$$

and those for the pentagon as:-

$$\sin\alpha_{53}\sin\theta''_3 = X_{612} \quad (4.62a)$$

$$\sin\alpha_{53}\cos\theta''_3 = Y_{612} \quad (4.62b)$$

$$\cos\alpha_{53} = Z_{612} \quad (4.62c)$$

In addition, from Figure 4.6(a) one has:-

$$\pi - \theta'_3 = \theta''_3 - \theta_3 \quad (4.63)$$

and hence, in exact analogy with the pentagon, the following three fundamental laws may be derived for the spherical hexagon:-

Sine Law $X_{6123} = \sin\alpha_{45}\sin\theta_4$ (4.64a)

Sine-Cosine Law $Y_{6123} = \sin\alpha_{45}\cos\theta_4$ (4.64b)

Cosine Law $Z_{6123} = \cos\alpha_{45}$ (4.64c)

where the terms X_{6123}, Y_{6123} and Z_{6123} are defined by:-

$$X_{6123} = X_{612}\cos\theta_3 - Y_{612}\sin\theta_3 \tag{4.65a}$$

$$Y_{6123} = \cos\alpha_{34}(X_{612}\sin\theta_3 + Y_{612}\cos\theta_3) - \sin\alpha_{34}Z_{612} \tag{4.65b}$$

$$Z_{6123} = \sin\alpha_{34}(X_{612}\sin\theta_3 + Y_{612}\cos\theta_3) + \cos\alpha_{34}Z_{612} \tag{4.65c}$$

and are each clearly functions of four angular displacements (i.e. θ_6, θ_1, θ_2 and θ_3). Furthermore, these definitions can be generalised for any four adjacent angles of a spherical polygon and a complete list of these and all cyclic permutations of equations (4.64a), (4.64b) and (4.64c) (the sine, sine-cosine and cosine laws for a spherical hexagon), which are fundamental formulae, is given in Appendix III. with the aid of Figure 4.6(b).

4.15 Identities for a Spherical Hexagon.

The following additional identities may be derived from the definitions (4.65):-

$$Z_{ijkl} \equiv Z_{lkji} \tag{4.66}$$

$$X^2_{ijkl} + Y^2_{ijkl} + Z^2_{ijkl} \equiv 1 \tag{4.67}$$

$$(\sin\alpha_{lm}Z_{ijkl} + \cos\alpha_{lm}Y_{ijkl}) \equiv (X_{ijk}\sin\theta_l + Y_{ijk}\cos\theta_l) \tag{4.68a}$$

$$(\cos\alpha_{lm}Z_{ijkl} - \sin\alpha_{lm}Y_{ijkl}) \equiv Z_{ijk} \tag{4.68b}$$

and the two self-evident identities are written:-

$$\cos\theta_l(X_{ijk}\cos\theta_l - Y_{ijk}\sin\theta_l) + \sin\theta_l(X_{ijk}\sin\theta_l + Y_{ijk}\cos\theta_l) \equiv X_{ijk} \tag{4.69a}$$

$$\sin\theta_l(X_{ijk}\cos\theta_l - Y_{ijk}\sin\theta_l) - \cos\theta_l(X_{ijk}\sin\theta_l + Y_{ijk}\cos\theta_l) \equiv -Y_{ijk} \tag{4.69b}$$

In each of the above identities i, j, k, l and m are positive integers in consecutive cyclic order.

4.16 Subsidiary Formulae for a Spherical Hexagon.

For the spherical hexagon it is possible to derive three distinct groups of subsidiary laws using procedures exactly analogous to those for the pentagon. Thus the subsidiary sine, sine-cosine and cosine laws in the first group may be written:-

$$(X_{612}\cos\theta_3 - Y_{612}\sin\theta_3) = \bar{\bar{X}}_4 \qquad (4.70a)$$

$$(X_{612}\sin\theta_3 + Y_{612}\cos\theta_3) = -\bar{\bar{Y}}_4 \qquad (4.70b)$$

$$Z_{612} = \bar{\bar{Z}}_4 \qquad (4.70c)$$

and these are obtained from equations (4.64) with the aid of identities (4.68). The laws in the second group are then obtained from equations (4.70), using (4.69a,b) and are:-

$$X_{612} = (\bar{\bar{X}}_4\cos\theta_3 - \bar{\bar{Y}}_4\sin\theta_3) \qquad (4.71a)$$

$$-Y_{612} = (\bar{\bar{X}}_4\sin\theta_3 + \bar{\bar{Y}}_4\cos\theta_3) \qquad (4.71b)$$

$$Z_{612} = \bar{\bar{Z}}_4 \qquad (4.71c)$$

Finally the laws in the third group are derived from equations (4.71) using (4.57a,b) and are written:-

$$(X_{61}\cos\theta_2 - Y_{61}\sin\theta_2) = X_{43} \qquad (4.72a)$$

$$(X_{61}\sin\theta_2 + Y_{61}\cos\theta_2) = -Y_{43} \qquad (4.72b)$$

$$Z_{61} = Z_{43} \qquad (4.72c)$$

All cyclic permutations of these laws are given in Appendix III.

4.17 The Spherical Heptagon.

The spherical heptagon (vertices 1, 2, 3, 4, 5, 6 and 7) defined by the seven intersecting unit line vectors $\underline{S}_1$, $\underline{S}_2$, $\underline{S}_3$, $\underline{S}_4$, $\underline{S}_5$, $\underline{S}_6$ and $\underline{S}_7$ is illustrated by Figure 4.7(a). The sides of the heptagon are designated α_{12}, α_{23}, α_{34}, α_{45}, α_{56}, α_{67}, α_{71} and the exterior angles are denoted by θ_1, θ_2, θ_3, θ_4, θ_5, θ_6 and θ_7 as shown. Constructing the heptagon by adding a spherical triangle, 456, to a spherical hexagon, 123467, and removing the common side α_{64}, one may write the laws for the triangle:-

$$\sin\alpha_{64}\sin\theta_4' = \sin\alpha_{56}\sin\theta_5 \tag{4.73a}$$

$$-(\cos\alpha_{64}\sin\alpha_{45} + \sin\alpha_{64}\cos\alpha_{45}\cos\theta_4') = \sin\alpha_{56}\cos\theta_5 \tag{4.73b}$$

$$(\cos\alpha_{64}\cos\alpha_{45} - \sin\alpha_{64}\sin\alpha_{45}\cos\theta_4') = \cos\alpha_{56} \tag{4.73c}$$

and for the hexagon:-

$$\sin\alpha_{64}\sin\theta_4'' = X_{7123} \tag{4.74a}$$

$$\sin\alpha_{64}\cos\theta_4'' = Y_{7123} \tag{4.74b}$$

$$\cos\alpha_{64} = Z_{7123} \tag{4.74c}$$

from which, together with the relationship (see Figure 4.7(a)):-

$$\pi - \theta_4' = \theta_4'' - \theta_4 \tag{4.75}$$

one may derive the three fundamental laws for a spherical heptagon. These are:-

Sine Law	$X_{71234} = \sin\alpha_{56}\sin\theta_5$	(4.76a)
Sine-Cosine Law	$Y_{71234} = \sin\alpha_{56}\cos\theta_5$	(4.76b)
Cosine Law	$Z_{71234} = \cos\alpha_{56}$	(4.76c)

where the terms X_{71234}, Y_{71234} and Z_{71234} are defined by:-

$$X_{71234} = X_{7123}\cos\theta_4 - Y_{7123}\sin\theta_4 \tag{4.77a}$$

$$Y_{71234} = \cos\alpha_{45}(X_{7123}\sin\theta_4 + Y_{7123}\cos\theta_4) - \sin\alpha_{45}Z_{7123} \tag{4.77b}$$

$$Z_{71234} = \sin\alpha_{45}(X_{7123}\sin\theta_4 + Y_{7123}\cos\theta_4) + \cos\alpha_{45}Z_{7123} \tag{4.77c}$$

and are each functions of five angular displacements (i.e. θ_7, θ_1, θ_2, θ_3, and θ_4). As with the other polygons, these definitions can be generalised for any five adjacent angles of a spherical polygon, and a complete list of cyclic permutations of the three laws (i.e. equations (4.76)) is given in Appendix III., with the aid of Figure 4.7(b).

4.18 Identities for a Spherical Heptagon.

The following additional identities may be derived from the definitions (4.77):-

$$Z_{ijklm} \equiv Z_{mlkji} \qquad (4.78)$$

$$X^2_{ijklm} + Y^2_{ijklm} + Z^2_{ijklm} \equiv 1 \qquad (4.79)$$

$$(\sin\alpha_{mn} Z_{ijklm} + \cos\alpha_{mn} Y_{ijklm}) \equiv (X_{ijkl}\sin\theta_m + Y_{ijkl}\cos\theta_m) \qquad (4.80a)$$

$$(\cos\alpha_{mn} Z_{ijklm} - \sin\alpha_{mn} Y_{ijklm}) \equiv Z_{ijkl} \qquad (4.80b)$$

and, in addition, the following two self-evident identities may be written:-

$$\cos\theta_m(X_{ijkl}\cos\theta_m - Y_{ijkl}\sin\theta_m) + \sin\theta_m(X_{ijkl}\sin\theta_m + Y_{ijkl}\cos\theta_m) \equiv X_{ijkl} \qquad (4.81a)$$

$$\sin\theta_m(X_{ijkl}\cos\theta_m - Y_{ijkl}\sin\theta_m) - \cos\theta_m(X_{ijkl}\sin\theta_m + Y_{ijkl}\cos\theta_m) \equiv -Y_{ijkl} \qquad (4.81b)$$

where i, j, k, l, m and n are positive integers in consecutive cyclic order.

4.19 Subsidiary Formulae for a Spherical Heptagon.

There are four distinct groups of subsidiary sine, sine-cosine and cosine laws that may be derived for a spherical heptagon from the three basic laws (equations (4.76)) using procedures exactly analogous to those for a spherical hexagon. The first group may be written with the aid of identities (4.80) as:-

$$(X_{7123}\cos\theta_4 - Y_{7123}\sin\theta_4) = \bar{X}_5 \qquad (4.82a)$$

$$(X_{7123}\sin\theta_4 + Y_{7123}\cos\theta_4) = -\bar{Y}_5 \qquad (4.82b)$$

$$Z_{7123} = \bar{Z}_5 \qquad (4.82c)$$

the second group, derived using (4.81a) and (4.81b), is:-

$$X_{7123} = (\bar{X}_5\cos\theta_4 - \bar{Y}_5\sin\theta_4) \qquad (4.83a)$$

$$-Y_{7123} = (\bar{X}_5\sin\theta_4 + \bar{Y}_5\cos\theta_4) \qquad (4.83b)$$

$$Z_{7123} = \bar{Z}_5 \qquad (4.83c)$$

whilst the third group may be written:-

$$(X_{712}\cos\theta_3 - Y_{712}\sin\theta_3) = X_{54} \qquad (4.84a)$$

$$(X_{712}\sin\theta_3 + Y_{712}\cos\theta_3) = -Y_{54} \qquad (4.84b)$$

$$Z_{712} = Z_{54} \qquad (4.84c)$$

using identities (4.68).

Finally the fourth group of laws is derived from (4.84a,b,c), using identities (4.69), and is written:-

$$X_{712} = (X_{54}\cos\theta_3 - Y_{54}\sin\theta_3) \qquad (4.85a)$$

$$-Y_{712} = (X_{54}\sin\theta_3 + Y_{54}\cos\theta_3) \qquad (4.85b)$$

$$Z_{712} = Z_{54} \qquad (4.85c)$$

All cyclic permutations of these laws are given in Appendix III.

4.20 Note on the Symbology.

The notation developed in the above sections is based on a system of three distinct but well-defined trigonometrical functions (the X, Y and Z expressions) which, apart from the single-suffix case, are all of the following form:-

$$X... = X..\cos\theta. - Y..\sin\theta. \qquad (4.86a)$$

$$Y... = \cos\alpha..(X..\sin\theta. + Y..\cos\theta.) - \sin\alpha..Z.. \qquad (4.86b)$$

$$Z... = \sin\alpha..(X..\sin\theta. + Y..\cos\theta.) + \cos\alpha..Z.. \qquad (4.86c)$$

(the number of suffices being determined from the context, but in each case representing the particular angular displacements involved). The expressions for a spherical dyad (i.e. X_j, Y_j, Z_j, $\bar{X}_j$, $\bar{Y}_j$ and $\bar{Z}_j$) can, however, be made to fit into this scheme by defining appropriate constants as follows:-

$$X = 0 \qquad \bar{X} = 0$$

$$Y = -\sin\alpha_{ij} \qquad \bar{Y} = -\sin\alpha_{jk}$$

$$Z = \cos\alpha_{ij} \qquad \bar{Z} = \cos\alpha_{jk} \qquad (4.87)$$

With these definitions, one may rewrite equations (4.6) and (4.7) as:-

$$X_j = X \cos\theta_j - Y \sin\theta_j \qquad (4.88a)$$
$$Y_j = \cos\alpha_{jk}(X \sin\theta_j + Y \cos\theta_j) - \sin\alpha_{jk} Z \qquad (4.88b)$$
$$Z_j = \sin\alpha_{jk}(X \sin\theta_j + Y \cos\theta_j) + \cos\alpha_{jk} Z \qquad (4.88c)$$

$$\bar{X}_j = \bar{X} \cos\theta_j - \bar{Y} \sin\theta_j \qquad (4.89a)$$
$$\bar{Y}_j = \cos\alpha_{ij}(\bar{X} \sin\theta_j + \bar{Y} \cos\theta_j) - \sin\alpha_{ij} \bar{Z} \qquad (4.89b)$$
$$\bar{Z}_j = \sin\alpha_{ij}(\bar{X} \sin\theta_j + \bar{Y} \cos\theta_j) + \cos\alpha_{ij} \bar{Z} \qquad (4.89c)$$

and these clearly fit into the scheme of (4.86). The sole drawback of this restatement of the dyad notation is that the constants (X, Y, Z, $\bar{X}$, $\bar{Y}$ and $\bar{Z}$) each depend on the value of j for their definition.

Nevertheless, the scheme of notation developed in this chapter is of fundamental importance as a means of presenting, in concise form, the loop equations for any spherical polygon, since it can clearly be extended and applied to polygons other than those considered here.

4.21 Direction Cosines.

If one considers the seven intersecting unit line vectors defining the spherical heptagon in Figure 4.8, it can be shown geometrically that, referred to the co-ordinate system illustrated, the x, y and z direction cosines of, say, $\underline{S}_6$ calculated in an anti-clockwise sense are X_{54321}, Y_{54321}, and Z_{54321} respectively, whilst those calculated in a clockwise sense are $\sin\alpha_{67} \sin\theta_7$, $\sin\alpha_{67} \cos\theta_7$ and $\cos\alpha_{67}$, respectively. (These are also the D.C.'s of the point, P, on the surface of the unit sphere, at which $\underline{S}_6$ intersects the latter). Hence equating corresponding direction cosines, one obtains the three equations:-

$$X_{54321} = \sin\alpha_{67} \sin\theta_7 \qquad (4.90a)$$
$$Y_{54321} = \sin\alpha_{67} \cos\theta_7 \qquad (4.90b)$$
$$Z_{54321} = \cos\alpha_{67} \qquad (4.90c)$$

which are merely the sine, sine-cosine and cosine laws for a spherical heptagon.

Listing the relevant direction cosines in this way for each of the seven unit line vectors in Figure 4.8, one obtains Table III. Notice that the D.C.'s of the line vector $\underline{S}_i$ (i = 1,...., 7) do not depend on θ_i (i.e. none of the X, Y, Z expressions has 'i' as a suffix).

The advantages of the X-Y-Z notation are thus apparent since all X terms represent the x-D.C. of some unit line vector through the origin (i.e. a pair axis), whilst Y or (X sinθ + Y cosθ) terms represent the y-D.C., and Z terms represent the z-D.C. Furthermore, the significance of the sine, sine-cosine and cosine laws, in arising from equating corresponding D.C.'s calculated in a clockwise sense with those calculated in an anti-clockwise sense, is now apparent. Also, it can be seen that the subsidiary laws are merely a restatement of the basic laws, referred to a different co-ordinate system.

Finally, it has been noted that the following identity is valid in general (see equations (4.9), (4.43), (4.56), (4.67) and (4.79)):-

$$X^2_{...} + Y^2_{...} + Z^2_{...} \equiv 1 \tag{4.91}$$

and in the present context this is nothing more than the identity satisfied by the three D.C.'s of any unit line vector at the origin. Note also that identities of the form:-

$$\begin{aligned}&(X_i\cos\theta_j - Y_i\sin\theta_j)^2\\&+(X_i\sin\theta_j + Y_i\cos\theta_j)^2\\&+ Z_i^2 \equiv 1\end{aligned} \tag{4.92}$$

are valid in general, which verifies that the terms on the R.H.S. of Table III. behave as D.C.'s.

4.22 Dual Laws for Spatial Polygons.

The equations and laws presented so far in this chapter describe the relationships between the various <u>real</u> angles of a spherical polygon and fall naturally into the classification of sine, sine-cosine and cosine laws adopted above. In addition, it is clear that, of all possible laws relating a given set of angles, only two such laws can be considered to be independent. (This can be seen from the derivations of the subsidiary formulae or from the fact that the various laws represent direction cosines and therefore equation (4.91) must be satisfied).

Now it has been explained in Chapter 3 that for each spherical law involving real angles there is a corresponding law for spatial polygons involving dual angles obtained by introducing the dual symbol and using the Principle of Transference. By applying the rules for manipulating functions of a dual variable (see Chapter 3) and equating primary and secondary parts, each spatial law may then be expanded to give two real equations. The first, which is termed the primary equation, is identical to the original spherical law, whilst the second is called the secondary equation corresponding to this primary, and is applicable only to the spatial polygon.

For the purposes of this expansion it is convenient to define, in the case of the spatial hexagon for example, the following dual notation (here $\hat{X}_{6123}$ is identified with $X(\hat{\theta}_6, \hat{\theta}_1, \hat{\theta}_2, \hat{\theta}_3)$):-

$$\begin{aligned}\hat{X}_{6123} &= X_{6123} + \epsilon\, X_{06123}\\ \hat{Y}_{6123} &= Y_{6123} + \epsilon\, Y_{06123}\\ \hat{Z}_{6123} &= Z_{6123} + \epsilon\, Z_{06123}\end{aligned} \tag{4.93}$$

in accordance with the usual practice. Thus, as an example, the three dual triangle laws corresponding to equations (4.12) are written:-

$$\begin{aligned}\hat{X}_1 &= \sin\hat{\alpha}_{23}\sin\hat{\theta}_2\\ \hat{Y}_1 &= \sin\hat{\alpha}_{23}\cos\hat{\theta}_2\\ \hat{Z}_1 &= \cos\hat{\alpha}_{23}\end{aligned} \tag{4.94}$$

where:-

$$\begin{aligned}\hat{X}_1 &= X_1 + \epsilon\, X_{01}\\ \hat{Y}_1 &= Y_1 + \epsilon\, Y_{01}\\ \hat{Z}_1 &= Z_1 + \epsilon\, Z_{01}\end{aligned} \tag{4.95}$$

and:-

$$\hat{\alpha}_{23} = \alpha_{23} + \epsilon a_{23}$$
$$\hat{\theta}_2 = \theta_2 + \epsilon S_2 \qquad (4.96)$$

Equations (4.94) then yield six real equations obtained by equating primary and secondary parts from each side. The primary equations are identical to the original triangle laws (equations (4.12)) whilst the secondary equations are derived by expansion of (4.94) and are:-

$$X_{01} = a_{23}\cos\alpha_{23}\sin\theta_2 + S_2\sin\alpha_{23}\cos\theta_2 \qquad (4.97a)$$
$$Y_{01} = a_{23}\cos\alpha_{23}\cos\theta_2 - S_2\sin\alpha_{23}\sin\theta_2 \qquad (4.97b)$$
$$Z_{01} = -a_{23}\sin\alpha_{23} \qquad (4.97c)$$

where:-

$$X_{01} = a_{31}\cos\alpha_{31}\sin\theta_1 + S_1\sin\alpha_{31}\cos\theta_1 \qquad (4.98a)$$

$$Y_{01} = -a_{12}(\cos\alpha_{12}\cos\alpha_{31} - \sin\alpha_{12}\sin\alpha_{31}\cos\theta_1) + a_{31}(\sin\alpha_{12}\sin\alpha_{31} - \cos\alpha_{12}\cos\alpha_{31}\cos\theta_1) + S_1\cos\alpha_{12}\sin\alpha_{31}\sin\theta_1 \qquad (4.98b)$$

$$Z_{01} = -a_{12}(\sin\alpha_{12}\cos\alpha_{31} + \cos\alpha_{12}\sin\alpha_{31}\cos\theta_1) - a_{31}(\cos\alpha_{12}\sin\alpha_{31} + \sin\alpha_{12}\cos\alpha_{31}\cos\theta_1) + S_1\sin\alpha_{12}\sin\alpha_{31}\sin\theta_1 \qquad (4.98c)$$

Such expansions are clearly of an extremely complex and laborious nature, and there is no advantage to be gained from listing numerous cyclic permutations. However, the expressions for X_{071234}, Y_{071234} and Z_{071234} are given in Appendix III. as representative for the spatial heptagon, together with various other 'Z_0' expressions, since these latter may be written in symmetrical form.

4.23 Dual Direction Cosines.

The significance of the X-Y-Z notation as real direction cosines for the intersecting unit line vectors defining a spherical polygon has already been delineated above. As a consequence of this and of the Principle of Transference (see Chapter 3) it now follows that the corresponding $\hat{X}$-$\hat{Y}$-$\hat{Z}$ symbols, defined by expressions of the form of (4.93) represent the dual direction cosines for the skew unit line vectors defining a spatial polygon. Thus, introducing the dual symbol into Table III. produces the corresponding dual direction cosines for the spatial heptagon shown in Figure 4.9, and it can be seen from Figure 4.10 that a particular unit line vector, say $\hat{\underline{S}}_5$, is represented uniquely in space by the three dual direction cosines, $\hat{X}_{4321}$, $\hat{Y}_{4321}$, $\hat{Z}_{4321}$.

Now the primary part of, say, $\hat{X}_{4321}$ (i.e. X_{4321}) is simply the cosine of the relative angle, β_x, between $\hat{\underline{S}}_5$ and the x-axis (see Figure 4.10), whilst the secondary part of $\hat{X}_{4321}$ (i.e. X_{04321}) must equal $-b_x.\sin\beta_x$, where b_x is the common perpendicular distance between $\hat{\underline{S}}_5$ and the x-axis (see also Chapter 3). In other words one may interpret $\hat{X}_{4321}$ as follows:-

$$\hat{X}_{4321} = \cos\hat{\beta}_x = \cos\beta_x - \epsilon\, b_x \sin\beta_x \tag{4.97}$$

where:-

$$\hat{X}_{4321} = X_{4321} + \epsilon\, X_{04321} \tag{4.98}$$

and:-

$$\hat{\beta}_x = \beta_x + \epsilon\, b_x \tag{4.99}$$

In a similar manner it is possible to interpret $\hat{Y}_{4321}$ and $\hat{Z}_{4321}$ as the dual D.C.'s of $\hat{\underline{S}}_5$ with the y and z axes respectively.

Now from equation (3.52) it is clear that:-

$$\hat{X}^2_{4321} + \hat{Y}^2_{4321} + \hat{Z}^2_{4321} \equiv \hat{1} \tag{4.100}$$

and hence equating primary parts one has:-

$$X^2_{4321} + Y^2_{4321} + Z^2_{4321} \equiv 1 \tag{4.101}$$

which is a cyclic permutation of (4.67); whilst equating secondary parts gives:-

$$X_{4321}.X_{04321} + Y_{4321}.Y_{04321} + Z_{4321}.Z_{04321} \equiv 0 \tag{4.102}$$

which may be proved directly. However, equations (4.101) and (4.102) are the conditions imposed on the components of a dual vector in order that it be a unit line vector (see Chapter 3 and Brand [4]), and so the significance of $\hat{X}_{4321}$, $\hat{Y}_{4321}$, and $\hat{Z}_{4321}$ as the dual-number components of a unit line vector is verified. Thus, with reference to Figure 4.10, in the co-ordinate system shown, one may write:-

$$\underline{\hat{S}}_5 = \underline{S}_5 + \epsilon\, \underline{S}_{05} \tag{4.103}$$

where:-

$$\underline{S}_5 = \begin{bmatrix} X_{4321} \\ Y_{4321} \\ Z_{4321} \end{bmatrix}, \qquad \underline{S}_{05} = \begin{bmatrix} X_{04321} \\ Y_{04321} \\ Z_{04321} \end{bmatrix} \tag{4.104}$$

and:-

$$\underline{S}_5 \cdot \underline{S}_5 = 1 \quad , \qquad \underline{S}_5 \cdot \underline{S}_{05} = 0 \tag{4.105}$$

Hence X_{04321}, Y_{04321} and Z_{04321} are the x, y and z components of the vector representing the moment of the unit line vector, $\underline{\hat{S}}_5$, about the origin (see Figure 4.10). Consequently one must have the condition:-

$$X^2_{04321} + Y^2_{04321} + Z^2_{04321} = h^2 \tag{4.106}$$

where h is the perpendicular distance from the origin to $\underline{\hat{S}}_5$.

Furthermore, the co-ordinates of any point, P, on the line defined by $\underline{\hat{S}}_5$ may now be written immediately. Thus with reference to equations (3.1) and (3.2) one may write in this case (see Figure 4.10):-

$$\underline{r} \times \underline{S}_5 = \underline{S}_{05}$$

$$\text{and} \qquad \underline{r} \cdot \underline{S}_5 = d \tag{4.107}$$

and the solution to these equations is given by (see Brand [3] and equation (3.2)):-

$$\underline{r} = (\underline{S}_5 \times \underline{S}_{05} + d\, \underline{S}_5)/(\underline{S}_5 \cdot \underline{S}_5) \tag{4.108}$$

Hence, from (4.108) the components of P are:-

$$P_x = (Y_{4321} \cdot Z_{04321} - Z_{4321} \cdot Y_{04321}) + d\, X_{4321}$$
$$P_y = (Z_{4321} \cdot X_{04321} - X_{4321} \cdot Z_{04321}) + d\, Y_{4321}$$
$$P_z = (X_{4321} \cdot Y_{04321} - Y_{4321} \cdot X_{04321}) + d\, Z_{4321} \qquad (4.109)$$

Clearly, these considerations and deductions may be applied in general to any spatial polygon in any suitable co-ordinate system of the type illustrated in Figure 4.10, and the relevant dual direction cosines, etc., may then be written down immediately.

Thus, having obtained the basic equations to describe spherical and spatial polygons, and realised their significance as real or dual direction cosine expressions, the problem now is to derive input-output equations, for various spatial linkages, from these laws in terms of just two angular displacements, by eliminating the extraneous or unwanted variables. The difficulties encountered in such elimination procedures will be discussed in Chapter 5.

Unit Line Vector of Heptagon	Direction Cosines Calculated in an Anti-Clockwise Sense			Direction Cosines Calculated in a Clockwise Sense		
	x-D.C.	y-D.C.	z-D.C.	x-D.C.	y-D.C.	z-D.C.
$\underline{S}_1$	0	$-\sin\alpha_{71}$	$\cos\alpha_{71}$	$(X_{23456}\cos\theta_7 - Y_{23456}\sin\theta_7)$	$-(X_{23456}\sin\theta_7 + Y_{23456}\cos\theta_7)$	Z_{23456}
$\underline{S}_2$	$\bar{X}_1$	$\bar{Y}_1$	$\bar{Z}_1$	$(X_{3456}\cos\theta_7 - Y_{3456}\sin\theta_7)$	$-(X_{3456}\sin\theta_7 + Y_{3456}\cos\theta_7)$	Z_{3456}
$\underline{S}_3$	X_{21}	Y_{21}	Z_{21}	$(X_{456}\cos\theta_7 - Y_{456}\sin\theta_7)$	$-(X_{456}\sin\theta_7 + Y_{456}\cos\theta_7)$	Z_{456}
$\underline{S}_4$	X_{321}	Y_{321}	Z_{321}	$(X_{56}\cos\theta_7 - Y_{56}\sin\theta_7)$	$-(X_{56}\sin\theta_7 + Y_{56}\cos\theta_7)$	Z_{56}
$\underline{S}_5$	X_{4321}	Y_{4321}	Z_{4321}	$(X_6\cos\theta_7 - Y_6\sin\theta_7)$	$-(X_6\sin\theta_7 + Y_6\cos\theta_7)$	Z_6
$\underline{S}_6$	X_{54321}	Y_{54321}	Z_{54321}	$\sin\alpha_{67}\sin\theta_7$	$\sin\alpha_{67}\cos\theta_7$	$\cos\alpha_{67}$
$\underline{S}_7$	X_{654321}	Y_{654321}	Z_{654321}	0	0	1

Table III. The Direction Cosines of the seven intersecting unit line vectors defining a spherical heptagon, in terms of the X-Y-Z notation.

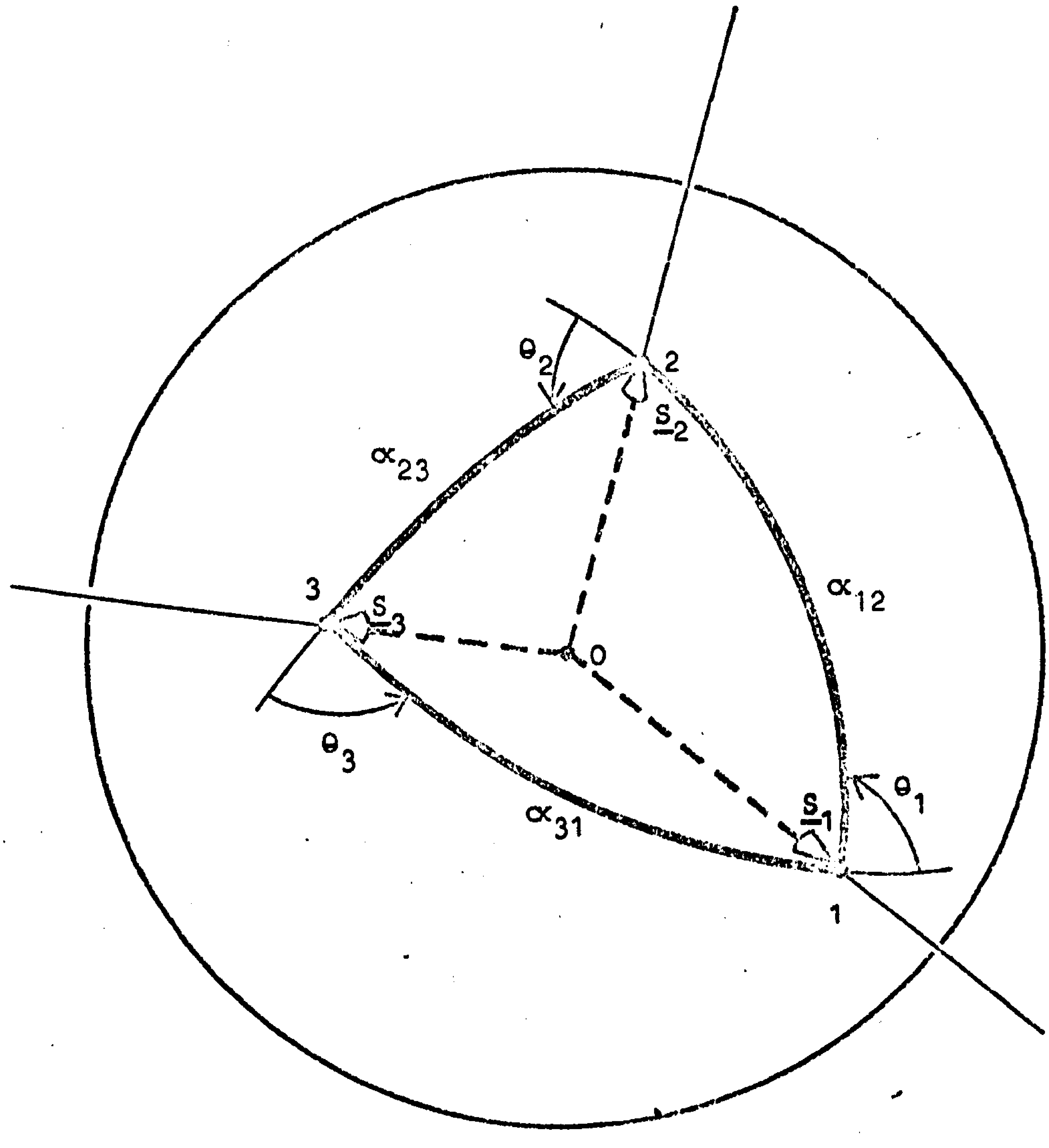
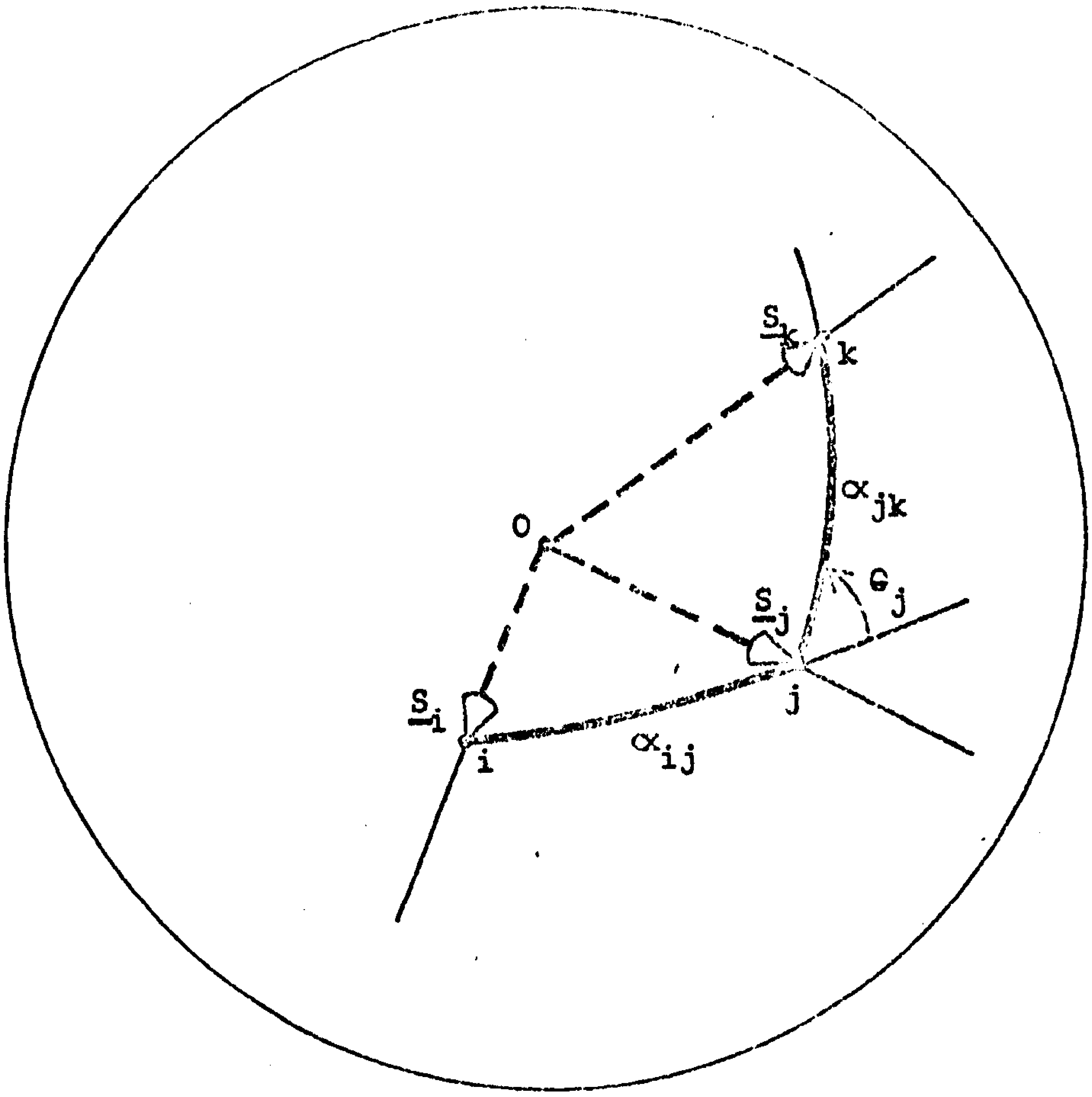
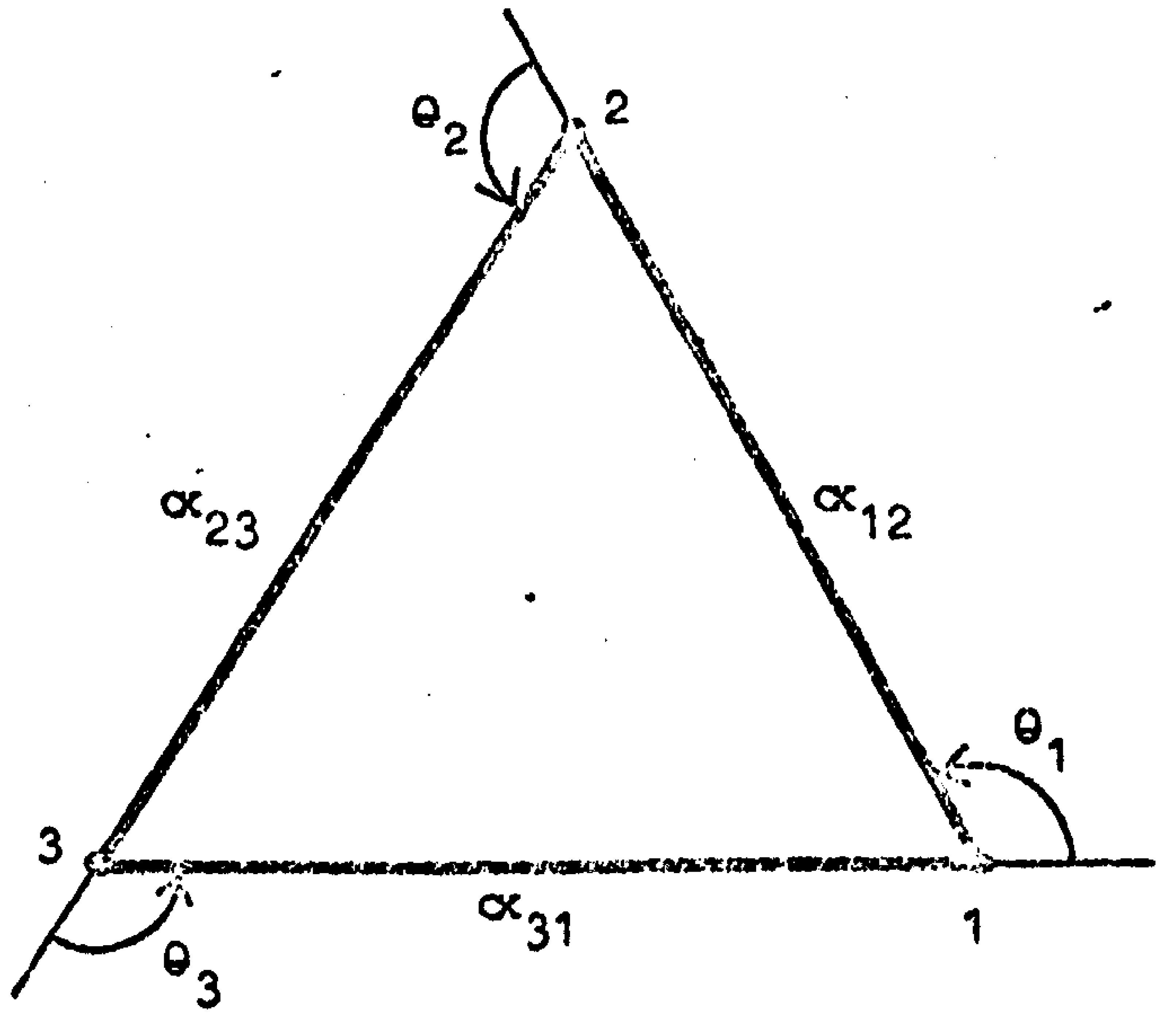


Figure 4.1 Representation of the Spherical Triangle.



(a) Notation for a Spherical Dyad.



(b) Planar Representation of the Spherical Triangle.

Figure 4.2 Notation and Representations for Spherical Polygons.

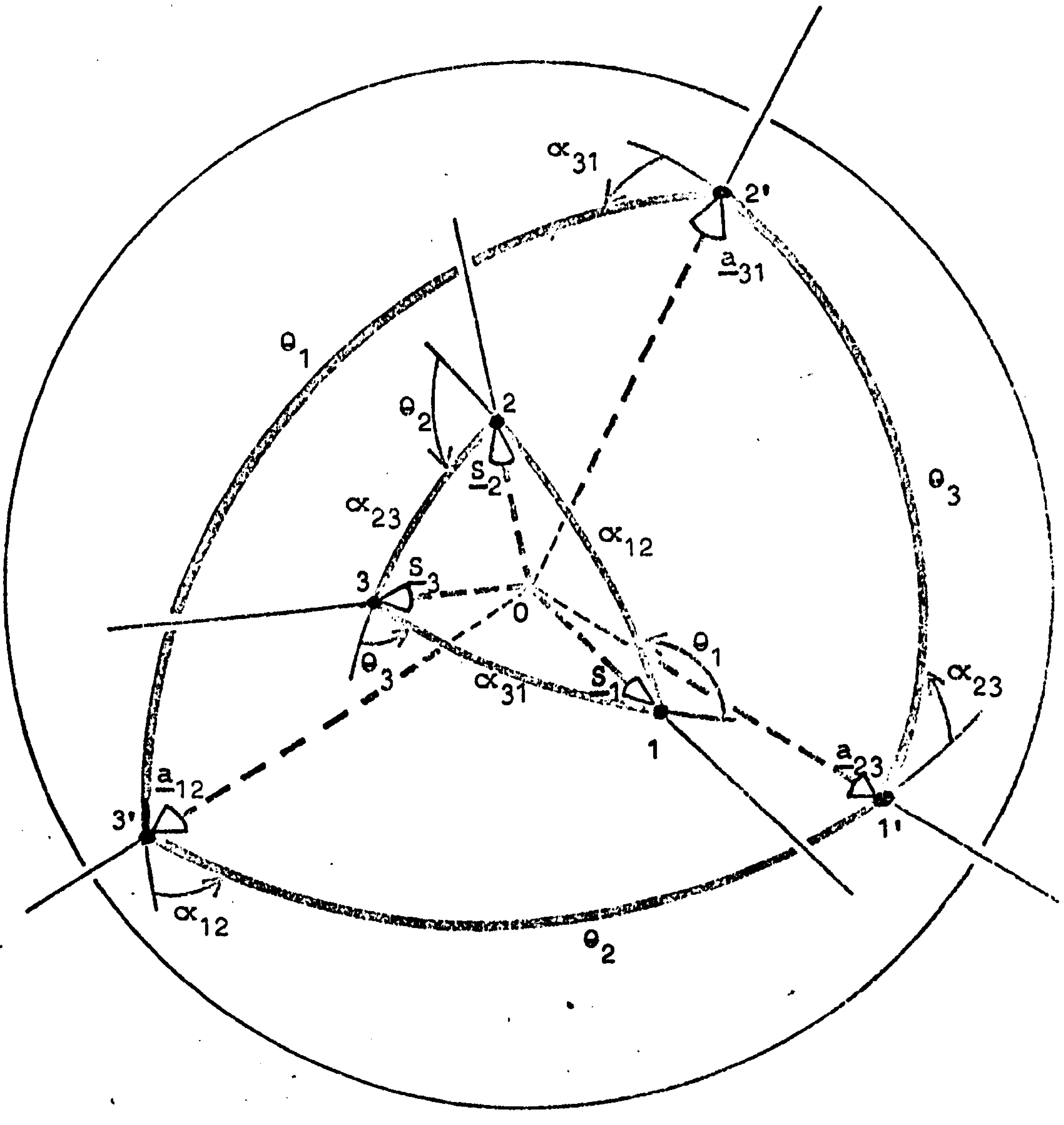
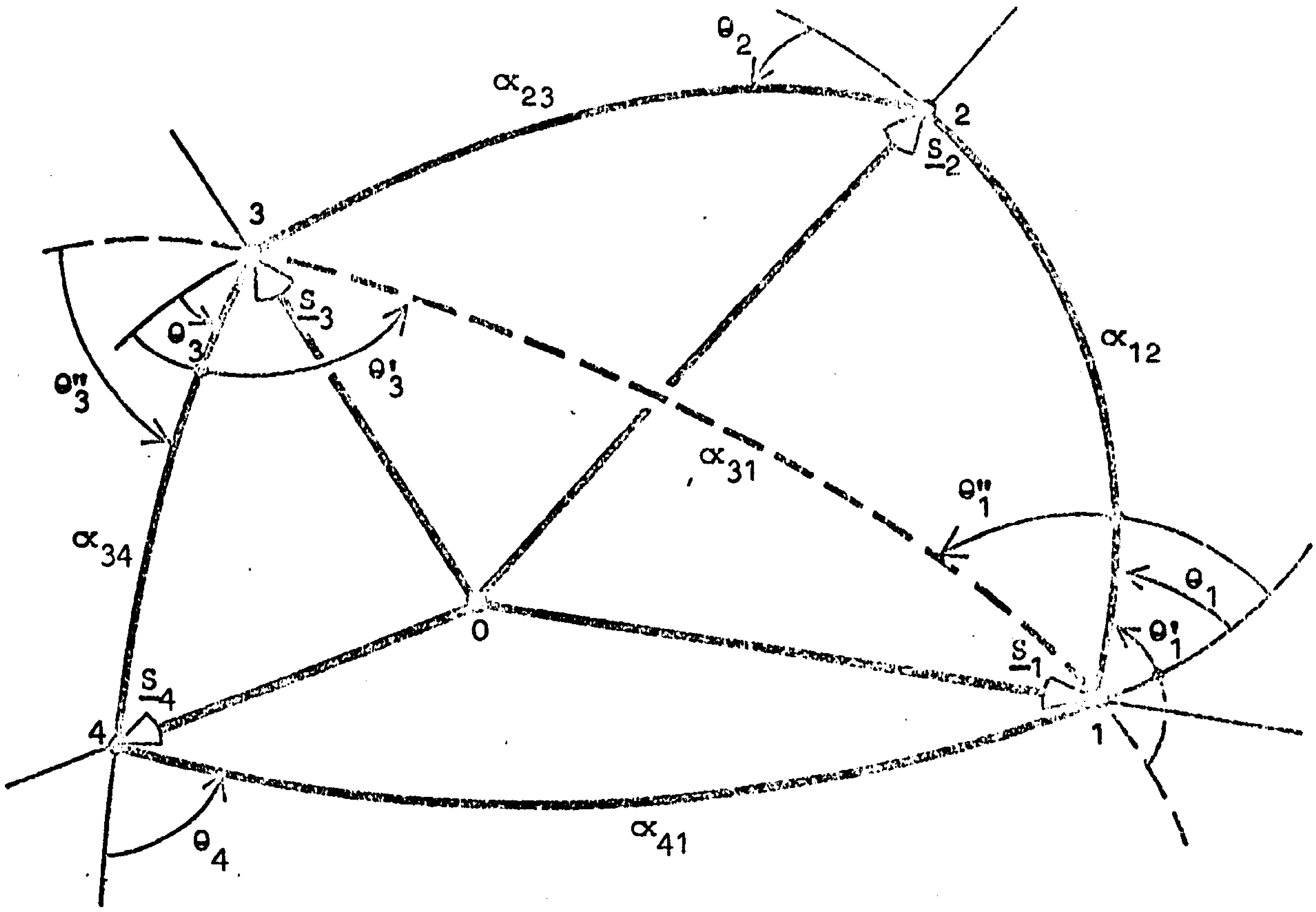
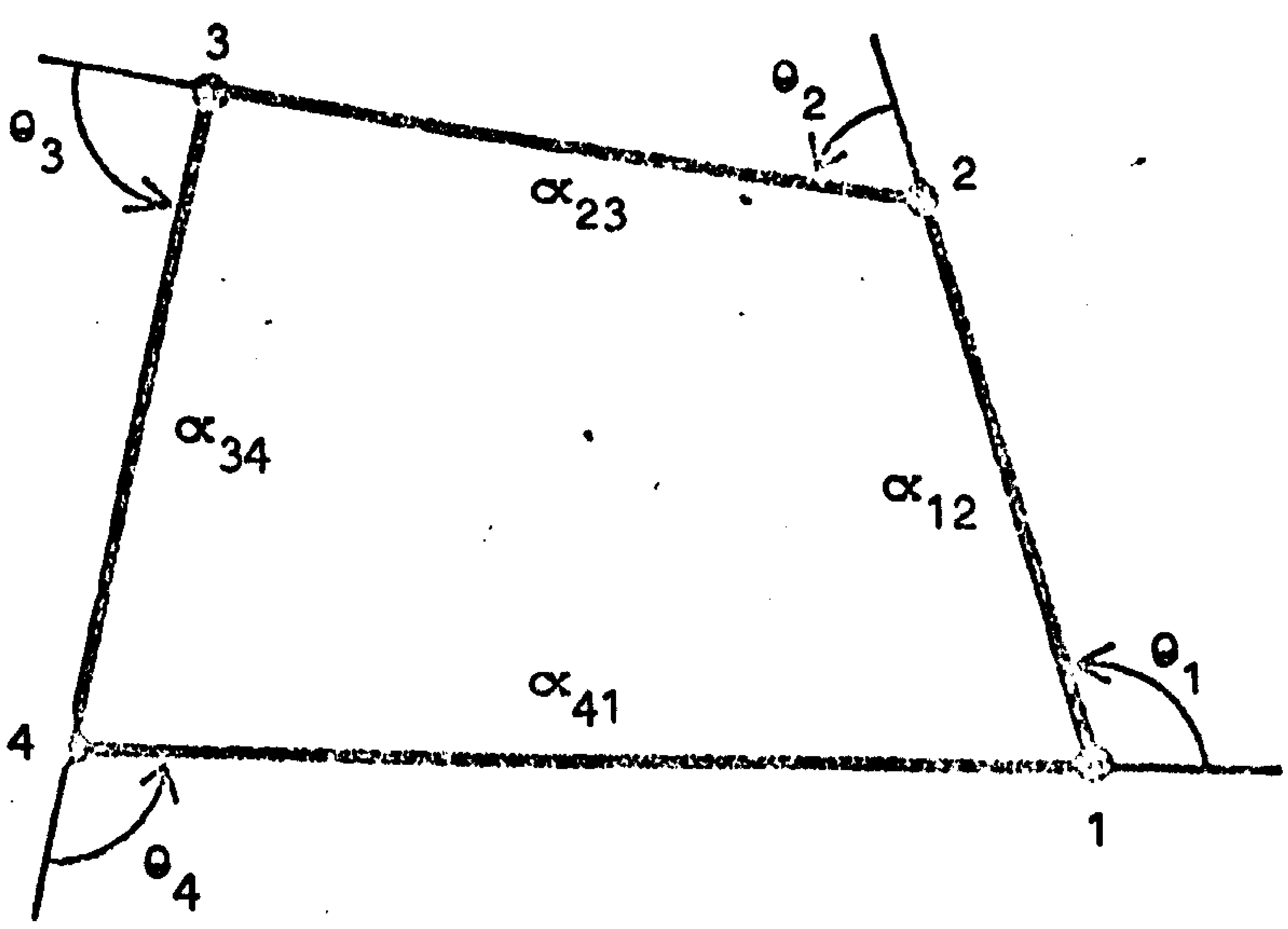


Figure 4.3 Representation of Polar Spherical Triangles.

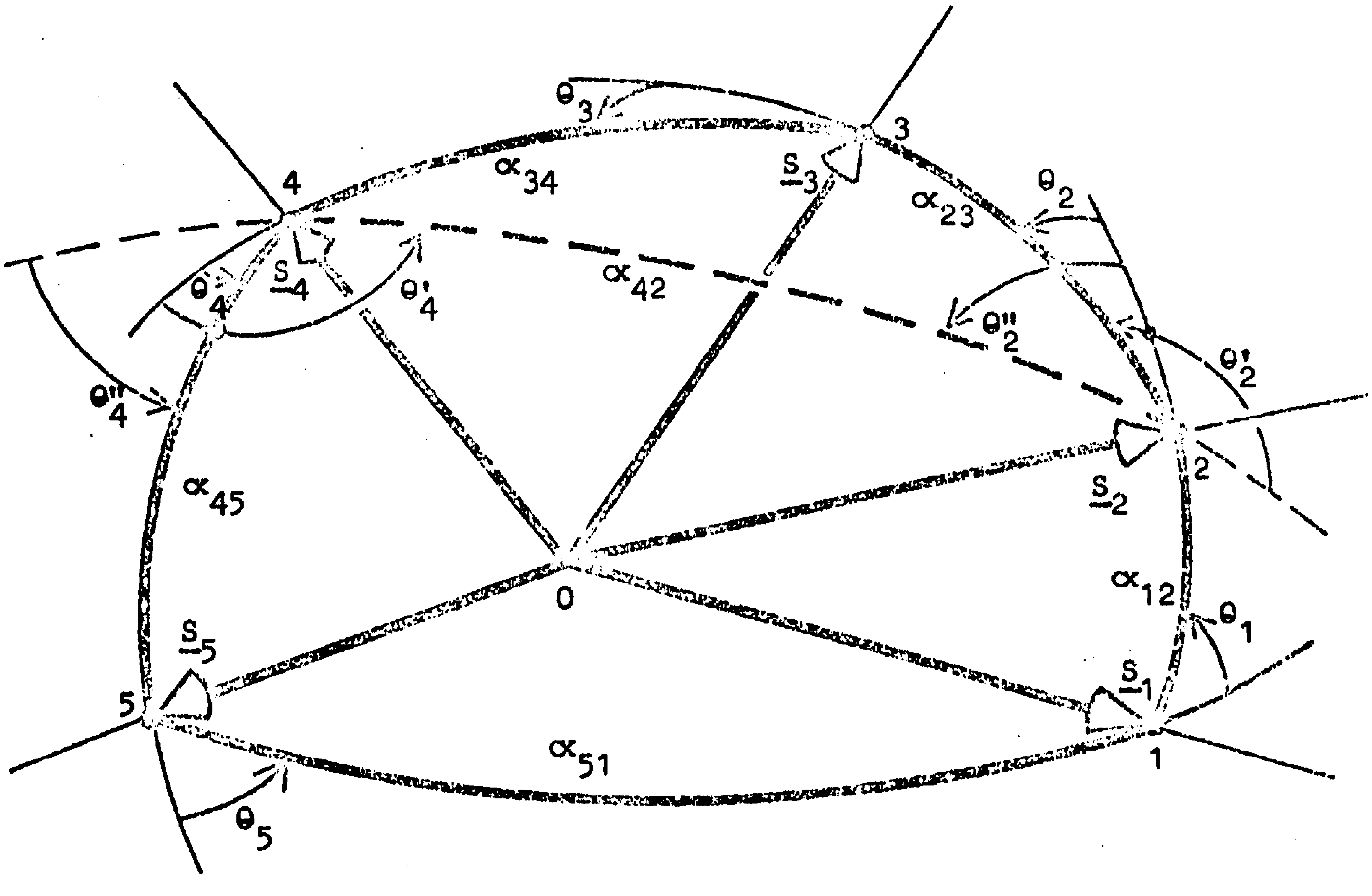


(a) Spherical Representation.

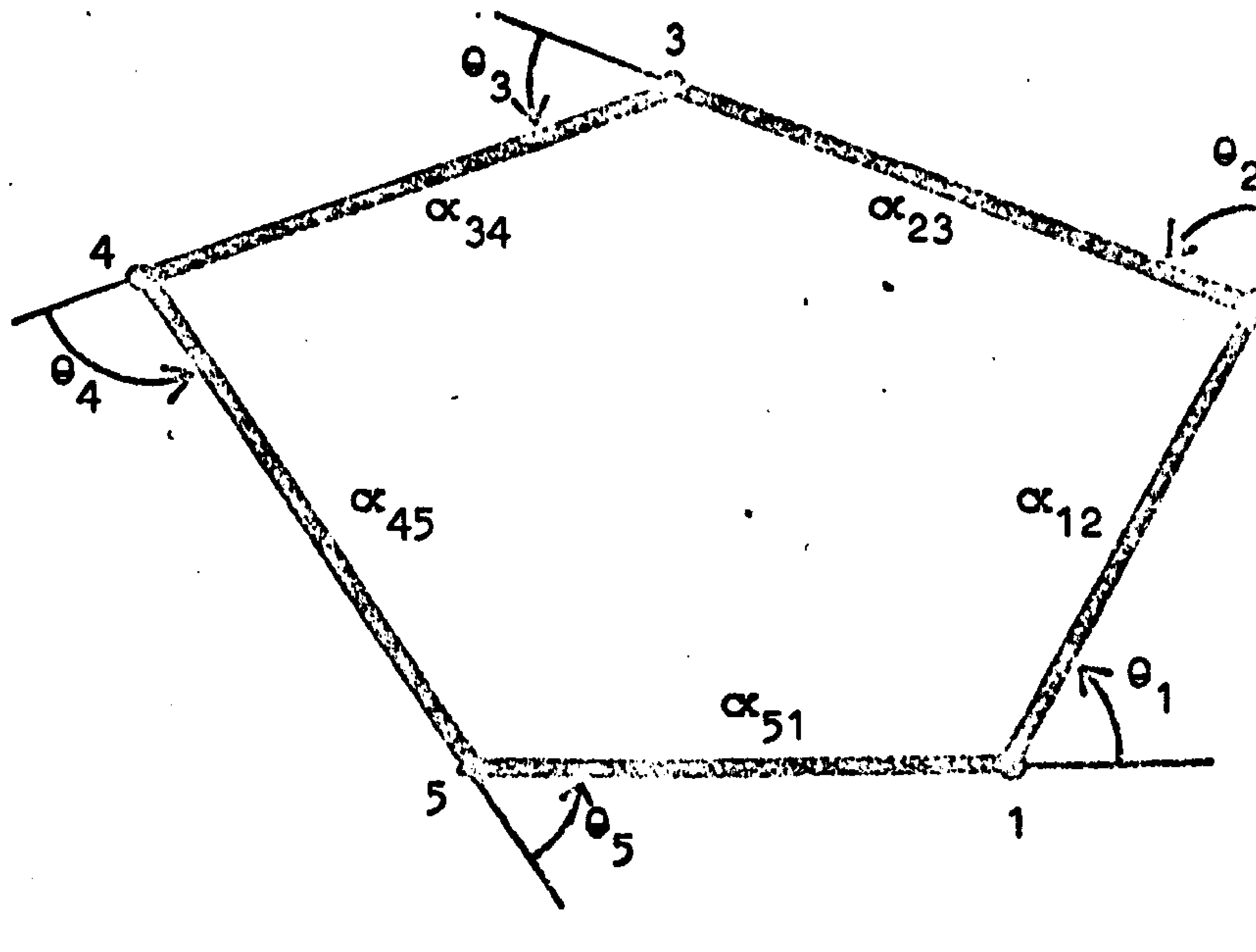


(b) Planar Representation.

Figure 4.4 Representation of the Spherical Quadrilateral.

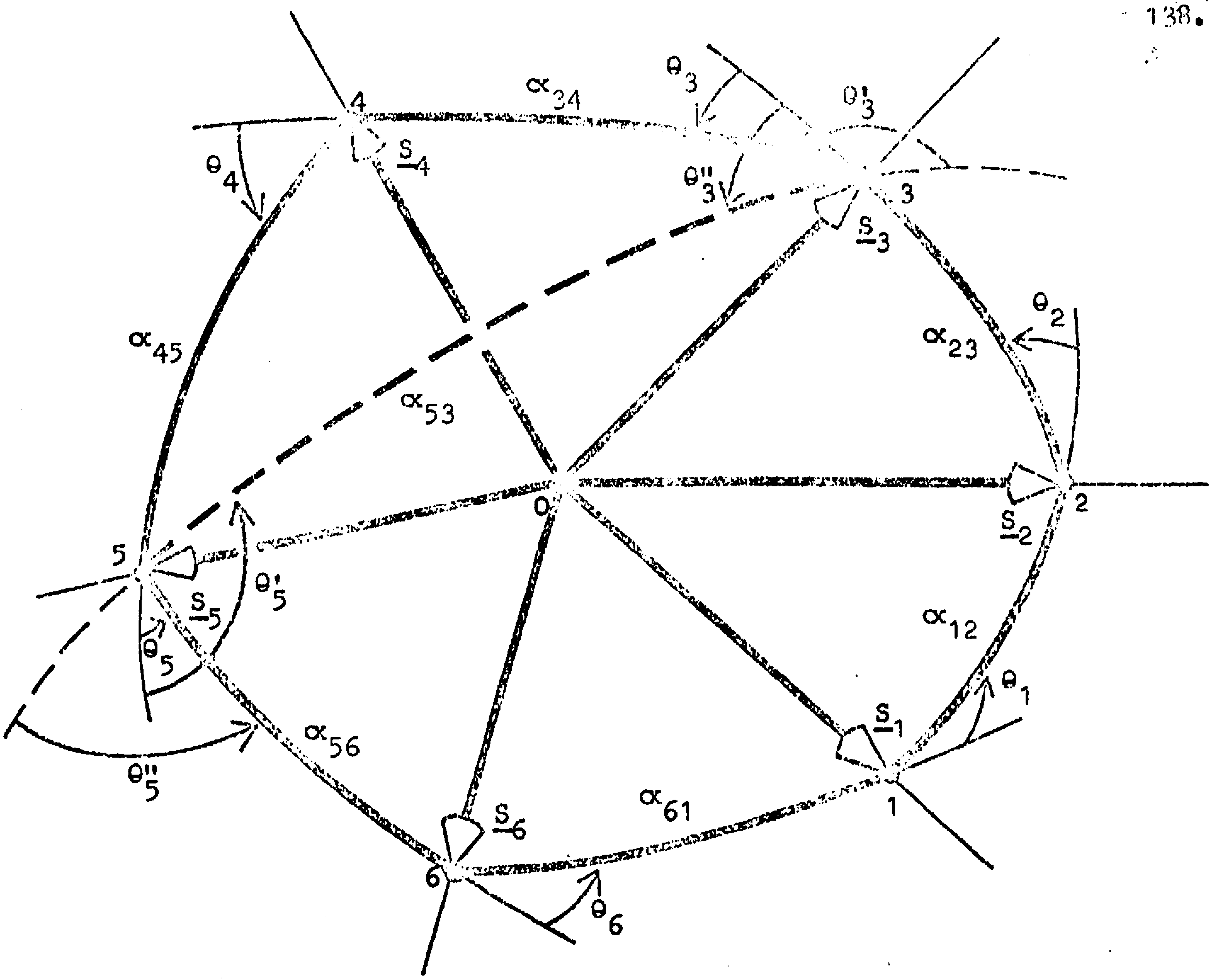


(a) Spherical Representation.

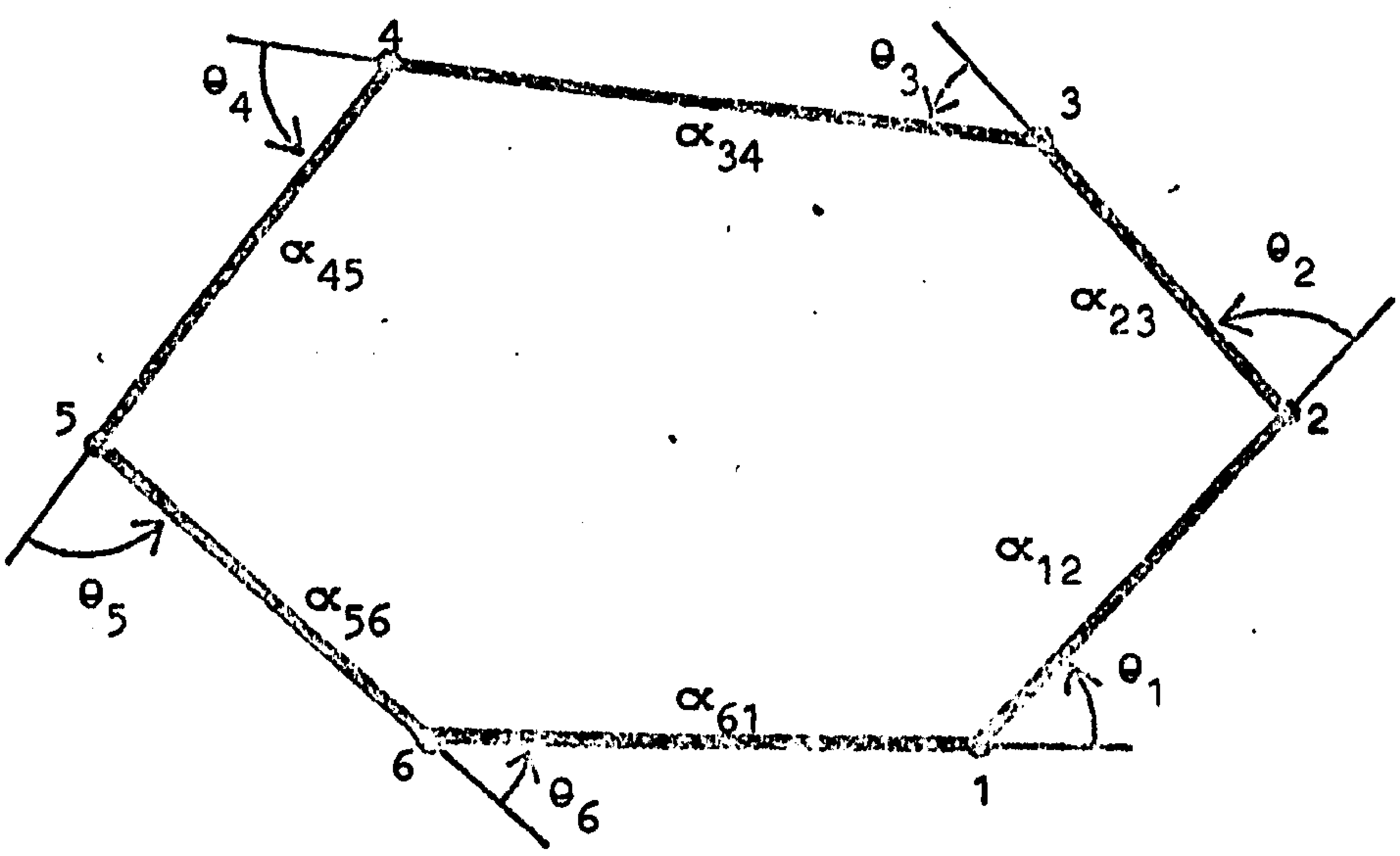


(b) Planar Representation.

Figure 4.5 Representation of the Spherical Pentagon.

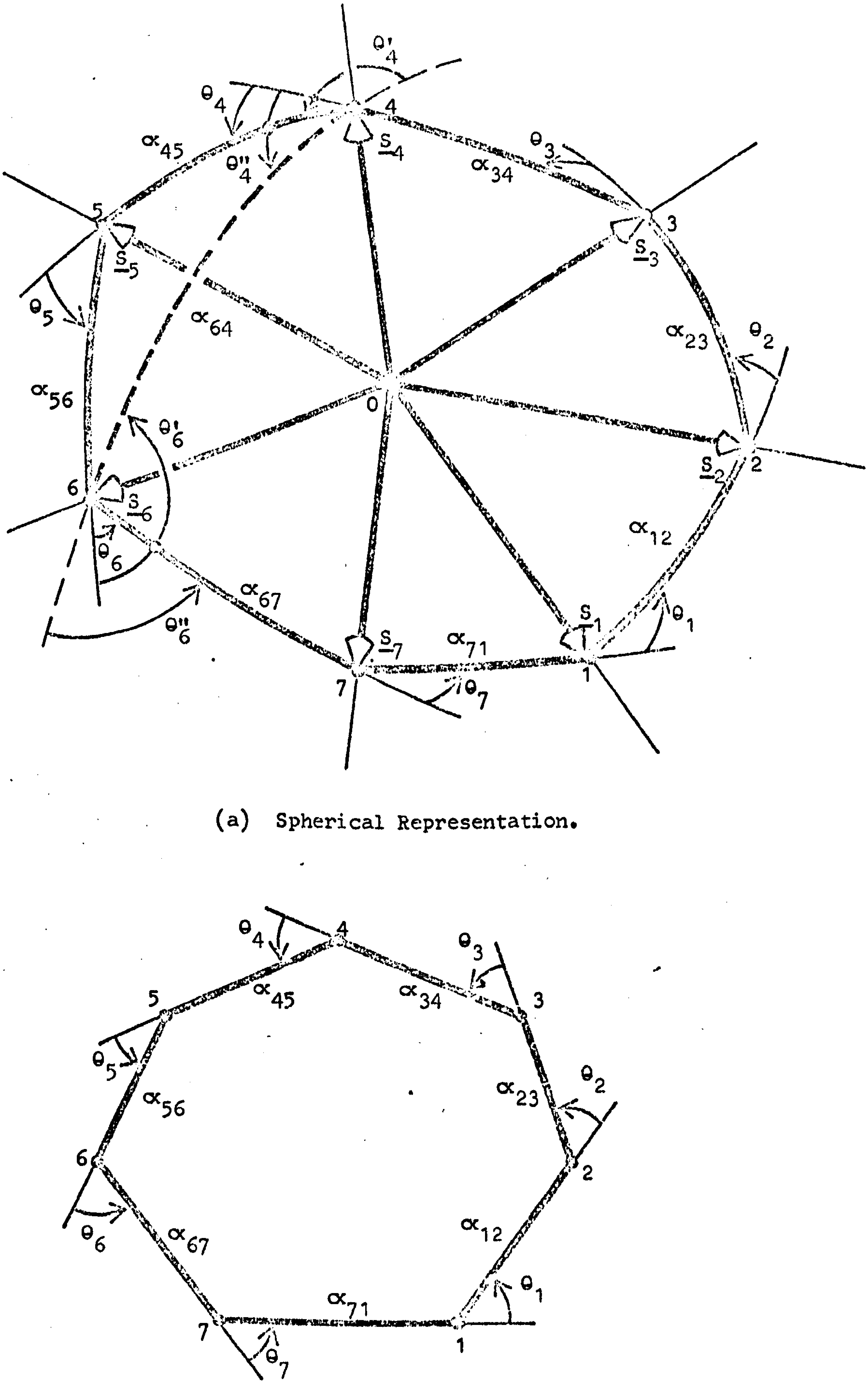


(a) Spherical Representation.



(b) Planar Representation.

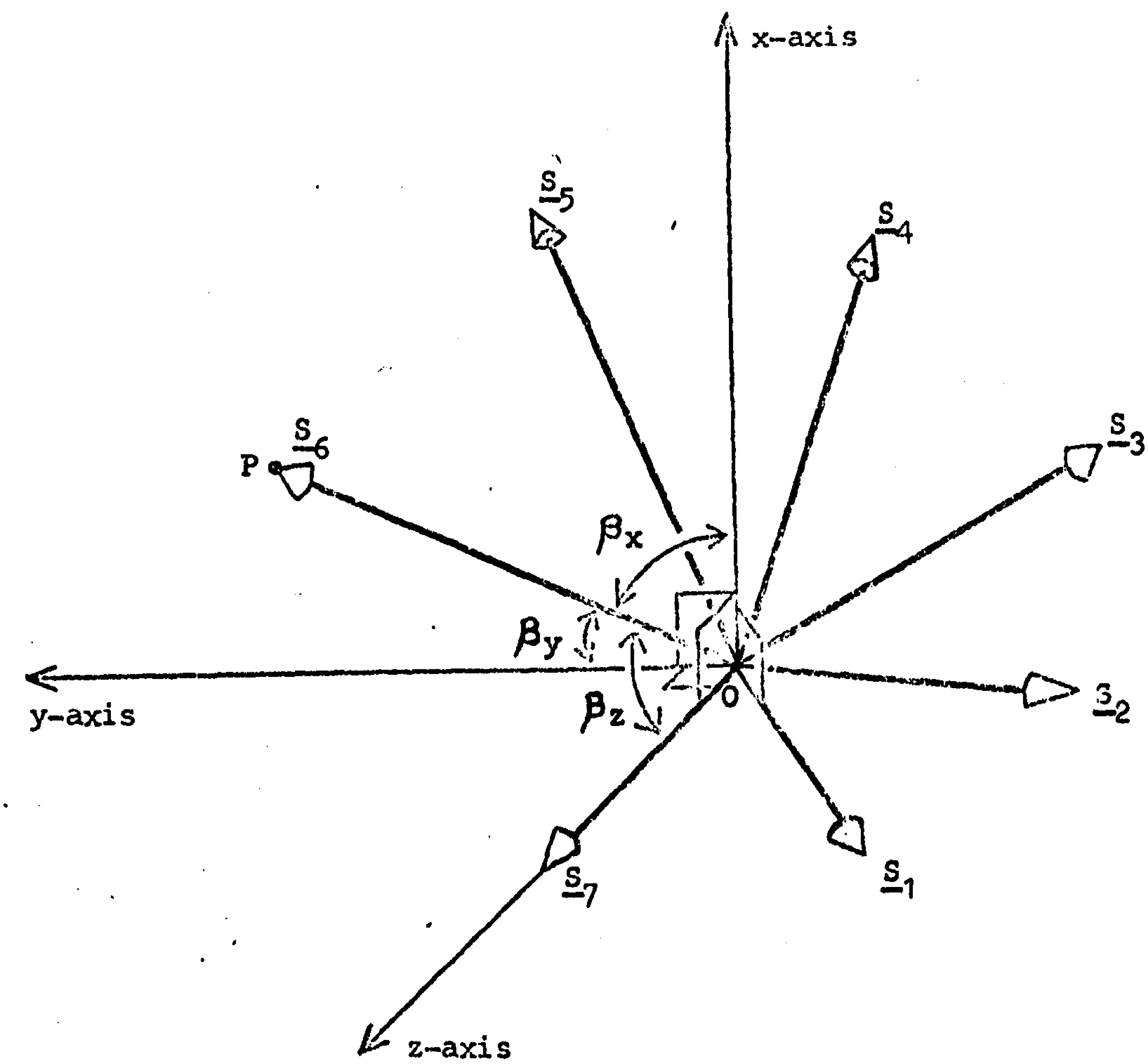
Figure 4.6 Representation of the Spherical Hexagon.



(a) Spherical Representation.

(b) Planar Representation.

Figure 4.7 Representation of the Spherical Heptagon.



$$\beta_z = \alpha_{67} \ .$$

Figure 4.8 Representation of the Intersecting Line Vectors Defining a Spherical Heptagon.

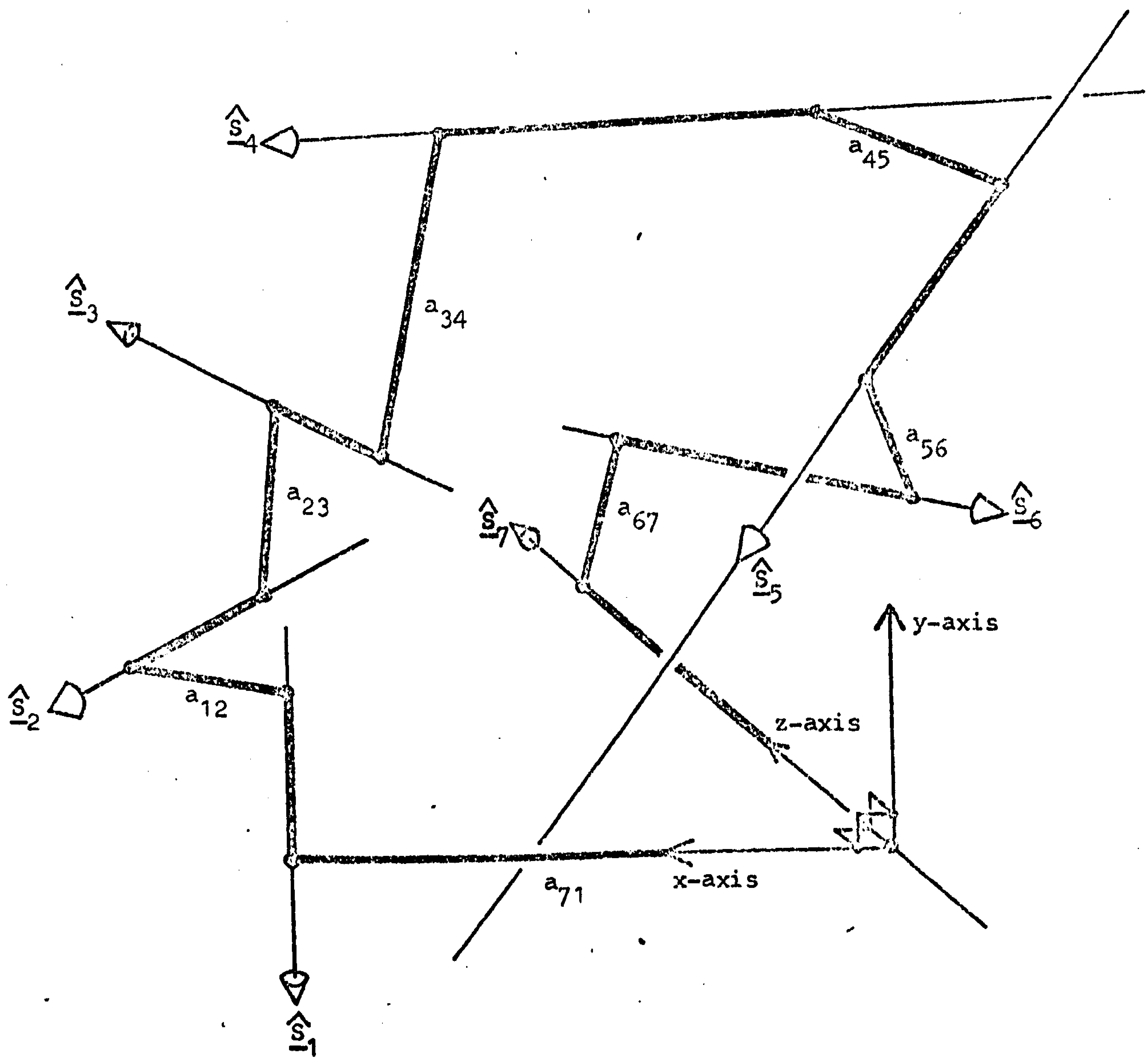
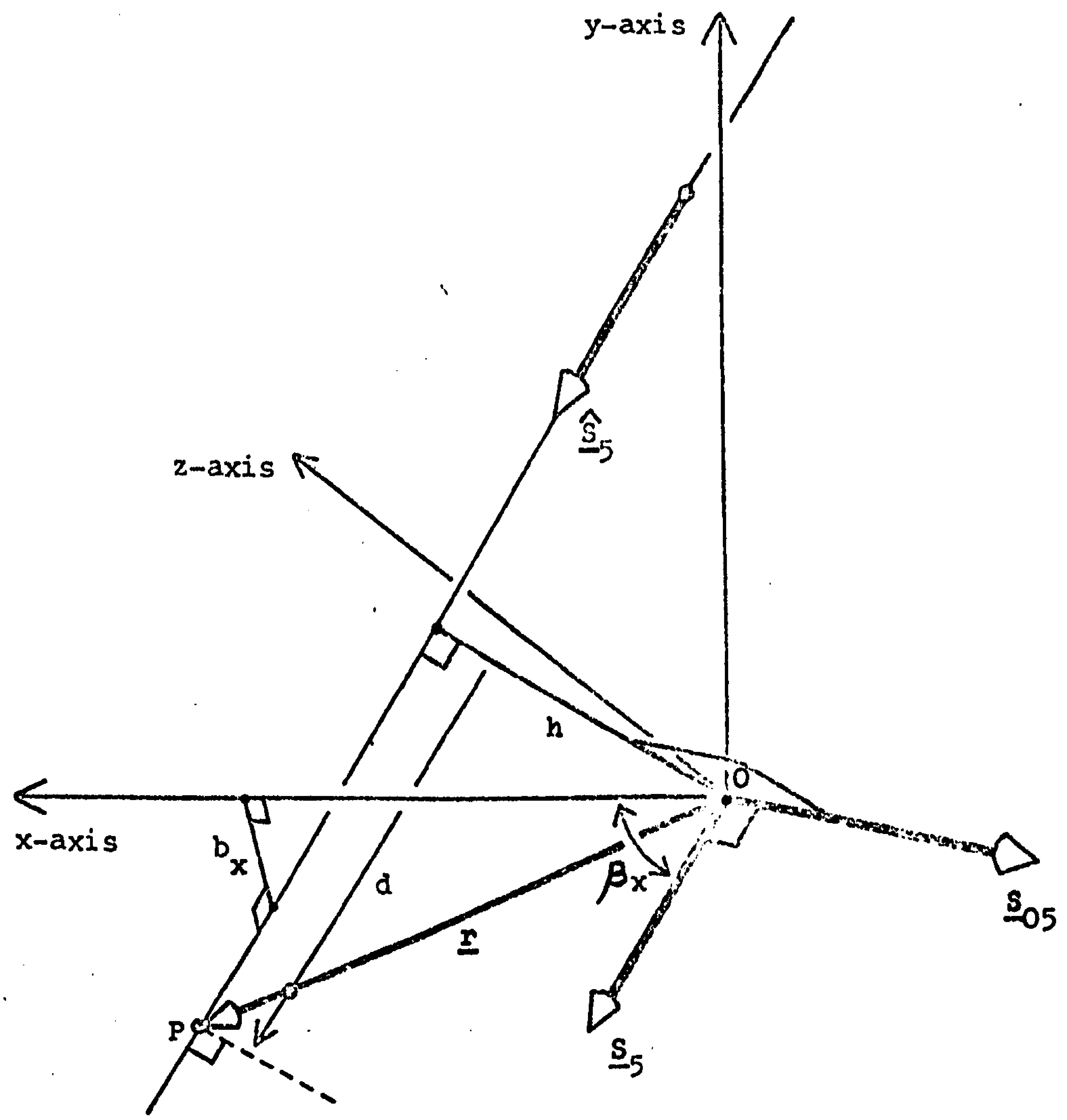


Figure 4.9 Representation of the Line Vectors Defining a Spatial Heptagon.



$$\hat{\underline{S}}_5 = \underline{S}_5 + \epsilon\, \underline{S}_{05}$$

where $\underline{S}_5 = (X_{4321}, Y_{4321}, Z_{4321})$

and $\underline{S}_{05} = (X_{04321}, Y_{04321}, Z_{04321})$

Figure 4.10 The Significance of the Dual Direction Cosines of a Unit Line Vector, $\hat{\underline{S}}_5$, of the Spatial Heptagon in Relation to the X-Y-Z Notation (see Figure 4.9).

CHAPTER 5

DERIVATION OF HALF-TANGENT LAWS

FOR

SPHERICAL AND SPATIAL POLYGONS

5.1 Introduction.

The major problem in the analysis of spatial linkages with more than four links is the derivation of input-output displacement equations. Thus it is necessary to derive, for the various mechanisms, an equation which relates the input angular displacement to the output angular (or sliding) displacement. In general, such input-output equations must contain all the mechanism dimensions.

The initial starting point in deriving the input-output equation for any particular spatial mechanism is the scheme of loop equations developed in Chapter 4 (i.e. the sine, sine-cosine and cosine laws, or combinations of these). The fundamental problem is then one of algebraic elimination for all except the RCCC mechanism, sine these transcendental loop equations can be converted into algebraic form by means of the following substitutions:-

$$\sin\theta_j = 2.x_j/(1 + x_j^2) \tag{5.1a}$$

$$\cos\theta_j = (1 - x_j^2)/(1 + x_j^2) \tag{5.1b}$$

Theories of algebraic elimination are given in Salmon [31], Bôcher [2] and Muir [28], and are outlined by many other authors such as Turnbull [38] etc.

For the majority of spatial mechanisms one is required to eliminate a single unknown between two non-homogeneous non-linear simultaneous equations, which are in polynomial form (the coefficients being each a function of the input and output variables), and the theory of dialytic elimination, as propounded by Sylvester, has been prominent in the field of mechanisms.

Briefly, Sylvester's method produces an eliminant, or condition on the coefficients of two polynomials, $f(x)$ and $g(x)$, (in one variable) such that f and g may have at least one root in common. If $f(x)$ is of degree m and $g(x)$ is of degree n then the eliminant, denoted by $E(f, g)$, is in the form of a determinant of order $(m + n)$, which is obtained by multiplying f by 1, x, x^2, $x^3, \ldots, x^{n-1}$; and g by 1, x, x^2, $x^3, \ldots, x^{m-1}$ in turn, and equating the determinant of coefficients of the resulting $(m + n)$ equations (considered homogeneous in the $(m + n)$ variables $1, x, \ldots, x^{m+n-1}$) to zero. This is the condition that the equations be satisfied by a common non-zero value of x. Thus, in the case of two quadratics one has:-

$$f(x) = a_2x^2 + a_1x + a_0 = 0 \tag{5.3}$$

$$g(x) = b_2x^2 + b_1x + b_0 = 0 \tag{5.4}$$

and multiplying f by 1, x; and g by 1, x gives:-

$$\begin{aligned} a_2x^3 + a_1x^2 + a_0x &= 0 \\ a_2x^2 + a_1x + a_0 &= 0 \\ b_2x^2 + b_1x + b_0 &= 0 \\ b_2x^3 + b_1x^2 + b_0x &= 0 \end{aligned} \tag{5.5}$$

Treating this system as four linear equations, homogeneous in x^3, x^2, x and 1, the condition for a non-trivial solution is:-

$$E(f, g) = \begin{vmatrix} a_2 & a_1 & a_0 & 0 \\ 0 & a_2 & a_1 & a_0 \\ 0 & b_2 & b_1 & b_0 \\ b_2 & b_1 & b_0 & 0 \end{vmatrix} = 0 \tag{5.6}$$

This determinantal form of the eliminant is called a <u>bigradient</u> (see Turnbull [38]).

Here the author prefers to use the method of Bézout which leads to a condensed form of the above eliminant, and, for a system of two quadratic equations, produces a symmetrical compound determinant of half the order of Sylvester's bigradient. Bézout's determinant is termed the <u>Bézoutian</u> and, in addition to the above mentioned reduction in order, it enables one to obtain various expressions for the common root directly.

The method may be demonstrated for the case of the two quadratics considered above (i.e. equations (5.3) and (5.4)) as follows. Thus, writing (5.3) and (5.4) in the two alternative forms:-

$$f(x) = a_2x^2 + (a_1x + a_0) = 0 \quad (5.3a)$$

$$g(x) = b_2x^2 + (b_1x + b_0) = 0 \quad (5.4a)$$

or:-

$$f(x) = (a_2x + a_1)x + a_0 = 0 \quad (5.3b)$$

$$g(x) = (b_2x + b_1)x + b_0 = 0 \quad (5.4b)$$

one may treat (5.3a) and (5.4a) as two non-homogeneous linear equations in x^2 and eliminate the latter to obtain the equation:-

$$(a_2b_1)x + (a_2b_0) = 0 \quad (5.7a)$$

Similarly, treating (5.3b) and (5.4b) as non-homogeneous in x and eliminating, one has, also:-

$$(a_2b_0)x + (a_1b_0) = 0 \quad (5.7b)$$

It is now a simple matter to eliminate x between (5.7a) and (5.7b) and obtain the desired Bézoutian of f and g as the following compound determinant:-

$$B(f, g) = \begin{vmatrix} (a_2b_1) & (a_2b_0) \\ (a_2b_0) & (a_1b_0) \end{vmatrix} \quad (5.8)$$

In equations (5.7a), (5.7b) and (5.8), the notation (a_2b_1), etc., has been adopted for brevity to represent a (2 x 2) determinant.

i.e.

$$(a_2b_1) = \begin{vmatrix} a_2 & a_1 \\ b_2 & b_1 \end{vmatrix} \text{, etc.} \quad (5.9)$$

The vanishing of B(f, g) is again the condition for f and g to have a common root and it is shown in Archbold [1] and in Turnbull [38] that the bigradient and the Bézoutian may be transformed one into the other by a suitable partitioning of their parent matrices. In fact it is clear that:-

$$B(f, g) = \pm E(f, g) \quad (5.10)$$

where **E**(f, g) is the bigradient of f and g. In the general case when f is of degree m and g of degree n, both the bigradient and Bézoutian produce an eliminant of degree n in the coefficients of f and of degree m in the coefficients of g, whilst the overall degree is $(m + n)$ in the coefficients.

However, it must be pointed out that if one required the condition that three quadratics share a common root, for example, then the desired eliminant is of a lesser degree than would be expected from the above discussion. Thus if f, g and h are given by:-

$$f(x) = a_2x^2 + a_1x + a_0 = 0 \tag{5.11}$$

$$g(x) = b_2x^2 + b_1x + b_0 = 0 \tag{5.12}$$

$$h(x) = c_2x^2 + c_1x + c_0 = 0 \tag{5.13}$$

the required eliminant, $E(f, g, h)$, is the determinant of the system treated as a set of three non-homogeneous equations linear in the two unknowns, x^2 and x. The vanishing of $E(f, g, h)$ is then the condition that (5.11), (5.12) and (5.13) have a common root.

i.e.
$$E(f, g, h) = \begin{vmatrix} a_2 & a_1 & a_0 \\ b_2 & b_1 & b_0 \\ c_2 & c_1 & c_0 \end{vmatrix} = 0 \tag{5.14}$$

This is a third order determinant and hence $E(f, g, h)$ is of overall degree three in the coefficients of f, g and h.

Clearly, the total degree of an eliminant of a system of equations depends on at least the following two major factors:-

(i) The degree of each equation.

(ii) The number of equations in the system.

As a final point it must be noted that an eliminant in Bézoutian form is easier to manipulate than is a bigradient, and, in addition, two alternative expressions for the common root of a pair of quadratics are obtainable immediately from equations (5.7a) and (5.7b).

5.2 Input-Output Equations and Extraneous Roots.

The fundamental problem in obtaining the input-output equation for the vast majority of spatial mechanisms is that of the elimination of a single unknown between two simultaneous non-linear equations and the theory of algebraic elimination for this problem is well established (see previous section). Nevertheless, several papers have already appeared, [16, 17, 49], which present input-output equations containing extraneous roots.

A contributing factor to the difficulties encountered in the elimination process is that the loop equations, as presented in Chapter 4, are each quadratic in the angular displacements. However, pairs of sine and sine-cosine laws can be expressed (using (5.1)) in terms of the half-tangent of an appropriate angular displacement, and expressions for the common root of these equations, lead to novel loop equations which are linear in a half-tangent. The author has defined these as fundamental half-tangent laws. Using these laws it has been possible to formulate the correct eliminant (and hence input-output equation), free from extraneous roots, for many mechanisms.

5.3 Fundamental Half-Tangent Laws.

The fundamental half-tangent laws, which are linear in the half-tangent of one angular displacement, may be derived in a straightforward manner for all the spherical polygons discussed in Chapter 4. However, the method is most conveniently explained with reference to the spherical quadrilateral. Thus, applying the half-tangent substitutions (5.1), (with j = 1) to the subsidiary sine and sine-cosine laws expressed by equations (4.47) and, rearranging, one obtains the following two quadratics:-

$$f(x_1) = (X_4 + \bar{X}_2)x_1^2 + 2.Y_4x_1 - (X_4 - \bar{X}_2) = 0 \qquad (5.15)$$

$$g(x_1) = (Y_4 - \bar{Y}_2)x_1^2 - 2.X_4x_1 - (Y_4 + \bar{Y}_2) = 0 \qquad (5.16)$$

where:- $x_1 = \tan(\theta_1/2)$ (5.17)

The eliminant of (5.15) and (5.16) may be obtained from the Bézoutian of f and g, which may be written, with reference to definition (5.8), as:-

$$E(f,\ g) = \begin{vmatrix} \begin{vmatrix} (X_4 + \bar{X}_2) & 2.Y_4 \\ (Y_4 - \bar{Y}_2) & -2.X_4 \end{vmatrix} & \begin{vmatrix} (X_4 + \bar{X}_2) & -(X_4 - \bar{X}_2) \\ (Y_4 - \bar{Y}_2) & -(Y_4 + \bar{Y}_2) \end{vmatrix} \\ \begin{vmatrix} (X_4 + \bar{X}_2) & -(X_4 - \bar{X}_2) \\ (Y_4 - \bar{Y}_2) & -(Y_4 + \bar{Y}_2) \end{vmatrix} & \begin{vmatrix} 2.Y_4 & -(X_4 - \bar{X}_2) \\ -2.X_4 & -(Y_4 + \bar{Y}_2) \end{vmatrix} \end{vmatrix} \qquad (5.18)$$

After expansion and simplification of this compound determinant one obtains:-

$$E(f,\ g) = 4.(X_4^2 + Y_4^2).[(X_4^2 + Y_4^2) - (\bar{X}_2^2 + \bar{Y}_2^2)] \qquad (5.19)$$

Now from identities (4.9a) and (4.9b) one has:-

$$X_4^2 + Y_4^2 \equiv 1 - Z_4^2 \qquad (5.20a)$$

and

$$\bar{X}_2^2 + \bar{Y}_2^2 \equiv 1 - \bar{Z}_2^2 \qquad (5.20b)$$

and hence (5.19) becomes:-

$$E(f,\ g) = 4.(1 - Z_4^2).(\bar{Z}_2^2 - Z_4^2) \qquad (5.21)$$

Thus from (5.21) and the subsidiary cosine law (4.47c) it is clear that $E(f,\ g)$ must be <u>identically zero</u> for all values of θ_2 and θ_4 and hence (5.15) and (5.16) must <u>always</u> possess a common root.

It is possible to obtain expressions for this common root by using the well-known formulae for the roots of a quadratic. Consequently the two roots of (5.15) may be written:-

$$x_1 = (-2.Y_4 \pm [\ 4.Y_4^2 + 4.(X_4^2 - \bar{X}_2^2)]^{\frac{1}{2}})/[2.(X_4 + \bar{X}_2)] \qquad (5.22)$$

whilst the two roots of (5.16) are:-

$$x_1 = (\ 2.X_4 \pm [\ 4.X_4^2 + 4.(Y_4^2 - \bar{Y}_2^2)]^{\frac{1}{2}})/[2.(Y_4 - \bar{Y}_2)] \qquad (5.23)$$

The discriminants in (5.22) and (5.23) may be simplified appreciably using the identities (5.20) together with the cosine law (4.47c) and thus (5.22) becomes:-

$$x_1 = (-Y_4 \pm \bar{Y}_2)/(X_4 + \bar{X}_2)$$

i.e.

$$(X_4 + \bar{X}_2)x_1 - (-Y_4 \pm \bar{Y}_2) = 0 \qquad (5.24)$$

whilst (5.23) may be written:-

$$x_1 = (X_4 \pm \bar{X}_2)/(Y_4 - \bar{Y}_2)$$

i.e. $$(Y_4 - \bar{Y}_2)x_1 - (X_4 \pm \bar{X}_2) = 0 \qquad (5.25)$$

The condition for one of the roots of (5.24) to be identically equal to one of those of (5.25), for all values of θ_2 and θ_4, is that the determinant of coefficients be identically zero. However, this will only occur for one of the four possible combinations of positive and negative signs and this is:-

$$\begin{vmatrix} (X_4 + \bar{X}_2) & -(-Y_4 - \bar{Y}_2) \\ (Y_4 - \bar{Y}_2) & -(X_4 - \bar{X}_2) \end{vmatrix} = (\bar{X}_2^2 + \bar{Y}_2^2) - (X_4^2 + Y_4^2) = 0 \qquad (5.26)$$

from (5.20a,b) and (4.47c).

Thus two distinct expressions for the common root of (5.15) and (5.16) may now be written, using (5.24) and (5.25), as:-

$$(X_4 + \bar{X}_2)x_1 + (Y_4 + \bar{Y}_2) = 0 \qquad (5.27)$$

and $$(Y_4 - \bar{Y}_2)x_1 - (X_4 - \bar{X}_2) = 0 \qquad (5.28)$$

and these expressions are both <u>linear</u> in the half-tangent of θ_1. The author has termed expressions of this form <u>fundamental half-tangent laws</u>, and their validity may be appreciated geometrically since, if <u>two</u> angular displacements (say θ_2 and θ_4) and the four sides of a spherical quadrilateral are specified, the remaining angular displacements must be <u>uniquely</u> determined.

In an analogous way, one may express the basic sine and sine-cosine laws (4.29) and (4.33) for the spherical quadrilateral in terms of the half-tangent of θ_2 and obtain, via a similar procedure to the above, a further two fundamental half-tangent laws, which may be written:-

$$X_{41}x_2 + (Y_{41} - \sin\alpha_{23}) = 0 \qquad (5.29)$$

$$(Y_{41} + \sin\alpha_{23})x_2 - X_{41} = 0 \qquad (5.30)$$

where $$x_2 = \tan(\theta_2/2) \qquad (5.31)$$

A complete list of all such fundamental half-tangent laws, linear in one half-tangent, is given in Appendix IV. for the spherical quadrilateral.

Similar laws to the above may be derived for any spherical polygon and, in particular, three distinct pairs exist for the hexagon, obtained respectively from the basic sine and sine-cosine laws (4.64a,b); the two subsidiary laws (4.70a) and (4.70b); and finally the two subsidiary laws (4.72a,b). These half-tangent laws may be listed as follows:-

$$X_{6123}x_4 + (Y_{6123} - \sin\alpha_{45}) = 0 \tag{5.32}$$

and
$$(Y_{6123} + \sin\alpha_{45})\,x_4 - X_{6123} = 0 \tag{5.33}$$

where
$$x_4 = \tan(\theta_4/2) \tag{5.34}$$

$$(X_{612} + \bar{X}_4)x_3 + (Y_{612} + \bar{Y}_4) = 0 \tag{5.35}$$

and
$$(Y_{612} - \bar{Y}_4)x_3 - (X_{612} - \bar{X}_4) = 0 \tag{5.36}$$

where
$$x_3 = \tan(\theta_3/2) \tag{5.37}$$

$$(X_{61} + X_{43})x_2 + (Y_{61} + Y_{43}) = 0 \tag{5.38}$$

and
$$(Y_{61} - Y_{43})x_2 - (X_{61} - X_{43}) = 0 \tag{5.39}$$

where
$$x_2 = \tan(\theta_2/2) \tag{5.40}$$

An exhaustive list of all cyclic permutations of these fundamental half-tangent laws is given in Appendix IV. for each spherical polygon up to and including the spherical heptagon.

5.4 Half-Tangent Laws for the Spherical Triangle.

For the spherical triangle there exists the following pair of fundamental half-tangent laws, (linear in the single angular displacement θ_2) which are derived from the basic sine and sine-cosine laws (4.12a) and (4.12b):-

$$X_1x_2 + (Y_1 - \sin\alpha_{23}) = 0 \tag{5.41}$$

and
$$(Y_1 + \sin\alpha_{23})\,x_2 - X_1 = 0 \tag{5.42}$$

where x_2 is given by (5.40).

However, in addition to these laws it is possible to derive a series of further half-tangent laws which are linear in two half-tangents for the spherical triangle. Thus from (5.41), (5.42) and the two cyclic permutations:-

$$\bar{X}_2 x_1 + (\bar{Y}_2 - \sin\alpha_{31}) = 0 \qquad (5.43)$$

$$(\bar{Y}_2 + \sin\alpha_{31})x_1 - \bar{X}_2 = 0 \qquad (5.44)$$

one may derive the following four further half-tangent laws, linear in x_1 and x_2:-

$$[\cos(\alpha_{12} - \alpha_{23}) - \cos\alpha_{31}]x_2 - [\cos(\alpha_{12} - \alpha_{31}) - \cos\alpha_{23}]x_1 = 0 \qquad (5.45)$$

$$[\cos(\alpha_{12} + \alpha_{31}) - \cos\alpha_{23}]x_2 - [\cos(\alpha_{12} + \alpha_{23}) - \cos\alpha_{31}]x_1 = 0 \qquad (5.46)$$

$$[\cos(\alpha_{12} - \alpha_{23}) - \cos\alpha_{31}]x_2 x_1 - \sin\alpha_{12}\sin\alpha_{31} = 0 \qquad (5.47)$$

$$[\cos(\alpha_{12} - \alpha_{31}) - \cos\alpha_{23}]x_2 x_1 + [\cos(\alpha_{12} + \alpha_{23}) - \cos\alpha_{31}] = 0 \qquad (5.48)$$

The derivations are rather tedious but basically one replaces the Y_1 terms in (5.41) and (5.42) with a Z_1 term by means of identity (4.11b) and this in turn may be replaced with $\cos\alpha_{23}$ from the cosine law (4.12c). Carrying out a similar procedure to remove $\bar{Y}_2$ in (5.43) and (5.44), one has two equations containing only X_1 and two containing only $\bar{X}_2$. Now since $X_1 = \bar{X}_2$ (from the basic sine law for a triangle (4.12a)), it is clear that these equations may be combined in four distinct ways to give the four further half-tangent laws listed above.

5.5 Further Half-Tangent Laws for Spherical Polygons.

For the spherical quadrilateral there exist six fundamental half-tangent laws in any three angular displacements, say θ_1, θ_2 and θ_3, and these may be written (see Appendix IV.) as follows:-

$$X_{12}x_3 + (Y_{12} - \sin\alpha_{34}) = 0 \quad (5.49)$$

$$(Y_{12} + \sin\alpha_{34})x_3 - X_{12} = 0 \quad (5.50)$$

$$(X_1 + \bar{X}_3)x_2 + (Y_1 + \bar{Y}_3) = 0 \quad (5.51)$$

$$(Y_1 - \bar{Y}_3)x_2 - (X_1 - \bar{X}_3) = 0 \quad (5.52)$$

$$X_{32}x_1 + (Y_{32} - \sin\alpha_{41}) = 0 \quad (5.53)$$

$$(Y_{32} + \sin\alpha_{41})x_1 - X_{32} = 0 \quad (5.54)$$

Using a similar procedure to that adopted for the triangle, above, it is possible to derive eight distinct further half-tangent laws in the three half-tangents x_1, x_2 and x_3 by taking various combinations of these laws. Thus, for example, from (5.50), (5.52) and (5.54) one obtains:-

$$\begin{aligned}&\sin\alpha_{12}[\cos(\alpha_{23} - \alpha_{34}) - Z_1]x_3\\ &+ [\sin\alpha_{12}\cos\alpha_{34} + \sin(\alpha_{23} - \alpha_{12}).Z_1 - \sin\alpha_{23}\cos\alpha_{41}]x_2\\ &- \sin\alpha_{23}[\cos(\alpha_{12} - \alpha_{41}) - Z_1]x_1 = 0\end{aligned} \quad (5.55)$$

which is analogous to the law (5.45) for the triangle.

However, unlike the triangle law, equation (5.55) is not linear in each half-tangent and is in fact cubic in x_1. The other seven possible further half-tangent laws are of a similar nature to (5.55).

Clearly the above discussion may be applied in general and it can be shown that from the 2.(n - 1) fundamental half-tangent laws (in one half-tangent) involving (n - 1) angular displacements, for an n-sided spherical polygon, it is possible to derive 2^{n-1} distinct further half-tangent laws, each in (n - 1) half-tangents. Appendix IV. contains a limited selection of such laws as examples, though their usefulness has so far proved to be minimal.

Nevertheless, in the process of deriving these further laws, certain intermediate half-tangent laws occur which contain less than (n - 1) half-tangents and, in particular, the following equation in two half-tangents for the spherical heptagon may be derived:-

$$\sin\alpha_{56} X_{1234} x_5 x_6 + (\sin\alpha_{56} Y_{1234} + \cos\alpha_{56} Z_{1234} - \cos\alpha_{67}) x_6 - [\cos(\alpha_{56} + \alpha_{67}) - Z_{1234}] = 0 \qquad (5.56)$$

It is the opinion of the author that equations of this form may prove useful in the formulation of the R^7 mechanism problem (see Chapter 11).

5.6 Dual Half-Tangent Laws.

Having obtained the fundamental half-tangent laws, which are linear in the half-tangent of one angular displacement, and from them the further half-tangent laws, one may inquire as to the form of the secondary parts of the corresponding dual half-tangent laws. For the fundamental half-tangent laws, these secondary equations may always be reduced to the same basic linear form as that of their primary equations.

In order to illustrate the point, consider equations (5.35) and (5.36) for the hexagon:-

$$(X_{612} + \bar{X}_4)x_3 + (Y_{612} + \bar{Y}_4) = 0 \qquad (5.35)$$

$$(Y_{612} - \bar{Y}_4)x_3 - (X_{612} - \bar{X}_4) = 0 \qquad (5.36)$$

These are the fundamental half-tangent laws derived from the subsidiary sine and sine-cosine laws (4.70a) and (4.70b). The latter may be written as follows with the aid of the half-tangent substitution, (5.1):-

$$(X_{612} + \bar{X}_4)x_3^2 + 2.Y_{612}x_3 - (X_{612} - \bar{X}_4) = 0 \qquad (5.57)$$

$$(Y_{612} - \bar{Y}_4)x_3^2 - 2.X_{612}x_3 - (Y_{612} + \bar{Y}_4) = 0 \qquad (5.58)$$

Now after introducing the dual symbol and expanding the resulting dual equations corresponding to (5.35) and (5.36) one obtains the two secondary equations:-

$$(X_{612} + \bar{X}_4)_0 x_3 + (X_{612} + \bar{X}_4)x_{03} + (Y_{612} + \bar{Y}_4)_0 = 0 \qquad (5.59)$$

$$(Y_{612} - \bar{Y}_4)_0 x_3 + (Y_{612} - \bar{Y}_4)x_{03} - (X_{612} - \bar{X}_4)_0 = 0 \qquad (5.60)$$

where the zero suffix signifies a secondary part.

However:-

$$
\begin{aligned}
x_{03} &= [\tan(\theta_3/2)]_0 \\
&= \tfrac{1}{2}.\sec^2(\theta_3/2).s_3 \qquad \text{(from Taylor's expansion)} \\
&= s_3[1 + \tan^2(\theta_3/2)]/2 \\
&= s_3(1 + x_3^2)/2 \qquad (5.61)
\end{aligned}
$$

and hence (5.59) and (5.60) become:-

$$s_3[(X_{612} + \bar{X}_4)x_3^2 + (X_{612} + \bar{X}_4)]/2 + (X_{612} + \bar{X}_4)_0 x_3 + (Y_{612} + \bar{Y}_4)_0 = 0 \quad (5.62)$$

$$s_3[(Y_{612} - \bar{Y}_4)x_3^2 + (Y_{612} - \bar{Y}_4)]/2 + (Y_{612} - \bar{Y}_4)_0 x_3 - (X_{612} - \bar{X}_4)_0 = 0 \quad (5.63)$$

At this point it can be seen that the coefficients of x_3^2 in (5.57) and (5.62) are proportional and hence (5.62) can be immediately reduced to an equation linear in x_3 by means of (5.57). A similar argument may be applied to (5.58) and (5.63) and the latter also reduces to an equation linear in x_3. (see also Chapter 10.).

Clearly this process is obviously general in the sense that the secondary equation corresponding to any fundamental half-tangent law may be reduced to a linear equation, using the quadratic primary equation from which it is derived.

In addition this type of reduction may be possible for the further half-tangent laws considered above, although the author has not attempted the task.

Thus, having developed a unified theory for the analysis of spatial mechanisms in Part I. of this dissertation, Part II. will be devoted to the application of the method to a series of specific mechanisms, and, in particular, to a group of five distinct six-link spatial mechanisms which in the author's knowledge have not been correctly analysed previously.

PART II

APPLICATION OF THE THEORY

TO

FOUR, FIVE, SIX AND SEVEN-LINK SPATIAL MECHANISMS

CHAPTER 6

A DISPLACEMENT ANALYSIS

OF

SPATIAL FOUR-LINK R-3C

AND

FIVE-LINK 3R-2C MECHANISMS

6.1 Introduction.

The objective of this chapter is to present an outline of the displacement analyses of spatial four and five-link mechanisms based on the unified theory developed in Part I. In recent years considerable progress has been made in this area and the four-link RC^3 linkage is well documented [6, 46]. For the five-link RCRCR mechanism Dimentberg [7] obtained initially an input-output equation of degree eight, but Yang [45] later succeeded in deriving a quartic equation for this mechanism. More recently this quartic was corrected by Yuan [47] and by Duffy and Habib-Olahi [11] who also derived input-output equations of degree four and eight respectively for the two inversions, RRCRC and RCRRC, of the RCRCR mechanism [13, 14]. Finally, Yuan [48] analysed the RRCCR five-link mechanism obtaining a degree eight equation, whilst Duffy and Habib-Olahi [14] derived the input-output equation for one of its inversions - the RRRCC mechanism.

In this chapter the basic problem of the elimination of a single unknown from two equations, which is central to the analysis of the majority of spatial mechanisms, is met with. The fundamental difficulty is the formation of the correct initial equations, since the elimination aspect presents no problem in this case (see Chapter 5), and the techniques adopted in this chapter will prove to be of general applicability. Finally, it was thought unnecessary to include numerical examples for the mechanisms dealt with in this chapter in view of the agreement obtained with earlier published results [11, 13, 14, 19, 45, 46, 48].

6.2 Description of the Four-Link RCCC Spatial Mechanism.

The four-link RCCC spatial mechanism is illustrated by Figure 2.2 and, in accordance with Chapters 1 and 3, it is modelled mathematically by a spatial quadrilateral with the following dual sides and dual angles:-

$$\hat{\alpha}_{12} = \alpha_{12} + \varepsilon\, a_{12}$$
$$\hat{\alpha}_{23} = \alpha_{23} + \varepsilon\, a_{23}$$
$$\hat{\alpha}_{34} = \alpha_{34} + \varepsilon\, a_{34}$$
$$\hat{\alpha}_{41} = \alpha_{41} + \varepsilon\, a_{41} \qquad (6.1)$$

$$\hat{\theta}_1 = \theta_1 + \varepsilon\, S_{11}$$
$$\hat{\theta}_2 = \theta_2 + \varepsilon\, S_2$$
$$\hat{\theta}_3 = \theta_3 + \varepsilon\, S_3$$
$$\hat{\theta}_4 = \theta_4 + \varepsilon\, S_4 \qquad (6.2)$$

where $\varepsilon^2 = 0$, and all constant mechanism dimensions have double or repeated suffices. The input variable is the angular displacement, θ_1, whilst the output variables are the angular displacement, θ_4, and the sliding displacement S_4.

6.2.1 Input-Output Displacement Equation.

For the RCCC mechanism no elimination procedure is required to obtain the input-output equation, since it is possible to write loop equations which contain only the input and output angular variables as unknowns. Thus from equation (4.37) (the cosine law for a spherical quadrilateral) one has, by introducing the dual symbol, the following cyclic permutation of the dual cosine law for a spatial quadrilateral:-

$$\hat{Z}_{14} = \cos\hat{\alpha}_{23} \qquad (6.3)$$

Expanding (6.3) into primary and secondary parts (see Chapter 3.) gives the respective primary and secondary equations:-

$$Z_{14} = \cos\alpha_{23} \qquad (6.4)$$

and

$$Z_{014} = -a_{23}\sin\alpha_{23} \qquad (6.5)$$

where:-

$$Z_{14} = \sin\alpha_{34}(\bar{X}_1\sin\theta_4 + \bar{Y}_1\cos\theta_4) + \cos\alpha_{34}\bar{Z}_1 \qquad (6.6)$$

and Z_{014} may be written in the symmetric form:-

$$\begin{aligned} Z_{014} = \; & a_{34}Y_{14} \\ & + S_4 \sin\alpha_{34} X_{14} \\ & + a_{41}(\cos\alpha_{12} Y_4 - \sin\alpha_{12} \cos\theta_1 Z_4) \\ & + S_{11} \sin\alpha_{12} X_{41} \\ & + a_{12} Y_{41} \end{aligned} \tag{6.7}$$

(Note that $Z_{014} \equiv Z_{041}$).

Equation (6.4) is clearly the input-output equation for both the spherical and the spatial four-link mechanisms, and may be expressed as a quadratic in the half-tangent of the output angular displacement. This verifies the fact that both the R^4 spherical and RC^3 spatial four-link mechanisms have two closures in general (see Chapter 2.). For a given set of $\hat{\alpha}_{ij}$, S_{11} and input angle θ_1, (6.4) may be solved to give in general two values for θ_4. By substituting the sets (θ_1, θ_4), obtained in this way, into (6.5) one obtains corresponding values for the output sliding displacement S_4.

In the notation adopted by Yang and Freudenstein [46] the input-output equation for the RCCC spatial mechanism would be written:-

$$A(\theta_1)\sin\theta_4 + B(\theta_1)\cos\theta_4 = C(\theta_1) \tag{6.8}$$

and hence from (6.4), (6.6) and (6.8) the relationship between the two notations is:-

$$\begin{aligned} A(\theta_1) &\equiv \sin\alpha_{34}\bar{\bar{X}}_1 \\ B(\theta_1) &\equiv \sin\alpha_{34}\bar{\bar{Y}}_1 \\ C(\theta_1) &\equiv \cos\alpha_{23} - \cos\alpha_{34}\bar{\bar{Z}}_1 \end{aligned} \tag{6.9}$$

6.2.2 Determination of θ_2 and θ_3.

Once the corresponding values of input and output angular displacements have been determined from equation (6.4), unique values for the angular variables θ_2 and θ_3 may be calculated using the fundamental half-tangent laws.

Thus θ_2 may be obtained from either of equations (5.29) or (5.30) by rearranging these as follows:-

$$x_2 = -(Y_{41} - \sin\alpha_{23})/X_{41} \tag{5.29a}$$

or $$x_2 = X_{41}/(Y_{41} + \sin\alpha_{23}) \tag{5.30a}$$

where $$x_2 \equiv \tan(\theta_2/2) \tag{5.31}$$

since:- $$X_{41} = X_4\cos\theta_1 - Y_4\sin\theta_1 \tag{4.28}$$

and $$Y_{41} = \cos\alpha_{12}(X_4\sin\theta_1 + Y_4\cos\theta_1) - \sin\alpha_{12}Z_4 \tag{4.32}$$

are uniquely determined for a given set, (θ_1, θ_4).

In a similar manner θ_3 may be found from the following cyclic permutations of (5.29a) or (5.30a):-

$$x_3 = -(Y_{14} - \sin\alpha_{23})/X_{14} \tag{6.10}$$

or $$x_3 = X_{14}/(Y_{14} + \sin\alpha_{23}) \tag{6.11}$$

where $$x_3 \equiv \tan(\theta_3/2) \tag{6.12}$$

and:- $$X_{14} = \bar{X}_1\cos\theta_4 - \bar{Y}_1\sin\theta_4 \tag{6.13}$$

$$Y_{14} = \cos\alpha_{34}(\bar{X}_1\sin\theta_4 + \bar{Y}_1\cos\theta_4) - \sin\alpha_{34}\bar{Z}_1 \tag{6.14}$$

6.2.3 Determination of S_2 and S_3.

Unique values for S_2 may be obtained from the secondary part of the dual cosine law:-

$$\hat{Z}_{12} = \cos\hat{\alpha}_{34} \tag{6.15}$$

which is written in the form:-

$$Z_{012} = -a_{34}\sin\alpha_{34} \tag{6.16}$$

where:

$$\begin{aligned} Z_{012} = \; & a_{23}Y_{12} \\ & + S_2\sin\alpha_{23}X_{12} \\ & + a_{12}[\cos\alpha_{41}\bar{Y}_2 - \sin\alpha_{41}\bar{Z}_2\cos\theta_1] \\ & + S_{11}\sin\alpha_{41}X_{21} \\ & + a_{41}Y_{21} \end{aligned} \tag{6.17}$$

since θ_1 and θ_2 are known.

Similarly, values for S_3 are obtained from the secondary part of the subsidiary dual cosine law:-

$$\hat{Z}_3 = \hat{\bar{Z}}_1 \tag{6.18}$$

which is written in the form:-

$$Z_{03} = \bar{Z}_{01} \tag{6.19}$$

where:-

$$Z_{03} = a_{34}Y_3 + S_3 \sin\alpha_{34} X_3 + a_{23}\bar{Y}_3 \tag{6.20}$$

and

$$\bar{Z}_{01} = a_{41}\bar{Y}_1 + S_{11}\sin\alpha_{41}\bar{X}_1 + a_{12}Y_1 \tag{6.21}$$

6.3 Note on the Derivation of Input-Output Displacement Equations for Spatial Mechanisms with More than Four Links.

In the case of spatial mechanisms with more than four-links it is not possible to write down the input-output displacement equation immediately (in contrast with the RCCC mechanism), since the loop equations contain more than two angular displacements. Generally the input-output equation must involve all the constant mechanism proportions, and it is impossible to include, say the fixed offsets, S_{ii}, of a mechanism, in the analysis without introducing the corresponding extraneous angular variables, θ_i. The difficulty is inherent in the loop equations derived in Chapters 4 and 5, since they are, of necessity, expressions in terms of the dual angles, $\hat{\theta}_i = \theta_i + \varepsilon S_{ii}$. The major problem in the analysis of such mechanisms is the derivation of input-output displacement equations free from extraneous or unwanted roots, via an elimination procedure.

For all five-link 3R-2C, six-link 4R-P-C and seven-link 5R-2P mechanisms one may write appropriate primary equations which contain the input,output and a single extraneous angular displacement, θ_i. Hence, it is necessary to form a second equation containing the input, output and θ_i in order to derive the input-output equation from an elimination of this θ_i between the two equations. The required second equation must be derived from that secondary equation which involves all the fixed offsets. This ensures that all the constant mechanism proportions are included in the analysis.

Now the second equation must not be of an unnecessarily high degree in the input, output or extraneous θ_i, since if it is, one has inadvertently performed an elimination, and the final eliminant will contain the required input-output equation multiplied by an extraneous factor, which is, practically, impossible to find. In general, more than one elimination introduces extraneous roots.

The derivation of the second equation is greatly facilitated by the following factors:-

(i) The scheme of notation introduced in Chapter 4.

(ii) The classification of the loop equations into sine, sine-cosine and cosine laws. (see Chapter 4).

(iii) The expected degree (as predicted by Chapter 2) of the input-output equation.

(iv) The discovery of the fundamental half-tangent laws, derived in Chapter 5.

Finally it must be noted that for six-link 5R-C and seven-link 6R-P and 7R mechanisms one is confronted with the problem of the elimination of more than one extraneous angular displacement from at least three equations, as can be seen from the relevant loop equations for the spatial hexagon and heptagon (see Chapter 4), and this presents formidable difficulties which will be discussed further in Chapters 10 and 11.

6.4 Description of the Five-Link RCRCR Spatial Mechanism.

The five-link RCRCR spatial mechanism is illustrated by Figure 2.24 and is represented mathematically by the following dual sides and angles:-

$$\hat{\alpha}_{12} = \alpha_{12} + \epsilon\, a_{12}$$
$$\hat{\alpha}_{23} = \alpha_{23} + \epsilon\, a_{23}$$
$$\hat{\alpha}_{34} = \alpha_{34} + \epsilon\, a_{34}$$
$$\hat{\alpha}_{45} = \alpha_{45} + \epsilon\, a_{45}$$
$$\hat{\alpha}_{51} = \alpha_{51} + \epsilon\, a_{51} \qquad (6.22)$$

$$\hat{\theta}_1 = \theta_1 + \epsilon\, S_{11}$$
$$\hat{\theta}_2 = \theta_2 + \epsilon\, S_2$$
$$\hat{\theta}_3 = \theta_3 + \epsilon\, S_{33}$$
$$\hat{\theta}_4 = \theta_4 + \epsilon\, S_4$$
$$\hat{\theta}_5 = \theta_5 + \epsilon\, S_{55} \qquad (6.23)$$

where $\epsilon^2 = 0$, and all fixed mechanism proportions have double or repeated suffices. The input and output angular displacements are respectively θ_1 and θ_5, and the frame may be considered to be the constant dual side, $\hat{\alpha}_{51}$.

6.4.1 Derivation of Input-Output Equation.

The input-output equation for the RCRCR five-link mechanism is derived by eliminating the extraneous angular variable, θ_3, between the primary and secondary parts of the following dual subsidiary cosine law (see Chapter 4):-

$$\hat{Z}_{15} = \hat{Z}_3 \qquad (6.24)$$

The primary component of equation (6.24) is written as:-

$$Z_{15} = Z_3 \qquad (6.25)$$

where:-

$$Z_{15} = \sin\alpha_{45}\,(\bar{X}_1\sin\theta_5 + \bar{Y}_1\cos\theta_5) + \cos\alpha_{45}\,\bar{Z}_1 \qquad (6.26)$$

$$Z_3 = \cos\alpha_{23}\cos\alpha_{34} - \sin\alpha_{23}\sin\alpha_{34}\cos\theta_3 \qquad (6.27)$$

whilst the secondary component of (6.24) is:-

$$Z_{015} = Z_{03} \qquad (6.28)$$

where Z_{015} and Z_{03} are obtained by expansion of (6.26) and (6.27) as explained in Chapters 3 and 4, and may be written in the following symmetrical form (see Appendix III):-

$$Z_{015} = a_{45}Y_{15}$$
$$+ S_{55}\sin\alpha_{45}X_{15}$$
$$+ a_{51}\operatorname{cosec}\alpha_{51}(\bar{Z}_1 Z_5 - \cos\alpha_{12}\cos\alpha_{45})$$
$$+ S_{11}\sin\alpha_{12}X_{51}$$
$$+ a_{12}Y_{51} \quad (6.29)$$

$$Z_{03} = a_{34}Y_3$$
$$+ S_{33}\sin\alpha_{34}X_3$$
$$+ a_{23}\bar{Y}_3 \quad (6.30)$$

Now equations (6.25) and (6.28) contain the input, output and θ_3 and so are in a suitable form for the elimination of the latter. Thus, making the substitution (5.1) for $\cos\theta_3$ in (6.27) and (6.25) and rearranging, one obtains the following quadratic in $x_3 (\equiv \tan(\theta_3/2))$, from (6.25):-

$$f(x_3) = a_2 x_3^2 + a_1 x_3 + a_0 = 0 \quad (6.31)$$

where:-

$$a_2 = Z_{15} - \cos(\alpha_{23} - \alpha_{34})$$
$$a_1 = 0$$
$$a_0 = Z_{15} - \cos(\alpha_{23} + \alpha_{34}) \quad (6.32)$$

In a similar manner, since X_3, Y_3 and $\bar{Y}_3$ are defined by:-

$$X_3 = \sin\alpha_{23}\sin\theta_3$$
$$Y_3 = -(\cos\alpha_{23}\sin\alpha_{34} + \sin\alpha_{23}\cos\alpha_{34}\cos\theta_3)$$
$$\bar{Y}_3 = -(\cos\alpha_{34}\sin\alpha_{23} + \sin\alpha_{34}\cos\alpha_{23}\cos\theta_3) \quad (6.33)$$

it is possible to obtain the following quadratic from (6.28), using (6.30):-

$$g(x_3) = b_2 x_3^2 + b_1 x_3 + b_0 = 0 \quad (6.34)$$

where:-

$$b_2 = Z_{015} + (a_{23} - a_{34})\sin(\alpha_{23} - \alpha_{34})$$
$$b_1 = -2.S_{33}\sin\alpha_{23}\sin\alpha_{34}$$
$$b_0 = Z_{015} + (a_{23} + a_{34})\sin(\alpha_{23} + \alpha_{34}) \quad (6.35)$$

Now, forming the Bézoutian of (6.31) and (6.34) (see Chapter 5), one obtains as the eliminant, the compound symmetric determinant:-

$$B(f, g) = \begin{vmatrix} (a_2b_1) & (a_2b_0) \\ (a_2b_0) & (a_1b_0) \end{vmatrix} \qquad (5.8)$$

which, when equated to zero, must give the desired input-output equation.

Since B(f, g) is of order 4 in the coefficients a_2, a_1,...etc., which are themselves quadratic in both the input and output variables, one might expect to obtain a degree eight input-output equation for the RCRCR. However, expanding (5.8) gives:-

$$B(f, g) = (a_2b_1 - a_1b_2)(a_1b_0 - a_0b_1) - (a_2b_0 - a_0b_2)^2 = 0 \qquad (6.36)$$

and from (6.32) and (6.35) it is clear that:-

$$\begin{aligned} a_2 &= a_0 + k_1 \\ a_1 &= 0 \end{aligned} \qquad (6.37)$$

$$\begin{aligned} b_2 &= b_0 + k_2 \\ b_1 &= S_{33}k_1 \end{aligned} \qquad (6.38)$$

where k_1 and k_2 are constants depending only on the fixed mechanism proportions and given by:-

$$\begin{aligned} k_1 &= -2\sin\alpha_{23}\sin\alpha_{34} \\ k_2 &= -2(a_{23}\cos\alpha_{23}\sin\alpha_{34} + a_{34}\sin\alpha_{23}\cos\alpha_{34}) \end{aligned} \qquad (6.39)$$

Hence (6.36) reduces to the form:-

$$-S_{33}^2k_1^2a_0(a_0 + k_1) - [(a_0 + k_1)b_0 - a_0(b_0 + k_2)]^2 = 0 \qquad (6.40)$$

which upon rearranging becomes:-

$$(S_{33}^2k_1^2 + k_2^2)a_0^2 + k_1^2b_0^2 + k_1(S_{33}^2k_1^2 - 2k_2b_0)a_0 = 0 \qquad (6.41)$$

Clearly, since only a_0 and b_0 are functions of the input and ouput variables and are quadratic in both, equation (6.41) is the biquartic input-output equation for the RCRCR mechanism, and may be arranged in the form:-

$$\begin{aligned}
&(p_{44}x_1^4 + p_{43}x_1^3 + p_{42}x_1^2 + p_{41}x_1 + p_{40})x_5^4 \\
&+ (p_{34}x_1^4 + p_{33}x_1^3 + p_{32}x_1^2 + p_{31}x_1 + p_{30})x_5^3 \\
&+ (p_{24}x_1^4 + p_{23}x_1^3 + p_{22}x_1^2 + p_{21}x_1 + p_{20})x_5^2 \\
&+ (p_{14}x_1^4 + p_{13}x_1^3 + p_{12}x_1^2 + p_{11}x_1 + p_{10})x_5 \\
&+ (p_{04}x_1^4 + p_{03}x_1^3 + p_{02}x_1^2 + p_{01}x_1 + p_{00}) = 0
\end{aligned} \tag{6.42}$$

where:-

$$x_1 \equiv \tan(\theta_1/2) \tag{6.43}$$

$$x_5 \equiv \tan(\theta_5/2) \tag{6.44}$$

and the coefficients, p_{ij}, are expressions in terms of the mechanism proportions only. This is in complete agreement with the predicted degree for the RCRCR (see Table II. Chapter 2), and with the algebraic results obtained by Yang [45], Yuan [47] and Duffy and Habib-Olahi [11].

6.4.2 Determination of θ_3.

For a given set, (θ_1, θ_5), of corresponding input and output angles obtained from a solution of (6.42), one may calculate the unique value of the angular displacement, θ_3, from either of the two expressions for the common root of (6.31) and (6.34), derived from the Bézoutian (5.8) (see equations (5.7a,b) of Chapter 5.). These expressions are:-

$$\begin{aligned}
x_3 &= -(a_2b_0)/(a_2b_1) \\
&= -(a_2b_0 - a_0b_2)/(a_2b_1 - a_1b_2)
\end{aligned} \tag{6.45a}$$

or

$$\begin{aligned}
x_3 &= -(a_1b_0)/(a_2b_0) \\
&= -(a_1b_0 - a_0b_1)/(a_2b_0 - a_0b_2)
\end{aligned} \tag{6.45b}$$

Thus from (6.32), (6.35), (6.37) and (6.38) one has:-

$$x_3 = (k_2a_0 - k_1b_0)/S_{33}k_1(a_0 + k_1) \tag{6.46a}$$

or

$$x_3 = S_{33}k_1a_0/(k_1b_0 - k_2a_0) \tag{6.46b}$$

where $x_3 \equiv \tan(\theta_3/2)$ and k_1, k_2 are given by (6.39).

6.4.3 Determination of θ_2 and θ_4.

Having determined corresponding values for θ_1, θ_5 and θ_3 it is now a simple matter to obtain the unique value of θ_2 from either of the two fundamental half-tangent laws (see Appendix IV):-

$$x_2 = -(Y_{51} + \bar{Y}_3)/(X_{51} + \bar{X}_3) \tag{6.47a}$$

or

$$x_2 = (X_{51} - \bar{X}_3)/(Y_{51} - \bar{Y}_3) \tag{6.47b}$$

where

$$x_2 \equiv \tan(\theta_2/2) \tag{6.48}$$

In a similar manner one may obtain the value of θ_4 from a cyclic permutation of (6.47a,b). Thus:-

$$x_4 = -(Y_{15} + Y_3)/(X_{15} + X_3) \tag{6.49a}$$

or

$$x_4 = (X_{15} - X_3)/(Y_{15} - Y_3) \tag{6.49b}$$

6.4.4 Determination of S_2 and S_4.

The sliding displacement S_2 may be determined from the secondary component of the dual cosine law:-

$$\hat{Z}_{512} = \cos\hat{\alpha}_{34} \tag{6.50}$$

which is:-

$$Z_{0512} = -a_{34}\sin\alpha_{34} \tag{6.51}$$

where:-

$$\begin{aligned} Z_{0512} = \; & a_{23}Y_{512} \\ & + S_2\sin\alpha_{23}X_{512} \\ & + a_{12}[(X_5\sin\theta_1 + Y_5\cos\theta_1)\bar{Z}_2 + X_5\bar{Y}_2] \\ & + S_{11}[(Y_5\bar{Y}_2 - X_5\bar{X}_2)\sin\theta_1 - (Y_5\bar{X}_2 + X_5\bar{Y}_2)\cos\theta_1] \\ & + a_{51}[(\bar{X}_2\sin\theta_1 + \bar{Y}_2\cos\theta_1)Z_5 + \bar{Z}_2Y_5] \\ & + S_{55}\sin\alpha_{45}X_{215} \\ & + a_{45}Y_{215} \end{aligned} \tag{6.52}$$

since X_{512}, Y_{512}, X_{215},....etc., are uniquely defined for a given set of θ_1, θ_5 and θ_2.

Similarly the displacement, S_4, is determined from the secondary component of the dual cosine law:-

$$\hat{Z}_{154} = \cos\hat{\alpha}_{23} \qquad (6.53)$$

which is:-

$$Z_{0154} = -a_{23}\sin\alpha_{23} \qquad (6.54)$$

where:-

$$\begin{aligned} Z_{0154} = \; & a_{34}Y_{154} \\ & + S_4\sin\alpha_{34}X_{154} \\ & + a_{45}[(\bar{X}_1\sin\theta_5 + \bar{Y}_1\cos\theta_5)Z_4 + \bar{Z}_1Y_4] \\ & + S_{55}[(\bar{Y}_1Y_4 - \bar{X}_1X_4)\sin\theta_5 - (\bar{Y}_1X_4 + \bar{X}_1Y_4)\cos\theta_5] \\ & + a_{51}[(X_4\sin\theta_5 + Y_4\cos\theta_5)\bar{Z}_1 + Z_4\bar{Y}_1] \\ & + S_{11}\sin\alpha_{12}X_{451} \\ & + a_{12}Y_{451} \end{aligned} \qquad (6.55)$$

6.5 Description of the Five-Link RRCCR Spatial Mechanism.

The five-link RRCCR spatial mechanism is illustrated by Figure 2.23 and is represented mathematically by the following dual sides and angles:-

$$\begin{aligned} \hat{\alpha}_{12} &= \alpha_{12} + \epsilon\, a_{12} \\ \hat{\alpha}_{23} &= \alpha_{23} + \epsilon\, a_{23} \\ \hat{\alpha}_{34} &= \alpha_{34} + \epsilon\, a_{34} \\ \hat{\alpha}_{45} &= \alpha_{45} + \epsilon\, a_{45} \\ \hat{\alpha}_{51} &= \alpha_{51} + \epsilon\, a_{51} \end{aligned} \qquad (6.56)$$

$$\begin{aligned} \hat{\theta}_1 &= \theta_1 + \epsilon\, S_{11} \\ \hat{\theta}_2 &= \theta_2 + \epsilon\, S_{22} \\ \hat{\theta}_3 &= \theta_3 + \epsilon\, S_3 \\ \hat{\theta}_4 &= \theta_4 + \epsilon\, S_4 \\ \hat{\theta}_5 &= \theta_5 + \epsilon\, S_{55} \end{aligned} \qquad (6.57)$$

where $\epsilon^2 = 0$, and all fixed mechanism proportions have double or repeated suffices. The input and output angular displacements are respectively θ_1 and θ_5, and the frame is the constant dual side $\hat{\alpha}_{51}$.

6.5.1 Derivation of Input-Output Equation.

The input-output equation for the RRCCR five-link spatial mechanism is derived by eliminating the extraneous angular variable, θ_2, between the primary and secondary parts of the following dual cosine law (see Chapter 4):-

$$\hat{Z}_{512} = \cos\hat{\alpha}_{34} \qquad (6.58)$$

The primary part of (6.58) is:-

$$Z_{512} = \cos\alpha_{34} \qquad (6.59)$$

where:-

$$Z_{512} = \sin\alpha_{23}(X_{51}\sin\theta_2 + Y_{51}\cos\theta_2) + \cos\alpha_{23}Z_{51} \qquad (6.60)$$

whilst the secondary part of (6.58) is given by equations (6.51) and (6.52):-

$$Z_{0512} = -a_{34}\sin\alpha_{34} \qquad (6.51)$$

where:-

$$\begin{aligned} Z_{0512} = \; & a_{23}Y_{512} \\ & + S_2\sin\alpha_{23}X_{512} \\ & + a_{12}[(X_5\sin\theta_1 + Y_5\cos\theta_1)\bar{Z}_2 + Z_5\bar{Y}_2] \\ & + S_{11}[(Y_5\bar{Y}_2 - X_5\bar{X}_2)\sin\theta_1 - (Y_5\bar{X}_2 + X_5\bar{Y}_2)\cos\theta_1] \\ & + a_{51}[(\bar{X}_2\sin\theta_1 + \bar{Y}_2\cos\theta_1)Z_5 + \bar{Z}_2Y_5] \\ & + S_{55}\sin\alpha_{45}X_{215} \\ & + a_{45}Y_{215} \end{aligned} \qquad (6.52)$$

However, for present purposes, it is more convenient to write Z_{0512} as:-

$$\begin{aligned} Z_{0512} = \; & a_{23}Y_{512} + S_{22}\sin\alpha_{23}X_{512} \\ & + \sin\alpha_{23}(X_{051}\sin\theta_2 + Y_{051}\cos\theta_2) + \cos\alpha_{23}Z_{051} \end{aligned} \qquad (6.61)$$

where:-

$$X_{512} = X_{51}\cos\theta_2 - Y_{51}\sin\theta_2 \qquad (6.62)$$

$$Y_{512} = \cos\alpha_{23}(X_{51}\sin\theta_2 + Y_{51}\cos\theta_2) - \sin\alpha_{23}Z_{51} \qquad (6.63)$$

since (6.61) follows immediately by expansion of (6.60).

Arranging (6.59) and (6.51) as quadratics in $x_2 (\equiv \tan(\theta_2/2))$, with the aid of (6.60) and (6.61), gives the following two equations:-

$$f(x_2) = a_2 x_2^2 + a_1 x_2 + a_0 = 0 \qquad (6.64)$$

$$g(x_2) = b_2 x_2^2 + b_1 x_2 + b_0 = 0 \qquad (6.65)$$

where:-

$$\begin{aligned} a_2 &= \cos\alpha_{23} Z_{51} - \sin\alpha_{23} Y_{51} - \cos\alpha_{34} \\ a_1 &= 2.\sin\alpha_{23} X_{51} \\ a_0 &= \cos\alpha_{23} Z_{51} + \sin\alpha_{23} Y_{51} - \cos\alpha_{34} \end{aligned} \qquad (6.66)$$

and:-

$$\begin{aligned} b_2 &= (\cos\alpha_{23} Z_{051} - \sin\alpha_{23} Y_{051}) - S_{22}\sin\alpha_{23} X_{51} \\ &\quad -a_{23}(\sin\alpha_{23} Z_{51} + \cos\alpha_{23} Y_{51}) + a_{34}\sin\alpha_{34} \\ b_1 &= 2(\sin\alpha_{23} X_{051} - S_{22}\sin\alpha_{23} Y_{51} + a_{23}\cos\alpha_{23} X_{51}) \\ b_0 &= (\cos\alpha_{23} Z_{051} + \sin\alpha_{23} Y_{051}) + S_{22}\sin\alpha_{23} X_{51} \\ &\quad -a_{23}(\sin\alpha_{23} Z_{51} - \cos\alpha_{23} Y_{51}) + a_{34}\sin\alpha_{34} \end{aligned} \qquad (6.67)$$

The Bézoutian of (6.64) and (6.65) is again given by (5.8), since they are quadratics and hence the input-output equation for the RRCCR is:-

$$B(f, g) = \begin{vmatrix} (a_2 b_1) & (a_2 b_0) \\ (a_2 b_0) & (a_1 b_0) \end{vmatrix} = 0 \qquad (5.8)$$

i.e.

$$B(f, g) = (a_2 b_1 - a_1 b_2)(a_1 b_0 - a_0 b_1) - (a_2 b_0 - a_0 b_2)^2 = 0 \qquad (6.68)$$

Unlike the case of the RCRCR, there do not exist relationships of the form (6.37) and (6.38) between the coefficients (6.66) and (6.67), and thus (6.68) does not reduce in order or degree. Consequently the input-output equation for the RRCCR is of degree eight in the input and output variables. This result is in complete agreement with the predicted degree for this mechanism (see Table II., Chapter 2.), and with the algebraic results obtained by Yuan [48] and Soni [33].

6.5.2 Determination of Remaining Mechanism Variables (θ_2, θ_3, θ_4, S_3 and S_4).

The remaining mechanism variables may be determined by following similar procedures to those used for the RCRCR. Thus θ_2 may be obtained from either of the following two expressions for the common root of (6.64) and (6.65):-

$$x_2 = -(a_2b_0)/(a_2b_1)$$
$$= -(a_2b_0 - a_0b_2)/(a_2b_1 - a_1b_2) \qquad (6.69a)$$

or $$x_2 = -(a_1b_0)/(a_2b_0)$$
$$= -(a_1b_0 - a_0b_1)/(a_2b_0 - a_0b_2) \qquad (6.69b)$$

where $x_2 \equiv \tan(\theta_2/2)$

Having determined corresponding values for θ_1, θ_5 and θ_2, one may obtain a unique value for θ_3 from either of the two fundamental half-tangent laws (see Appendix IV.):-

$$x_3 = -(Y_{512} - \sin\alpha_{34})/X_{512} \qquad (6.70a)$$

or $$x_3 = X_{512}/(Y_{512} + \sin\alpha_{34}) \qquad (6.70b)$$

where $x_3 \equiv \tan(\theta_3/2)$.

Similarly θ_4 may be determined from either of:-

$$x_4 = -(Y_{215} - \sin\alpha_{34})/X_{215} \qquad (6.71a)$$

or $$x_4 = X_{215}/(Y_{215} + \sin\alpha_{34}) \qquad (6.71b)$$

where $x_4 \equiv \tan(\theta_4/2)$

The sliding displacement, S_3, is then obtained from the secondary component of the dual subsidiary cosine law (6.24), which is:-

$$Z_{015} = Z_{03} \qquad (6.72)$$

where:-

$$Z_{015} = a_{45}Y_{15}$$
$$+ S_{55}\sin\alpha_{45}X_{15}$$
$$+ a_{51}\mathrm{cosec}\alpha_{51}(\bar{Z}_1Z_5 - \cos\alpha_{12}\cos\alpha_{45})$$
$$+ S_{11}\sin\alpha_{12}X_{51}$$
$$+ a_{12}Y_{51} \qquad (6.73)$$

and:-

$$Z_{03} = a_{34}Y_3 + S_3\sin\alpha_{34}X_3 + a_{23}\bar{Y}_3 \tag{6.74}$$

Finally, S_4 may be determined from the secondary component of the dual cosine law (6.53), which is:-

$$Z_{0154} = -a_{23}\sin\alpha_{23} \tag{6.75}$$

where:-

$$\begin{aligned} Z_{0154} = \; & a_{34}Y_{154} \\ & + S_4\sin\alpha_{34}X_{154} \\ & + a_{45}[(\bar{X}_1\sin\theta_5 + \bar{Y}_1\cos\theta_5)Z_4 + \bar{Z}_1Y_4] \\ & + S_{55}[(\bar{Y}_1Y_4 - \bar{X}_1X_4)\sin\theta_5 - (\bar{Y}_1X_4 + \bar{X}_1Y_4)\cos\theta_5] \\ & + a_{51}[(X_4\sin\theta_5 + Y_4\cos\theta_5)\bar{Z}_1 + Z_4\bar{Y}_1] \\ & + S_{11}\sin\alpha_{12}X_{451} \\ & + a_{12}Y_{451} \end{aligned} \tag{6.76}$$

6.6 Derivation of the Input-Output Displacement Equations for the two Inversions, RRCRC and RCRRC, of the RCRCR Mechanism and the Inversion, RRRCC, of the RRCCR Mechanism.

There exist two inversions of the RCRCR five-link spatial mechanism, namely the RRCRC and the RCRRC, whilst the RRCCR has but a single inversion, the RRRCC. Now these three mechanisms share the property that they all have a cylindric output pair and this presents difficulties in the derivation of their input-output displacement equations. Thus, although one may write a suitable primary equation relating input, output and a single extraneous variable in each case, the corresponding secondary equations involve, in addition, the respective output sliding displacements, and are unsuitable for the elimination of a single unknown. In fact it is necessary to utilize that secondary equation which includes all the fixed mechanism proportions and involves only those angular displacements related to revolute pairs. The problem is then to manipulate this secondary equation into a form suitable for the elimination of a single angular variable without unnecessarily increasing its degree.

6.7 Description of the Five-Link RRCRC Spatial Mechanism.

The five-link RRCRC spatial mechanism is illustrated by Figure 2.24 since it is an inversion of the RCRCR. It is represented mathematically by the following dual sides and angles:-

$$\begin{aligned}
\hat{\alpha}_{12} &= \alpha_{12} + \epsilon\, a_{12} \\
\hat{\alpha}_{23} &= \alpha_{23} + \epsilon\, a_{23} \\
\hat{\alpha}_{34} &= \alpha_{34} + \epsilon\, a_{34} \\
\hat{\alpha}_{45} &= \alpha_{45} + \epsilon\, a_{45} \\
\hat{\alpha}_{51} &= \alpha_{51} + \epsilon\, a_{51}
\end{aligned} \qquad (6.77)$$

$$\begin{aligned}
\hat{\theta}_1 &= \theta_1 + \epsilon\, S_{11} \\
\hat{\theta}_2 &= \theta_2 + \epsilon\, S_2 \\
\hat{\theta}_3 &= \theta_3 + \epsilon\, S_{33} \\
\hat{\theta}_4 &= \theta_4 + \epsilon\, S_4 \\
\hat{\theta}_5 &= \theta_5 + \epsilon\, S_{55}
\end{aligned} \qquad (6.78)$$

where $\epsilon^2 = 0$ and all fixed mechanism proportions have double or repeated suffices. The input angular displacement is θ_1 whilst the output angular displacement and sliding displacement are respectively θ_2 and S_2. The frame is the constant dual side, $\hat{\alpha}_{12}$.

6.7.1 Derivation of Input-Output Equation for the RRCRC.

The input-output equation for the RRCRC five-link mechanism, unlike the RCRCR, cannot be derived directly by eliminating, say θ_5, between the primary and secondary parts of the dual cosine law (6.58), since Z_{0512} (given by equation (6.52)) involves the sliding output variable S_2 in addition to the output angular displacement θ_2. Consequently an alternative derivation is required. Thus, consider the two half-tangent laws:-

$$(X_{51} + \bar{X}_3)x_2 + (Y_{51} + \bar{Y}_3) = 0 \tag{6.79}$$

$$(Y_{51} - \bar{Y}_3)x_2 - (X_{51} - \bar{X}_3) = 0 \tag{6.80}$$

which are expressions for the common root between the subsidiary sine and sine-cosine laws (4.59a) and (4.59b).

The $\bar{X}_3$ terms can be removed from (6.79) and (6.80) by means of the secondary equation (6.28), which conveniently involves all the mechanism proportions and hence introduces these into the analysis.

For this purpose it is necessary to write (6.28) in the form:-

$$\begin{aligned} Z_{015} - a_{23}\bar{Y}_3 - a_{34}Y_3 &= S_{33}\sin\alpha_{34}X_3 \\ &= S_{33}\sin\alpha_{23}\bar{X}_3 \end{aligned} \tag{6.81}$$

and with the aid of (6.81), one may rewrite (6.79) and (6.80) as:-

$$\begin{aligned} &(S_{33}\sin\alpha_{23}X_{51} + Z_{015} - a_{23}\bar{Y}_3 - a_{34}Y_3)x_2 \\ &+ S_{33}\sin\alpha_{23}(Y_{51} + \bar{Y}_3) = 0 \end{aligned} \tag{6.82}$$

$$\begin{aligned} &S_{33}\sin\alpha_{23}(Y_{51} - \bar{Y}_3)x_2 \\ &-(S_{33}\sin\alpha_{23}X_{51} - Z_{015} + a_{23}\bar{Y}_3 + a_{34}Y_3) = 0 \end{aligned} \tag{6.83}$$

Finally, the Y_3 and $\bar{Y}_3$ terms may easily be replaced by means of the identities (4.10b) and (4.11b) together with the subsidiary cosine law (4.59c). Thus:-

$$\begin{aligned} \sin\alpha_{23}\bar{Y}_3 &\equiv \cos\alpha_{23}\bar{Z}_3 - \cos\alpha_{34} \\ &= \cos\alpha_{23}Z_{51} - \cos\alpha_{34} \end{aligned} \tag{6.84}$$

and:-

$$\begin{aligned} \sin\alpha_{34}Y_3 &\equiv \cos\alpha_{34}Z_3 - \cos\alpha_{23} \\ &= \cos\alpha_{34}Z_{15} - \cos\alpha_{23} \\ &= \cos\alpha_{34}Z_{51} - \cos\alpha_{23} \end{aligned} \tag{6.85}$$

and hence (6.82) and (6.83) become:-

$$[S_{33}\sin\alpha_{23}X_{51} + Z_{015} - (a_{23}\cot\alpha_{23} + a_{34}\cot\alpha_{34})Z_{51}$$
$$+ (a_{23}\cos\alpha_{34}\,\mathrm{cosec}\,\alpha_{23} + a_{34}\cos\alpha_{23}\,\mathrm{cosec}\,\alpha_{34})]x_2$$
$$+ S_{33}[\sin\alpha_{23}Y_{51} + \cos\alpha_{23}Z_{51} - \cos\alpha_{34}] = 0 \qquad (6.86)$$

and:-

$$S_{33}[\sin\alpha_{23}Y_{51} - \cos\alpha_{23}Z_{51} + \cos\alpha_{34}]x_2$$
$$- [S_{33}\sin\alpha_{23}X_{51} - Z_{015} + (a_{23}\cot\alpha_{23} + a_{34}\cot\alpha_{34})Z_{51}$$
$$- (a_{23}\cos\alpha_{34}\,\mathrm{cosec}\,\alpha_{23} + a_{34}\cos\alpha_{23}\,\mathrm{cosec}\,\alpha_{34})] = 0 \qquad (6.87)$$

Now (6.86) and (6.87) are of the form:-

$$f(x_5) = a_2x_5^2 + a_1x_5 + a_0 = 0 \qquad (6.88)$$
$$g(x_5) = b_2x_5^2 + b_1x_5 + b_0 = 0 \qquad (6.89)$$

where:- $x_5 \equiv \tan(\theta_5/2)$

and each of a_2, $a_1,\ldots,b_0$ is of degree two in x_1(input) and linear in x_2 (output). Hence, eliminating θ_5 by means of the Bézoutian (5.8), one obtains an input-output equation which is of the eighth degree in x_1 and quartic in x_4. This result is in complete agreement with the predicted degree for the RRCRC (see Chapter 2), and with the algebraic results obtained by Duffy and Habib-Olahi [14]. Note also that (6.86) and (6.87) are in the form:-

$$f(x_2) = a_1'x_2 + a_0' = 0 \qquad (6.90)$$
$$g(x_2) = b_1'x_2 + b_0' = 0. \qquad (6.91)$$

where:- $x_2 \equiv \tan(\theta_2/2)$

and a_1', a_0', b_1', b_0' are each of degree 2 in both x_1 and x_5. Hence eliminating x_2 between (6.90) and (6.91) gives a biquartic relationship between x_1 and x_5, and this is an alternative form for the input-output displacement equation of the RCRCR mechanism (see equation (6.42)).

The remaining variables for the RRCRC mechanism (i.e. S_2, θ_3, θ_4, S_4, θ_5) may now be calculated using similar procedures to those used for the RCRCR and RRCCR mechanisms.

6.8 Description of the Five-Link RCRRC Spatial Mechanism.

The five-link RCRRC spatial mechanism is illustrated by Figure 2.21 since it is an inversion of the RCRCR. It is represented mathematically by the following dual sides and angles:-

$$
\begin{aligned}
\hat{\alpha}_{12} &= \alpha_{12} + \epsilon\, a_{12} \\
\hat{\alpha}_{23} &= \alpha_{23} + \epsilon\, a_{23} \\
\hat{\alpha}_{34} &= \alpha_{34} + \epsilon\, a_{34} \\
\hat{\alpha}_{45} &= \alpha_{45} + \epsilon\, a_{45} \\
\hat{\alpha}_{51} &= \alpha_{51} + \epsilon\, a_{51}
\end{aligned} \tag{6.92}
$$

$$
\begin{aligned}
\hat{\theta}_1 &= \theta_1 + \epsilon\, S_{11} \\
\hat{\theta}_2 &= \theta_2 + \epsilon\, S_2 \\
\hat{\theta}_3 &= \theta_3 + \epsilon\, S_{33} \\
\hat{\theta}_4 &= \theta_4 + \epsilon\, S_4 \\
\hat{\theta}_5 &= \theta_5 + \epsilon\, S_{55}
\end{aligned} \tag{6.93}
$$

where $\epsilon^2 = 0$, and all fixed mechanism proportions have double or repeated suffices. The input angular displacement is θ_3 whilst the output angular displacement and sliding displacement are respectively θ_2 and S_2. The frame is the constant dual side, $\hat{\alpha}_{23}$.

6.8.1 Derivation of Input-Output Equation for the RCRRC.

The input-output equation for the RCRRC five-link mechanism cannot be derived directly by eliminating say θ_1 between the primary and secondary parts of the dual cosine law:-

$$\hat{Z}_{321} = \cos\hat{\alpha}_{45} \tag{6.94}$$

since Z_{0321} involves the output sliding variable S_2.

i.e.
$$
\begin{aligned}
Z_{0321} = \; & a_{51}Y_{321} \\
& + S_{11}\sin\alpha_{51}X_{321} \\
& + a_{12}[(\bar{X}_3\sin\theta_2 + \bar{Y}_3\cos\theta_2)Z_1 + \bar{Z}_3Y_1] \\
& + S_2[(\bar{Y}_3Y_1 - \bar{X}_3X_1)\sin\theta_2 - (\bar{Y}_3X_1 + \bar{X}_3Y_1)\cos\theta_2] \\
& + a_{23}[(X_1\sin\theta_2 + Y_1\cos\theta_2)\bar{Z}_3 + Z_1\bar{Y}_3] \\
& + S_{33}\sin\alpha_{34}X_{123} \\
& + a_{34}Y_{123}
\end{aligned}
\tag{6.95}
$$

However, it is possible to use the primary part of (6.94), which is written in the form of a quadratic in $x_1(\equiv \tan(\theta_1/2))$, by means of the half-tangent substitutions (5.1), as follows:-

$$
f(x_1) = a_2x_1^2 + a_1x_1 + a_0 = 0 \tag{6.96}
$$

where:-

$$
\begin{aligned}
a_2 &= \cos\alpha_{51}Z_{32} - \sin\alpha_{51}Y_{32} - \cos\alpha_{45} \\
a_1 &= 2.\sin\alpha_{51}X_{32} \\
a_0 &= \cos\alpha_{51}Z_{32} + \sin\alpha_{51}Y_{32} - \cos\alpha_{45}
\end{aligned}
\tag{6.97}
$$

A second equation of the form (6.96) is now required from which to eliminate x_1, and for this purpose one must use equation (6.28) in order to include in the analysis all the fixed mechanism proportions. Thus, writing (6.28) in full with the aid of definitions (6.29) and (6.30), one has, after rearranging:-

$$
\begin{aligned}
& a_{45}Y_{15} + S_{55}\sin\alpha_{45}X_{15} \\
& + a_{51}\operatorname{cosec}\alpha_{51}(\bar{Z}_1Z_5 - \cos\alpha_{12}\cos\alpha_{45}) \\
& + a_{12}Y_{51} + S_{11}\sin\alpha_{12}X_{51} \\
& - a_{23}\bar{Y}_3 - a_{34}Y_3 - S_{33}\sin\alpha_{34}X_3 \qquad = 0
\end{aligned}
\tag{6.98}
$$

In its present form (6.98) cannot be used since it involves the additional variable θ_5. Nevertheless, it is possible to replace every such term in (6.98) with expressions involving only θ_3, θ_2 and θ_1 without increasing the degree of the equation by means of the following identities and relationships:-

$$Y_{15} \equiv Z_{15}\cot\alpha_{45} - \bar{Z}_1\operatorname{cosec}\alpha_{45} \text{ (see (4.45b))}$$
$$= Z_3\cot\alpha_{45} - \bar{Z}_1\operatorname{cosec}\alpha_{45} \text{ (see (4.59c))} \qquad (6.99)$$

$$Z_5 = Z_{32} \text{ (subsidiary cosine law)} \qquad (6.100)$$

$$Y_{51} \equiv Z_{51}\cot\alpha_{12} - Z_5\operatorname{cosec}\alpha_{12} \text{ (see (4.44b))}$$
$$= \bar{Z}_3\cot\alpha_{12} - Z_{32}\operatorname{cosec}\alpha_{12} \text{ (using (4.59c) and (6.100))} \qquad (6.101)$$

$$X_{51} = X_{32} \text{ (subsidiary sine law)} \qquad (6.102)$$

The remaining unwanted term involving θ_5 contains the expression X_{15} and this presents certain difficulties. However, by using the definition of X_{15} (see Chapter 4), together with the various laws for a spherical pentagon, it is possible to reduce this term to one involving only θ_3, θ_2, and θ_1 in the following sequence of steps:-

$$\sin\alpha_{45}X_{15} = \bar{X}_1(\sin\alpha_{45}\cos\theta_5) - \bar{Y}_1(\sin\alpha_{45}\sin\theta_5)$$
$$= \bar{X}_1Y_{321} - \bar{Y}_1X_{321}$$

$$= \sin\alpha_{12}\sin\theta_1[\cos\alpha_{51}(X_{32}\sin\theta_1 + Y_{32}\cos\theta_1) - \sin\alpha_{51}Z_{32}]$$
$$+ (\cos\alpha_{12}\sin\alpha_{51} + \sin\alpha_{12}\cos\alpha_{51}\cos\theta_1)(X_{32}\cos\theta_1 - Y_{32}\sin\theta_1)$$

$$= -X_{32}Y_1 - (\cos\alpha_{12}Y_{32} + \sin\alpha_{12}Z_{32})X_1$$

$$= -[X_{32}Y_1 + (\bar{X}_3\sin\theta_2 + \bar{Y}_3\cos\theta_2)X_1] \qquad (6.103)$$

Thus, (6.98) may now be rewritten, using equations (6.99), (6.100), (6.101), (6.102) and (6.103), as follows:-

$$
\begin{aligned}
&(a_{51}\operatorname{cosec}\alpha_{51} Z_{32} - a_{45}\operatorname{cosec}\alpha_{45}) Z_1 \\
&- (S_{55}X_{32})Y_1 - S_{55}(\bar{X}_3\sin\theta_2 + \bar{Y}_3\cos\theta_2)X_1 \\
&- a_{12}\operatorname{cosec}\alpha_{12} Z_{32} + S_{11}\sin\alpha_{12} X_{32} \\
&+ (a_{12}\cot\alpha_{12} + a_{45}\cot\alpha_{45}) Z_3 \\
&- a_{23}\bar{Y}_3 - a_{34}Y_3 - S_{33}\sin\alpha_{34} X_3 \\
&- a_{51}\cos\alpha_{12}\cos\alpha_{45}\operatorname{cosec}\alpha_{51} = 0 \qquad (6.104)
\end{aligned}
$$

This can now be written in the form:-

$$g(x_1) = b_2 x_1^2 + b_1 x_1 + b_0 = 0 \qquad (6.105)$$

where:-

$$x_1 \equiv \tan(\theta_1/2)$$

and:-

$$
\begin{aligned}
b_2 = \; & [a_{51}\operatorname{cosec}\alpha_{51}\cos(\alpha_{51} - \alpha_{12}) - a_{12}\operatorname{cosec}\alpha_{12}]Z_{32} \\
& + [S_{11}\sin\alpha_{12} - S_{55}\sin(\alpha_{51} - \alpha_{12})]X_{32} \\
& + [(a_{12}\cot\alpha_{12} + a_{45}\cot\alpha_{45}) Z_3 - a_{23}\bar{Y}_3 - a_{34}Y_3 - S_{33}\sin\alpha_{34} X_3] \\
& - [a_{45}\operatorname{cosec}\alpha_{45}\cos(\alpha_{51} - \alpha_{12}) + a_{51}\operatorname{cosec}\alpha_{51}\cos\alpha_{12}\cos\alpha_{45}] \\
b_1 = \; & - 2.S_{55}\sin\alpha_{51}(\bar{X}_3\sin\theta_2 + \bar{Y}_3\cos\theta_2) \\
b_0 = \; & [a_{51}\operatorname{cosec}\alpha_{51}\cos(\alpha_{51} + \alpha_{12}) - a_{12}\operatorname{cosec}\alpha_{12}]Z_{32} \\
& + [S_{11}\sin\alpha_{12} + S_{55}\sin(\alpha_{51} + \alpha_{12})]X_{32} \\
& + [(a_{12}\cot\alpha_{12} + a_{45}\cot\alpha_{45}) Z_3 - a_{23}\bar{Y}_3 - a_{34}Y_3 - S_{33}\sin\alpha_{34} X_3] \\
& - [a_{45}\operatorname{cosec}\alpha_{45}\cos(\alpha_{51} + \alpha_{12}) + a_{51}\operatorname{cosec}\alpha_{51}\cos\alpha_{12}\cos\alpha_{45}] \qquad (6.106)
\end{aligned}
$$

Clearly equations (6.96) and (6.105) are of degree two in both input (θ_3) and output (θ_2), and hence eliminating x_1 between these two equations by forming the Bézoutian, (5.8), produces an input-output equation for the RCRRC, which is of degree eight in both the input and the output variables. This result is consistent with that obtained by Duffy and Habib-Olahi [13] and with the predictions of Chapter 2.

The remaining variables for the RCRRC mechanism (i.e. $S_2, \theta_1, \theta_5, \theta_4$ and S_4) may now be calculated using similar procedures to those used for the RCRCR mechanism above.

6.9 Description of the Five-Link RRRCC Spatial Mechanism.

The five-link RRRCC spatial mechanism is illustrated by Figure 2.23 since it is an inversion of the RRCCR. It is represented mathematically by the following dual sides and angles:-

$$\begin{aligned}
\hat{\alpha}_{12} &= \alpha_{12} + \epsilon\, a_{12} \\
\hat{\alpha}_{23} &= \alpha_{23} + \epsilon\, a_{23} \\
\hat{\alpha}_{34} &= \alpha_{34} + \epsilon\, a_{34} \\
\hat{\alpha}_{45} &= \alpha_{45} + \epsilon\, a_{45} \\
\hat{\alpha}_{51} &= \alpha_{51} + \epsilon\, a_{51}
\end{aligned} \qquad (6.107)$$

$$\begin{aligned}
\hat{\theta}_1 &= \theta_1 + \epsilon\, S_{11} \\
\hat{\theta}_2 &= \theta_2 + \epsilon\, S_{22} \\
\hat{\theta}_3 &= \theta_3 + \epsilon\, S_3 \\
\hat{\theta}_4 &= \theta_4 + \epsilon\, S_4 \\
\hat{\theta}_5 &= \theta_5 + \epsilon\, S_{55}
\end{aligned} \qquad (6.108)$$

where $\epsilon^2 = 0$, and all fixed mechanism proportions have double or repeated suffices. The input angular displacement is θ_2 whilst the output angular displacement and sliding displacement are respectively θ_3 and S_3. The frame is the constant dual side, $\hat{\alpha}_{23}$.

6.9.1 Derivation of Input-Output Equation for the RRRCC.

The input-output equation for the RRRCC five-link mechanism cannot be derived directly by eliminating, say θ_1, between the primary and secondary parts of the dual cosine law (6.94), since Z_{0321} involves the output sliding variable S_3 in addition to θ_3, θ_2 and θ_1. i.e.:-

$$\hat{Z}_{321} = \cos\hat{\alpha}_{45} \qquad (6.94)$$

and:-

$$
\begin{aligned}
Z_{0321} = \; & a_{51}Y_{321} \\
& + S_{11}\sin\alpha_{51}X_{321} \\
& + a_{12}[(\bar{X}_3\sin\theta_2 + \bar{Y}_3\cos\theta_2)Z_1 + \bar{Z}_3Y_1] \\
& + S_{22}[(\bar{Y}_3Y_1 - \bar{X}_3X_1)\sin\theta_2 - (\bar{Y}_3X_1 + \bar{X}_3Y_1)\cos\theta_2] \\
& + a_{23}[(X_1\sin\theta_2 + Y_1\cos\theta_2)\bar{Z}_3 + Z_1\bar{Y}_3] \\
& + S_3\sin\alpha_{34}X_{123} \\
& + a_{34}Y_{123}
\end{aligned}
\qquad (6.109)
$$

Consequently, an alternative procedure is required. In practice it is not possible to use the primary part of (6.94) as one of the equations from which to eliminate θ_1, and hence it is necessary to use the primary part of the following dual cosine law (a cyclic permutation of the above):-

$$\hat{Z}_{234} = \cos\hat{\alpha}_{51} \qquad (6.110)$$

which is:-

$$Z_{234} = \cos\alpha_{51} \qquad (6.111)$$

and to seek a second equation of the form of (6.111), from which to eliminate θ_4. This has proved to be the correct method.

Thus, making the half-tangent substitution, (5.1), for $x_4 (\equiv \tan(\theta_4/2))$, in equation (6.111), one has:-

$$f(x_4) = a_2x_4^2 + a_1x_4 + a_0 = 0 \qquad (6.112)$$

where:-

$$
\begin{aligned}
a_2 &= \cos\alpha_{45}Z_{23} - \sin\alpha_{45}Y_{23} - \cos\alpha_{51} \\
a_1 &= 2.\sin\alpha_{45}X_{23} \\
a_0 &= \cos\alpha_{45}Z_{23} + \sin\alpha_{45}Y_{23} - \cos\alpha_{51}
\end{aligned}
\qquad (6.113)
$$

and a further such equation is now required.

Now, as outlined previously, it is clear that one must take, as the second equation, the secondary part of the dual cosine law:-

$$\hat{Z}_{215} = \cos\hat{\alpha}_{34} \qquad (6.114)$$

which is:-

$$Z_{0215} = -a_{34}\sin\alpha_{34} \qquad (6.115)$$

where:-

$$\begin{aligned} Z_{0215} = \; & a_{45}Y_{215} \\ & + S_{55}\sin\alpha_{45}X_{215} \\ & + a_{51}[(\bar{X}_2\sin\theta_1 + \bar{Y}_2\cos\theta_1)Z_5 + \bar{Z}_2Y_5] \\ & + S_{11}[(Y_5\bar{Y}_2 - X_5\bar{X}_2)\sin\theta_1 - (Y_5\bar{X}_2 + X_5\bar{Y}_2)\cos\theta_1] \\ & + a_{12}[(X_5\sin\theta_1 + Y_5\cos\theta_1)\bar{Z}_2 + Z_5\bar{Y}_2] \\ & + S_{22}\sin\alpha_{23}X_{512} \\ & + a_{23}Y_{512} \end{aligned} \qquad (6.116)$$

since this contains all the mechanism proportions. However, it is necessary to convert this equation into one involving only θ_2, θ_3 and θ_4 by means of substitutions which do not raise the degree unnecessarily. The most convenient means of achieving this end is to examine the coefficients of each of the terms, a_{45}, S_{55}, a_{51},...etc., of (6.116) in turn, in order to convert these to the required form. Thus the coefficients of a_{45}, S_{55}, S_{22} and a_{23} may be easily rewritten using the following sine and sine-cosine laws for a spherical pentagon (see Chapter 4 and Appendix III.):-

$$\begin{aligned} X_{215} &= \sin\alpha_{34}\sin\theta_4 \\ Y_{215} &= \sin\alpha_{34}\cos\theta_4 \end{aligned} \qquad (6.117)$$

$$\begin{aligned} X_{512} &= \sin\alpha_{34}\sin\theta_3 \\ Y_{512} &= \sin\alpha_{34}\cos\theta_3 \end{aligned} \qquad (6.11'0$$

The remaining three coefficients present a certain amount of difficulty however, and are most conveniently dealt with individually. Thus, using the subsidiary sine-cosine and cosine laws:-

$$\begin{aligned} (\bar{X}_2\sin\theta_1 + \bar{Y}_2\cos\theta_1) &= -Y_{45} \\ \bar{Z}_2 &= Z_{45} \end{aligned} \qquad (6.119)$$

one may rewrite the coefficient of a_{51} in (6.116) as:-

$$(\text{coefficient of } a_{51}) = Z_{45}Y_5 - Y_{45}Z_5 \qquad (6.120)$$

and, expanding and regrouping this expression, one obtains, with the aid of identity (4.10a):-

$$Z_{45}Y_5 - Y_{45}Z_5 = \sin\alpha_{34}(\cos\theta_4\cos\theta_5 - \sin\theta_4\sin\theta_5\cos\alpha_{45}) \qquad (6.121)$$

Similarly, by symmetry one has for the coefficient of a_{12} in (6.116):-

$$\begin{aligned}(\text{coefficient of } a_{12}) &= Z_{32}\bar{Y}_2 - Y_{32}\bar{Z}_2 \\ &= \sin\alpha_{34}(\cos\theta_3\cos\theta_2 - \sin\theta_3\sin\theta_2\cos\alpha_{23}) \qquad (6.122)\end{aligned}$$

Finally, using the subsidiary sine and sine-cosine laws:-

$$\begin{aligned}(X_5\cos\theta_1 - Y_5\sin\theta_1) &= X_{32} \\ (X_5\sin\theta_1 + Y_5\cos\theta_1) &= -Y_{32} \qquad (6.123)\end{aligned}$$

together with identity (4.11a) one may write the coefficient of S_{11} in (6.116) as:-

$$\begin{aligned}(\text{coefficient of } S_{11}) &= Y_{32}\bar{X}_2 - X_{32}\bar{Y}_2 \\ &= \sin\alpha_{34}X_{23} \qquad (6.124)\end{aligned}$$

Hence from (6.117), (6.118), (6.121), (6.122) and (6.124) it is possible to rewrite (6.115) in the following form:-

$$\begin{aligned}\sin\alpha_{34}[&a_{45}\cos\theta_4 + S_{55}\sin\alpha_{45}\sin\theta_4 + a_{51}(\cos\theta_4\cos\theta_5 - \sin\theta_4\sin\theta_5\cos\alpha_{45}) \\ &+ a_{23}\cos\theta_3 + S_{22}\sin\alpha_{23}\sin\theta_3 + a_{12}(\cos\theta_3\cos\theta_2 - \sin\theta_3\sin\theta_2\cos\alpha_{23}) \\ &+ S_{11}X_{23} + a_{34}] = 0 \qquad (6.125)\end{aligned}$$

from which one may cancel $\sin\alpha_{34}$, since, in general, $\alpha_{34} \neq 0$ or π. Now equation (6.125) is almost in the required form and it only remains to remove the $\sin\theta_5$ and $\cos\theta_5$ terms. The latter may be removed by means of the subsidiary cosine law:-

$$Z_5 = Z_{32} \qquad (6.126)$$

since this can be rearranged as:-

$$\cos\theta_5 = (\cos\alpha_{45}\cos\alpha_{51} - Z_{32})\operatorname{cosec}\alpha_{45}\operatorname{cosec}\alpha_{51} \tag{6.127}$$

In addition, the $\sin\theta_5$ term may be replaced using the half-tangent law:-

$$x_4(Y_{23} - \bar{Y}_5) - (X_{23} - \bar{X}_5) = 0 \tag{6.128}$$

which, using the identity:-

$$\begin{aligned}\sin\alpha_{45}\bar{Y}_5 &\equiv \cos\alpha_{45}\bar{Z}_5 - \cos\alpha_{51} \\ &= \cos\alpha_{45}Z_{23} - \cos\alpha_{51}\end{aligned} \tag{6.129}$$

may be rearranged in the form:-

$$-\sin\theta_5 = [x_4(\sin\alpha_{45}Y_{23} - \cos\alpha_{45}Z_{23} + \cos\alpha_{51}) - \sin\alpha_{45}X_{23}]\operatorname{cosec}\alpha_{45}\operatorname{cosec}\alpha_{51} \tag{6.130}$$

Hence substituting into (6.125) for $\sin\theta_5$ and $\cos\theta_5$ obtained from (6.127) and (6.130), gives the following equation in θ_2, θ_3 and θ_4, which is of the desired form:-

$$\begin{aligned}&\cos\theta_4[a_{45} + a_{51}(\cos\alpha_{45}\cos\alpha_{51} - Z_{32})\operatorname{cosec}\alpha_{45}\operatorname{cosec}\alpha_{51}] \\ &+ \sin\theta_4\Big[S_{55}\sin\alpha_{45} + a_{51}\cos\alpha_{45}[x_4(\sin\alpha_{45}Y_{23} - \cos\alpha_{45}Z_{23} + \cos\alpha_{51}) \\ &\qquad - \sin\alpha_{45}X_{23}]\operatorname{cosec}\alpha_{45}\operatorname{cosec}\alpha_{51}\Big] \\ &+ [a_{23}\cos\theta_3 + S_{22}\sin\alpha_{23}\sin\theta_3 + S_{11}X_{23} + a_{34} \\ &\qquad + a_{12}(\cos\theta_3\cos\theta_2 - \sin\theta_3\sin\theta_2\cos\alpha_{23})] = 0\end{aligned} \tag{6.131}$$

Finally, making the substitution (5.1), for $x_4(\equiv \tan(\theta_4/2))$ in (6.131) gives the following quadratic:-

$$g(x_4) = b_2x_4^2 + b_1x_4 + b_0 = 0 \tag{6.132}$$

where:-

$$\begin{aligned}b_2 = \ & a_{23}\cos\theta_3 + S_{22}\sin\alpha_{23}\sin\theta_3 + S_{11}X_{23} + a_{34} \\ &+ a_{12}(\cos\theta_3\cos\theta_2 - \sin\theta_3\sin\theta_2\cos\alpha_{23}) \\ &- a_{45} - a_{51}(\cos\alpha_{45}\cos\alpha_{51} - Z_{32})\operatorname{cosec}\alpha_{45}\operatorname{cosec}\alpha_{51} \\ &+ 2.a_{51}(\sin\alpha_{45}Y_{23} - \cos\alpha_{45}Z_{23} + \cos\alpha_{51})\cot\alpha_{45}\operatorname{cosec}\alpha_{51} \\ b_1 = \ & 2.(S_{55}\sin\alpha_{45} - a_{51}\cos\alpha_{45}\operatorname{cosec}\alpha_{51}X_{23})\end{aligned}$$

$$b_0 = a_{23}\cos\theta_3 + S_{22}\sin\alpha_{23}\sin\theta_3 + S_{11}X_{23} + a_{34}$$
$$+ a_{12}(\cos\theta_3\cos\theta_2 - \sin\theta_3\sin\theta_2\cos\alpha_{23})$$
$$+ a_{45} + a_{51}(\cos\alpha_{45}\cos\alpha_{51} - Z_{32})\operatorname{cosec}\alpha_{45}\operatorname{cosec}\alpha_{51} \qquad (6.133)$$

Clearly equations (6.112) and (6.132) are of degree two in both input (θ_2) and output (θ_3), and hence eliminating x_4 between these two equations by forming the Bézoutian, (5.8), produces an input-output equation for the RRRCC, which is of degree eight in both the input and the output variables. This result is in agreement with the predictions of Chapter 2 and is an improvement on that produced algebraically by Habib-Olahi [19]. The latter obtained an input-output equation for this mechanism which was of degree twelve in the input angular displacement and of degree eight in the output angular displacement.

The determination of the remaining variables for the RRRCC mechanism (i.e. S_3, θ_1, θ_5, θ_4, S_4) is now a relatively simple matter using similar procedures to those used for the RRCCR mechanism above.

CHAPTER 7

A DISPLACEMENT ANALYSIS

OF THE

SPATIAL SIX-LINK RCRPRR MECHANISM

7.1 Introduction.

There exist three distinct 4R-P-C mechanisms, which differ in the separation of their cylindric and prismatic pairs and they may be listed as follows together with their inversions:-

Mechanism.	Inversions.
RCRPRR	RPRCRR, RPRRRC, RRRPRC, RRRCRP, RCRRRP.
RCRRPR	RRCRRP, RRPRRC.
RRRPCR	RRRCPR, RRPCRR, RRRRCP, RRRRPC.

In this and the following two chapters, novel eighth degree polynomials are derived for the input-output relationships of the RCRPRR, RCRRPR, and RRRPCR mechanisms respectively. In addition, these equations collectively contain, as special cases, the input-output displacement equations for all spatial five-link 3R-2C mechanisms (see Chapter 6.), since a combination of revolute and prismatic pairs ($\overline{RP}$ or $\overline{PR}$) may be used to simulate a cylindric pair. Thus, for example, the input-output equation for the RCRPRR six-link mechanism reduces to the input-output equations for the five-link RCRCR (RCR$\overline{PR}$R) and the RCCRR (RC$\overline{RP}$RR) mechanisms, and similar reductions are possible for the other two six-link mechanisms considered here. Further reduction and specialization of these six-link mechanisms yields solutions for the spatial four-link RPSC (RP$\overline{RRR}$C), RCSP (RC$\overline{RRR}$P), RSCP (R$\overline{RRR}$CP) and RSPC (R$\overline{RRR}$PC) mechanisms. (Three revolute pairs may be suitably combined to simulate a spherical pair, S).

In this chapter, a degree eight input-output displacement equation is derived for the spatial six-link RCRPRR mechanism. Following this, a procedure is presented which determines uniquely the remaining linkage variables, the method of analysis verifying the closures. This result implies that the RPRCRR six-link mechanism also has a degree eight input-output equation since it is an inversion of the RCRPRR mechanism and both of these reduce to the same basic five-link RRCRP structure (which has eight assembly configurations) for a given fixed value of their respective input angular displacements. This is an improvement on the degree sixteen input-output displacement equation obtained by Yuan [49] recently for the spatial six-link RPRCRR mechanism and, indeed, Yuan concluded in his paper that his equation must contain extraneous roots since a number of the roots of the degree sixteen equation did not satisfy the closure conditions for the mechanism.

In Chapters 8 and 9, degree eight input-output displacement equations are derived in detail for the spatial six-link RCRRPR and RRRPCR mechanisms respectively, and procedures are presented for determining the remaining linkage variables in each case. Although the basic problem (i.e. the elimination of a single extraneous unknown between two equations) is common to all three 4R-P-C mechanisms, the derivations of their input-output displacement equations differ significantly from one another and, therefore, warrant special attention. In this respect the most difficult result to obtain was the degree eight equation for the spatial six-link RRRPCR analysed in Chapter 9. In each case, the closures were checked numerically, which afforded a verification of the accuracy of the results.

7.2 Description of the Six-Link RCRPRR Mechanism.

The six-link RCRPRR spatial mechanism is illustrated by Figure 2.25 and is represented mathematically by the following six dual sides and six dual angles:-

$$\hat{\alpha}_{12} = \alpha_{12} + \epsilon a_{12}$$
$$\hat{\alpha}_{23} = \alpha_{23} + \epsilon a_{23}$$
$$\hat{\alpha}_{34} = \alpha_{34} + \epsilon a_{34}$$
$$\hat{\alpha}_{45} = \alpha_{45} + \epsilon a_{45}$$
$$\hat{\alpha}_{56} = \alpha_{56} + \epsilon a_{56}$$
$$\hat{\alpha}_{61} = \alpha_{61} + \epsilon a_{61} \qquad (7.1)$$

$$\hat{\theta}_1 = \theta_1 + \varepsilon S_{11}$$
$$\hat{\theta}_2 = \theta_2 + \varepsilon S_2$$
$$\hat{\theta}_3 = \theta_3 + \varepsilon S_{33}$$
$$\hat{\theta}_4 = \theta_{44} + \varepsilon S_4$$
$$\hat{\theta}_5 = \theta_5 + \varepsilon S_{55}$$
$$\hat{\theta}_6 = \theta_6 + \varepsilon S_{66} \qquad (7.2)$$

where $\varepsilon^2 = 0$, and all fixed mechanism proportions have double or repeated suffices. The input and output angular displacements are respectively θ_1 and θ_6, and the frame is considered to be the constant dual side, $\hat{\alpha}_{61}$.

A relationship between the input and output angular displacements only, of the lowest possible degree in both, is required.

7.3 Derivation of Input-Output Equation for the RCRPRR Mechanism.

As with the five-link 3R-2C mechanisms analysed in Chapter 6., it is possible to write appropriate primary equations for the 4R-P-C mechanisms, which contain the input, output and a single extraneous angular displacement. The problem is then to derive a second equation of the same form, which contains all the fixed mechanism proportions, in order to eliminate the extraneous angular displacement and obtain the input-output equation.

Hence the strategy in all three cases may be outlined in three steps as follows:-

(i) Transform the relevant primary equation into the most suitable form for eliminating the extraneous angular displacement.

(ii) Derive the required second equation from that secondary equation which involves all the fixed mechanism parameters.

(iii) Perform the elimination procedure using the Bézoutian, (5.8), to eliminate the single extraneous angular displacement.

Using this procedure one may now analyse the RCRPRR mechanism.

7.3.1 First Equation in θ_1, θ_6 and θ_5.

For the RCRPRR mechanism, the primary part of the dual cosine law:-

$$\hat{Z}_{1654} = \cos\hat{\alpha}_{23} \qquad (7.3)$$

which is:-

$$Z_{1654} = \cos\alpha_{23} \tag{7.4}$$

where:-

$$Z_{1654} = \sin\alpha_{34}(X_{165}\sin\theta_{44} + Y_{165}\cos\theta_{44}) + \cos\alpha_{34}Z_{165} \tag{7.5}$$

involves only the input angle (θ_1), output angle (θ_6) and a single extraneous angular variable (θ_5), since the angle, θ_{44}, is a constant mechanism proportion. Hence, one can rewrite (7.4) in the form:-

$$L_{416}\sin\theta_5 + M_{416}\cos\theta_5 + N_{416} = \cos\alpha_{23} \tag{7.6}$$

where:-

$$\begin{aligned} L_{416} &= -(Y_4X_{16} + X_4Y_{16}) \\ M_{416} &= (X_4X_{16} - Y_4Y_{16}) \\ N_{416} &= Z_4Z_{16} \end{aligned} \tag{7.7}$$

The terms X_{16}, Y_{16} and Z_{16} of equation (7.7) are defined in Appendix III., whilst X_4, Y_4 and Z_4 are given by:-

$$\begin{aligned} X_4 &= \sin\alpha_{34}\sin\theta_{44} \\ Y_4 &= -(\cos\alpha_{34}\sin\alpha_{45} + \sin\alpha_{34}\cos\alpha_{45}\cos\theta_{44}) \\ Z_4 &= (\cos\alpha_{34}\cos\alpha_{45} - \sin\alpha_{34}\sin\alpha_{45}\cos\theta_{44}) \end{aligned} \tag{7.8}$$

It is convenient to transform (7.6) into a form suitable for the elimination of θ_5 by making the substitutions (5.1) and rearranging terms. Thus:-

$$\begin{aligned} \sin\theta_5 &\equiv 2x_5/(1 + x_5^2) \\ \cos\theta_5 &\equiv (1 - x_5^2)/(1 + x_5^2) \\ \text{where} \quad x_5 &\equiv \tan(\theta_5/2) \end{aligned} \tag{5.1}$$

and (7.6) becomes:-

$$f(x_5) = a_2x_5^2 + a_1x_5 + a_0 = 0 \tag{7.9}$$

where:-

$$a_2 = N_{416} - M_{416} - \cos\alpha_{23}$$
$$a_1 = 2.L_{416}$$
$$a_0 = N_{416} + M_{416} - \cos\alpha_{23} \qquad (7.10)$$

7.3.2 Second Equation in θ_1, θ_6 and θ_5.

In accordance with the procedure outlined above, the secondary part of the subsidiary dual cosine law (see Chapter 4.):-

$$\hat{Z}_{165} = \hat{Z}_3 \qquad (7.11)$$

is transformed into an equation involving only θ_1, θ_6 and θ_5 as variables. Thus the secondary part of (7.11) is:-

$$Z_{0165} = Z_{03} \qquad (7.12)$$

and it is convenient to write Z_{0165} and Z_{03} in the following forms (see Appendix III):-

$$\begin{aligned} Z_{0165} = \; & a_{45}[\cos\alpha_{45}(X_{16}\sin\theta_5 + Y_{16}\cos\theta_5) - \sin\alpha_{45} Z_{16}] \\ & + S_{55}\sin\alpha_{45}(X_{16}\cos\theta_5 - Y_{16}\sin\theta_5) \\ & + [\sin\alpha_{45}(X_{016}\sin\theta_5 + Y_{016}\cos\theta_5) + \cos\alpha_{45} Z_{016}] \end{aligned} \qquad (7.13)$$

and

$$\begin{aligned} Z_{03} = \; & a_{34}Y_3 \\ & + S_{33}\sin\alpha_{34}X_3 \\ & + a_{23}\bar{Y}_3 \end{aligned} \qquad (7.14)$$

Now, using the following subsidiary sine, sine-cosine and cosine laws for a spherical hexagon:-

$$X_3 = (X_{165}\cos\theta_{44} - Y_{165}\sin\theta_{44})$$
$$-Y_3 = (X_{165}\sin\theta_{44} + Y_{165}\cos\theta_{44})$$
$$Z_3 = Z_{165} \qquad (7.15)$$

together with the identity (see equation (4.11b)):-

$$\bar{Y}_3 \equiv \cot\alpha_{23}\bar{Z}_3 - \cos\alpha_{34}\,\mathrm{cosec}\,\alpha_{23}$$
$$= \cot\alpha_{23} Z_{165} - \cos\alpha_{34}\,\mathrm{cosec}\,\alpha_{23} \qquad (7.16)$$

one may rewrite (7.12) as follows:-

$$P_{416}\sin\theta_5 + Q_{416}\cos\theta_5 + R_{416} = -a_{23}\cos\alpha_{34}\,\mathrm{cosec}\,\alpha_{23} \qquad (7.17)$$

where:-

$$\begin{aligned} P_{416} = \; & \big[\cos\alpha_{45}(S_{33}\sin\alpha_{34}\sin\theta_{44} + a_{34}\cos\theta_{44} + a_{45}) \\ & \quad - a_{23}\sin\alpha_{45}\cot\alpha_{23}\big]X_{16} \\ & + (S_{33}\sin\alpha_{34}\cos\theta_{44} - a_{34}\sin\theta_{44} - S_{55}\sin\alpha_{45})Y_{16} \\ & + \sin\alpha_{45}X_{016} \end{aligned}$$

$$\begin{aligned} Q_{416} = \; & \big[\cos\alpha_{45}(S_{33}\sin\alpha_{34}\sin\theta_{44} + a_{34}\cos\theta_{44} + a_{45}) \\ & \quad - a_{23}\sin\alpha_{45}\cot\alpha_{23}\big]Y_{16} \\ & - (S_{33}\sin\alpha_{34}\cos\theta_{44} - a_{34}\sin\theta_{44} - S_{55}\sin\alpha_{45})X_{16} \\ & + \sin\alpha_{45}Y_{016} \end{aligned}$$

$$\begin{aligned} R_{416} = - & \big[\sin\alpha_{45}(S_{33}\sin\alpha_{34}\sin\theta_{44} + a_{34}\cos\theta_{44} + a_{45}) \\ & \quad + a_{23}\cos\alpha_{45}\cot\alpha_{23}\big]Z_{16} \\ & + \cos\alpha_{45}Z_{016} \end{aligned} \qquad (7.18)$$

The terms X_{16}, Y_{16} and Z_{16} in the above equations are defined in Appendix III., whilst X_{016}, Y_{016} and Z_{016} are given in the most convenient form by:-

$$\begin{aligned} X_{016} = \; & (\bar{X}_{01}\cos\theta_6 - \bar{Y}_{01}\sin\theta_6) \\ & - S_{66}(\bar{X}_1\sin\theta_6 + \bar{Y}_1\cos\theta_6) \end{aligned}$$

$$\begin{aligned} Y_{016} = \; & \big[\cos\alpha_{56}(\bar{X}_{01}\sin\theta_6 + \bar{Y}_{01}\cos\theta_6) - \sin\alpha_{56}\bar{Z}_{01}\big] \\ & + S_{66}(\bar{X}_1\cos\theta_6 - \bar{Y}_1\sin\theta_6)\cos\alpha_{56} \\ & - a_{56}\big[\sin\alpha_{56}(\bar{X}_1\sin\theta_6 + \bar{Y}_1\cos\theta_6) + \cos\alpha_{56}\bar{Z}_1\big] \end{aligned}$$

$$\begin{aligned} Z_{016} = \; & \big[\sin\alpha_{56}(\bar{X}_{01}\sin\theta_6 + \bar{Y}_{01}\cos\theta_6) + \cos\alpha_{56}\bar{Z}_{01}\big] \\ & + S_{66}(\bar{X}_1\cos\theta_6 - \bar{Y}_1\sin\theta_6)\sin\alpha_{56} \\ & + a_{56}\big[\cos\alpha_{56}(\bar{X}_1\sin\theta_6 + \bar{Y}_1\cos\theta_6) - \sin\alpha_{56}\bar{Z}_1\big] \end{aligned} \qquad (7.19)$$

where:-

$$\bar{X}_{01} = a_{12}\cos\alpha_{12}\sin\theta_1 + S_{11}\sin\alpha_{12}\cos\theta_1$$
$$\bar{Y}_{01} = a_{12}(\sin\alpha_{12}\sin\alpha_{61} - \cos\alpha_{12}\cos\alpha_{61}\cos\theta_1)$$
$$- a_{61}(\cos\alpha_{12}\cos\alpha_{61} - \sin\alpha_{12}\sin\alpha_{61}\cos\theta_1)$$
$$+ S_{11}\cos\alpha_{61}\sin\alpha_{12}\sin\theta_1$$
$$\bar{Z}_{01} = - a_{12}(\cos\alpha_{61}\sin\alpha_{12} + \sin\alpha_{61}\cos\alpha_{12}\cos\theta_1)$$
$$- a_{61}(\sin\alpha_{61}\cos\alpha_{12} + \cos\alpha_{61}\sin\alpha_{12}\cos\theta_1)$$
$$+ S_{11}\sin\alpha_{61}\sin\alpha_{12}\sin\theta_1 \qquad (7.20)$$

It is now possible to make the substitutions (5.1) in equation (7.17) and obtain, upon rearranging, the following equation which is in the most suitable form for eliminating x_5:-

$$g(x_5) = b_2x_5^2 + b_1x_5 + b_0 = 0 \qquad (7.21)$$

where:-

$$b_2 = R_{416} - Q_{416} + a_{23}\cos\alpha_{34}\operatorname{cosec}\alpha_{23}$$
$$b_1 = 2.P_{416}$$
$$b_0 = R_{416} + Q_{416} + a_{23}\cos\alpha_{34}\operatorname{cosec}\alpha_{23} \qquad (7.22)$$

7.3.3 Elimination Procedure.

It is now possible to eliminate x_5 between equations (7.9) and (7.21), using the Bézoutian, (5.8), for two quadratics, and obtain as the eliminant the input-output equation for the RCRPRR mechanism in the form:-

$$(a_2b_0 - a_0b_2)^2 - (a_2b_1 - a_1b_2)(a_1b_0 - a_0b_1) = 0 \qquad (7.23)$$

where a_2, a_1, a_0 and b_2, b_1, b_0 are defined by (7.10) and (7.22) respectively. Clearly these coefficients may be expressed in terms of the half-tangent of the output angular displacement (θ_6), by means of the substitution (5.1) as follows:-

$$a_2 = P_{22}x_6^2 + P_{12}x_6 + P_{02}$$
$$a_1 = P_{21}x_6^2 + P_{11}x_6 + P_{01}$$
$$a_0 = P_{20}x_6^2 + P_{10}x_6 + P_{00} \qquad (7.24)$$

and:-

$$b_2 = q_{22}x_6^2 + q_{12}x_6 + q_{02}$$
$$b_1 = q_{21}x_6^2 + q_{11}x_6 + q_{01}$$
$$b_0 = q_{20}x_6^2 + q_{10}x_6 + q_{00} \qquad (7.25)$$

where the terms p_{ij} and q_{ij} are each a function of the input angular displacement (θ_1) only, and are listed in Appendix V.

Since all of the terms a_2, a_1, a_0 and b_2, b_1, b_0 are quadratic expressions in x_6, it is clear from (7.23) that the input-output displacement equation for the RCRPRR mechanism is of degree eight in the half-tangent of the output angular displacement. In addition the coefficients a_2, a_1, a_0 and b_2, b_1, b_0 may be expressed alternatively as quadratics in the half-tangent of the input variable, x_1, and so (7.23) is also of degree eight in x_1. These results are in agreement with the predicted degree for this six-link mechanism (see Chapter 2).

7.4 Displacement Analysis.

Solving the input-output equation (7.23) for x_6, one obtains, in general, eight distinct real values for the output angular displacement (i.e. θ_{61}, θ_{62}, θ_{63}, θ_{64}, θ_{65}, θ_{66}, θ_{67}, θ_{68}), for each value of the input angular displacement, θ_1. The eight ordered pairs (θ_1, θ_{61}), (θ_1, θ_{62}), (θ_1, θ_{63}),.....,(θ_1, θ_{68}) thus produced will then each give rise to corresponding values for the remaining linkage variables (θ_5, S_4, θ_3, θ_2, S_2) using procedures outlined below.

Thus θ_5 may be determined from either of the two expressions for the common root of (7.9) and (7.21), which are derived from the Bézoutian (5.8) (see Chapter 5.). These expressions are:-

$$x_5 = -(a_2b_0)/(a_2b_1)$$
$$= -(a_2b_0 - a_0b_2)/(a_2b_1 - a_1b_2) \qquad (7.26a)$$

or

$$x_5 = -(a_1b_0)/(a_2b_0)$$
$$= -(a_1b_0 - a_0b_1)/(a_2b_0 - a_0b_2) \qquad (7.26b)$$

where:-

$$x_5 \equiv \tan(\theta_5/2)$$

Having determined corresponding values for θ_1, θ_6 and θ_5 it is then a relatively simple matter to obtain the unique value of θ_3 from either of the two fundamental half-tangent laws (see Appendix IV):-

$$x_3 = -(Y_{1654} - \sin\alpha_{23})/X_{1654} \tag{7.27a}$$

or $$x_3 = X_{1654}/(Y_{1654} + \sin\alpha_{23}) \tag{7.27b}$$

where $x_3 \equiv \tan(\theta_3/2)$

which are cyclic permutations of (5.32) and (5.33).

Here:-

$$\begin{aligned} X_{1654} &= X_{165}\cos\theta_{44} - Y_{165}\sin\theta_{44} \\ Y_{1654} &= \cos\alpha_{34}(X_{165}\sin\theta_{44} + Y_{165}\cos\theta_{44}) - \sin\alpha_{34}Z_{165} \end{aligned} \tag{7.28}$$

whilst X_{165}, Y_{165} and Z_{165} are given in Appendix III.

In a similar manner one may obtain the value of θ_2 from a cyclic permutation of (7.27a,b) once θ_1, θ_6 and θ_5 are known. Thus:-

$$x_2 = -(Y_{4561} - \sin\alpha_{23})/X_{4561} \tag{7.29a}$$

or $$x_2 = X_{4561}/(Y_{4561} + \sin\alpha_{23}) \tag{7.29b}$$

where $x_2 \equiv \tan(\theta_2/2)$

and:-

$$\begin{aligned} X_{4561} &= X_{456}\cos\theta_1 - Y_{456}\sin\theta_1 \\ Y_{4561} &= \cos\alpha_{12}(X_{456}\sin\theta_1 + Y_{456}\cos\theta_1) - \sin\alpha_{12}Z_{456} \end{aligned} \tag{7.30}$$

The sliding displacement S_4 may be determined from the secondary component of the dual subsidiary cosine law:-

$$\hat{Z}_{16} = \hat{Z}_{34} \tag{7.31}$$

which is:-

$$Z_{016} = Z_{034} \tag{7.32}$$

where Z_{016} and Z_{034} are defined as follows, in their symmetric form:-

$$
\begin{aligned}
Z_{016} = \;& a_{56}Y_{16} \\
&+ S_{66}\sin\alpha_{56}X_{16} \\
&+ a_{61}\operatorname{cosec}\alpha_{61}(\bar{Z}_1Z_6 - \cos\alpha_{12}\cos\alpha_{56}) \\
&+ S_{11}\sin\alpha_{12}X_{61} \\
&+ a_{12}Y_{61}
\end{aligned} \tag{7.33}
$$

and:-

$$
\begin{aligned}
Z_{034} = \;& a_{45}Y_{34} \\
&+ S_4\sin\alpha_{45}X_{34} \\
&+ a_{34}\operatorname{cosec}\alpha_{34}(Z_3\bar{Z}_4 - \cos\alpha_{23}\cos\alpha_{45}) \\
&+ S_{33}\sin\alpha_{23}X_{43} \\
&+ a_{23}Y_{43}
\end{aligned} \tag{7.34}
$$

The terms X_{16}, Y_{16}, X_{61}, Y_{61}, X_{34}, Y_{34}, X_{43}, Y_{43}, $\bar{Z}_1$, Z_6, Z_3 and $\bar{Z}_4$ are again defined in Appendix III. and are each uniquely specified for a given set of θ_1, θ_6, θ_3 and θ_{44}.

In a similar manner the sliding displacement S_2 may be calculated from the secondary component of the dual cosine law:-

$$\hat{Z}_{6123} = \cos\hat{\alpha}_{45} \tag{7.35}$$

which is:-

$$Z_{06123} = -\, a_{45}\sin\alpha_{45} \tag{7.36}$$

where:-

$$
\begin{aligned}
Z_{06123} = \;& a_{34}Y_{6123} \\
&+ S_{33}\sin\alpha_{34}X_{6123} \\
&+ a_{23}[(X_{61}\sin\theta_2 + Y_{61}\cos\theta_2)\bar{Z}_3 + Z_{61}\bar{Y}_3] \\
&- S_2[(X_{61}\sin\theta_2 + Y_{61}\cos\theta_2)\bar{X}_3 + X_{612}\bar{Y}_3] \\
&+ a_{12}\operatorname{cosec}\alpha_{12}[Z_{61}Z_{32} - Z_6\bar{Z}_3] \\
&- S_{11}[(X_{32}\sin\theta_1 + Y_{32}\cos\theta_1)X_6 + X_{321}Y_6] \\
&+ a_{61}[(X_{32}\sin\theta_1 + Y_{32}\cos\theta_1)Z_6 + Z_{32}Y_6] \\
&+ S_{66}\sin\alpha_{56}X_{3216} \\
&+ a_{56}Y_{3216}
\end{aligned} \tag{7.37}
$$

and the terms X_{6123}, Y_{6123},....etc., in these equations are defined in Appendix III.

7.5 Numerical Results.

A computer program was developed by the author to solve the input-output equation for the RCRPRR mechanism (see equation (7.23)) and to obtain the remaining linkage variables for a given set of mechanism proportions. With the aid of this program, graphs of the output angle, θ_6, and remaining variables θ_5, θ_3, θ_2, S_4 and S_2 against the input, θ_1, were plotted (see Figures 7.3, 7.4, 7.5, 7.6, 7.7 and 7.8 respectively).

In addition, since a combination of revolute and prismatic pairs may be used to simulate a cylindric pair, the input-output equation (7.23) for the RCRPRR mechanism may be used to generate input-output relationships for five-link RCCRR or RCRCR spatial mechanisms. Figures 7.1 and 7.2. show plots for these two mechanisms.

The following sets of data for the mechanism proportions were chosen in each case:-

7.5.1 RCCRR Mechanism.

a_{12} = 2.0 ins.	α_{12} = 60 deg.	S_{11} = 3.0 ins.	
a_{23} = 2.5 ins.	α_{23} = 45 deg.	S_{33} = 0.0 ins.	
a_{34} = 0.0 ins.	α_{34} = 0 deg.	S_{55} = -6.0 ins.	
a_{45} = 3.5 ins.	α_{45} = 45 deg.	S_{66} = 1.0 ins.	
a_{56} = 1.0 ins.	α_{56} = 30 deg.		
a_{61} = 3.0 ins.	α_{61} = 20 deg.	θ_{44} = 0 deg.	(7.38)

Here the third revolute pair has been superimposed on the fourth sliding pair by selecting the proportions, $a_{34} = \alpha_{34} = S_{33} = \theta_{44} = 0$. The remaining proportions were selected to give the same RCCRR mechanism as that previously analysed by Yuan [48]. Figure 7.1 is identical to the input-output relationship presented in the latter reference.

7.5.2 RCRCR Mechanism.

a_{12} = 2.5 ins.	α_{12} = 60 deg.	S_{11} = 3.0 ins.
a_{23} = 3.0 ins.	α_{23} = 45 deg.	S_{33} = 2.5 ins.
a_{34} = 4.0 ins.	α_{34} = 35 deg.	S_{55} = 0.0 ins.
a_{45} = 0.0 ins.	α_{45} = 0 deg.	S_{66} = 0.0 ins.
a_{56} = 1.0 ins.	α_{56} = 30 deg.	
a_{61} = 3.2 ins.	α_{61} = 10 deg.	θ_{44} = 0 deg. (7.39)

Here the fourth prismatic pair has been superimposed on the fifth revolute pair by selecting the proportions, $a_{45} = \alpha_{45} = S_{55} = \theta_{44} = 0$. The remaining proportions were selected to give the same RCRCR mechanism as that previously analysed in [11]. Figure 7.2 is identical to the input-output relationship presented in [11].

7.5.3 RCRPRR Mechanism.

a_{12} = 3.0 ins.	α_{12} = 73 deg.	S_{11} = -2.4 ins.
a_{23} = 1.6 ins.	α_{23} = 264 deg.	S_{33} = 6.4 ins.
a_{34} = 2.0 ins.	α_{34} = 175 deg.	S_{55} = -5.4 ins.
a_{45} = 2.0 ins.	α_{45} = 100 deg.	S_{66} = 3.8 ins.
a_{56} = 2.7 ins.	α_{56} = 286 deg.	
a_{61} = 1.4 ins.	α_{61} = 242 deg.	θ_{44} = 30 deg. (7.40)

These proportions were chosen to yield eight real closures for various ranges of the input angular displacement since they are similar to a set used by Habib-Olahi [19] for a spatial five-link RCCRR mechanism which possessed eight such real closures. The direct correspondence of the results, illustrated in Figure 7.3 for the two mechanisms (RCCRR-broken lines, RCRPRR-continuous lines), is a consequence of the fact that it is possible to replace the third cylindric pair of the RCCRR with an $\overline{RP}$ combination of joints by replacing a side with a spatial triangle (apex angle θ_{44}), thereby converting the five-link mechanism into a six-link mechanism. By careful choice of the proportions, the two mechanisms will both have the same number of closures and similar input-output relationships.

Finally, it must be noted that for ease in identifying the different circuits, the turning points of the various curves (twelve in number, for this set of proportions) have been labelled 1-12, as shown in Figures 7.3-7.8 inclusive.

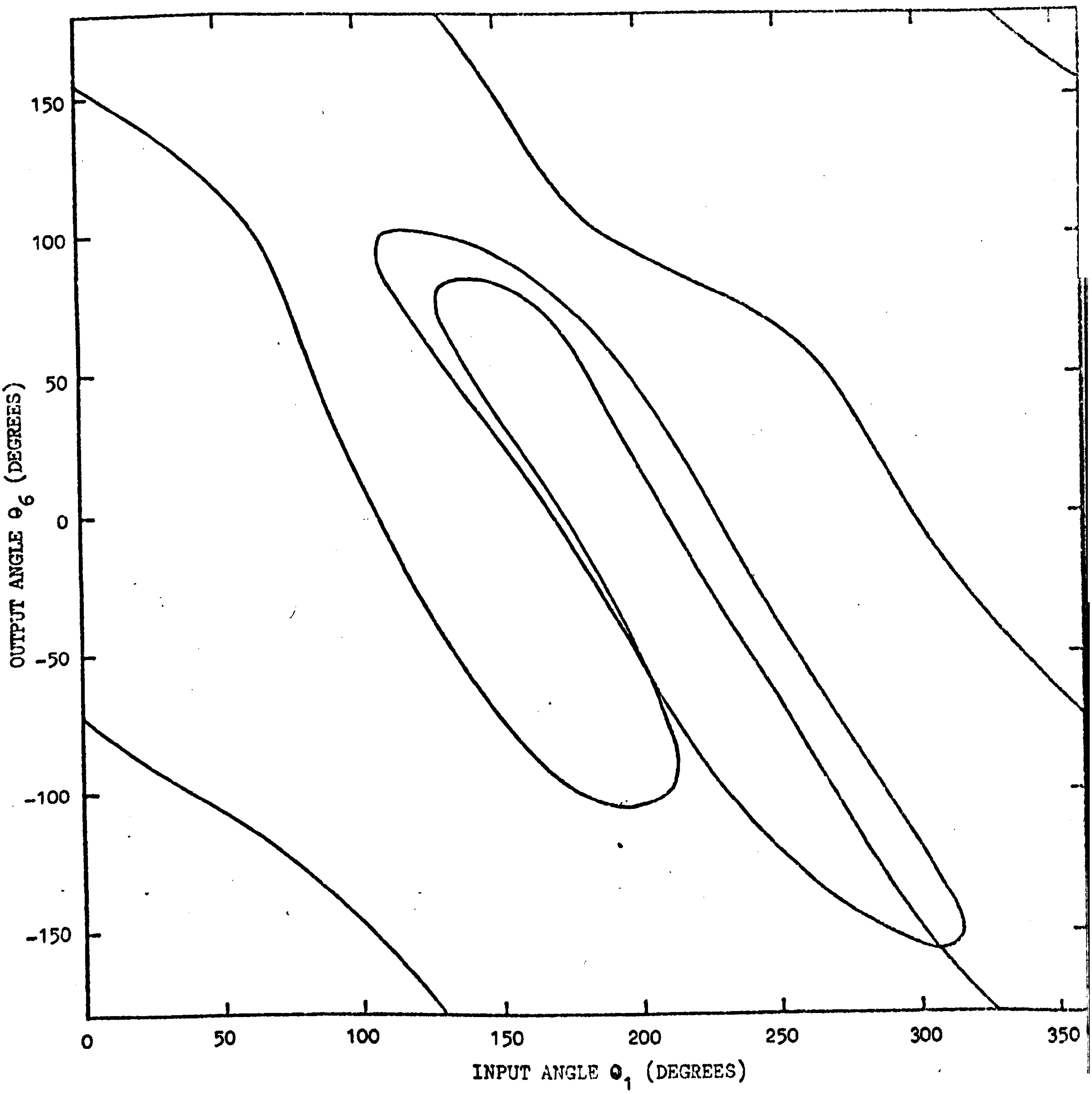


Figure 7.1 Graph of Input-Output Relationship (i.e. θ_6 vs θ_1) for the Six-Link RCRPRR Mechanism with Proportions chosen to Reduce the Latter to the Five-Link RCCRR Mechanism.

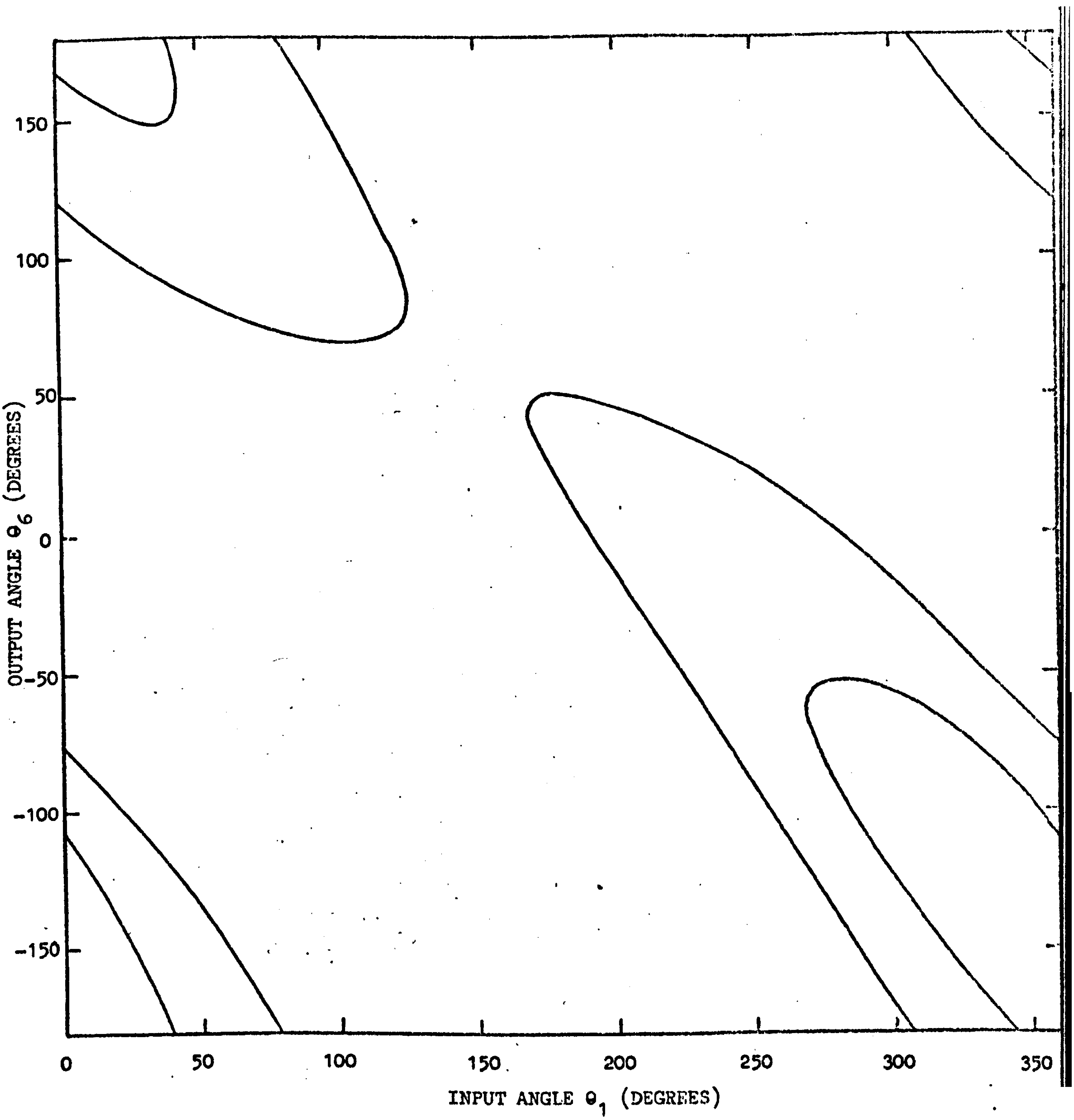


Figure 7.2 Graph of Input-Output Relationship (i.e. θ_6 vs θ_1) for the Six-Link RCRPRR Mechanism with Proportions chosen to Reduce the Latter to the Five-Link RCRCR Mechanism.

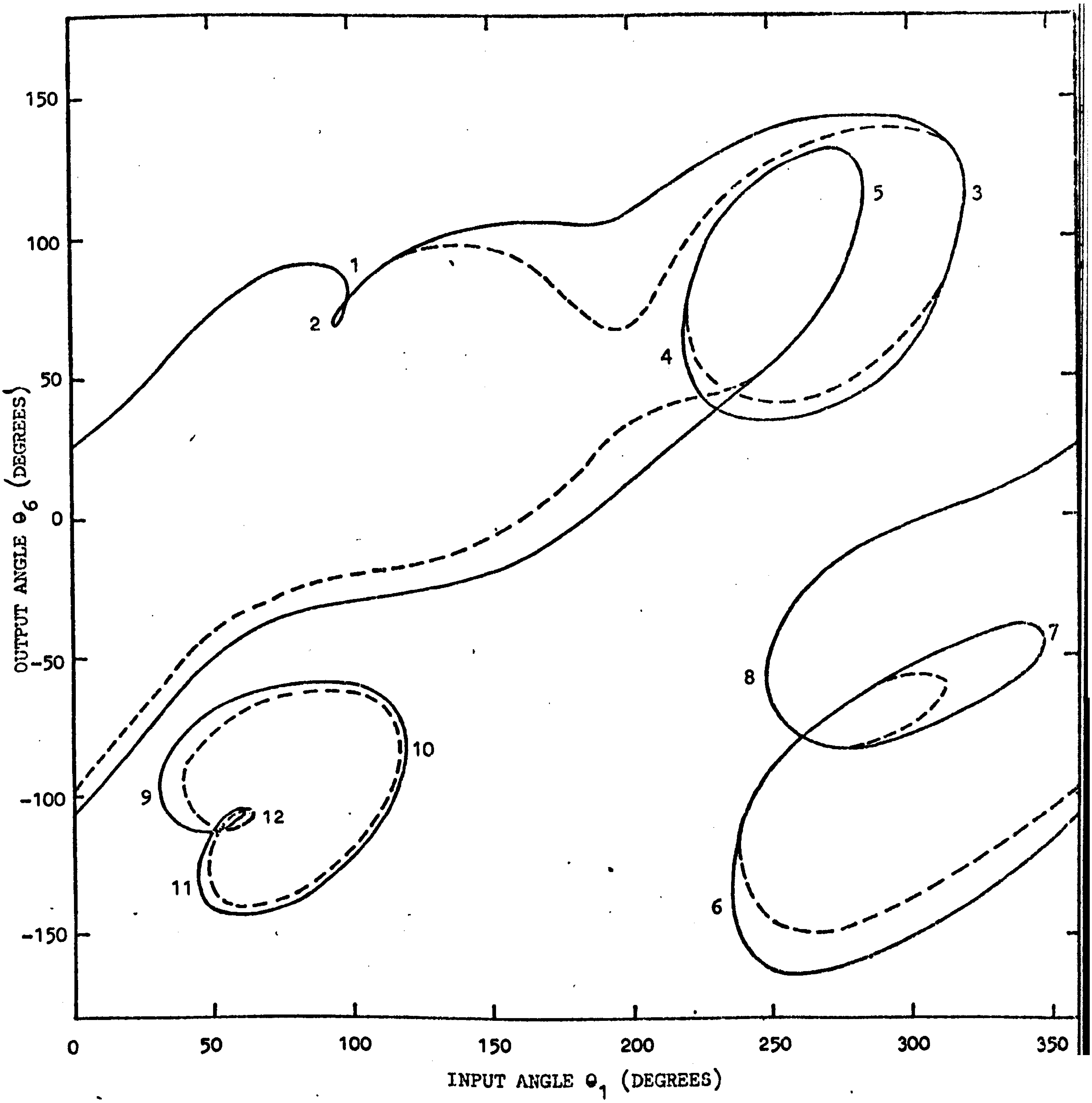


Figure 7.3 Graph of Input-Output Relationship (i.e. θ_6 vs θ_1) for the Six-Link RCRPRR Mechanism. (The Broken Curves Represent the Input-Output Relationship of a Five-Link RCCRR Mechanism with Similar Proportions).

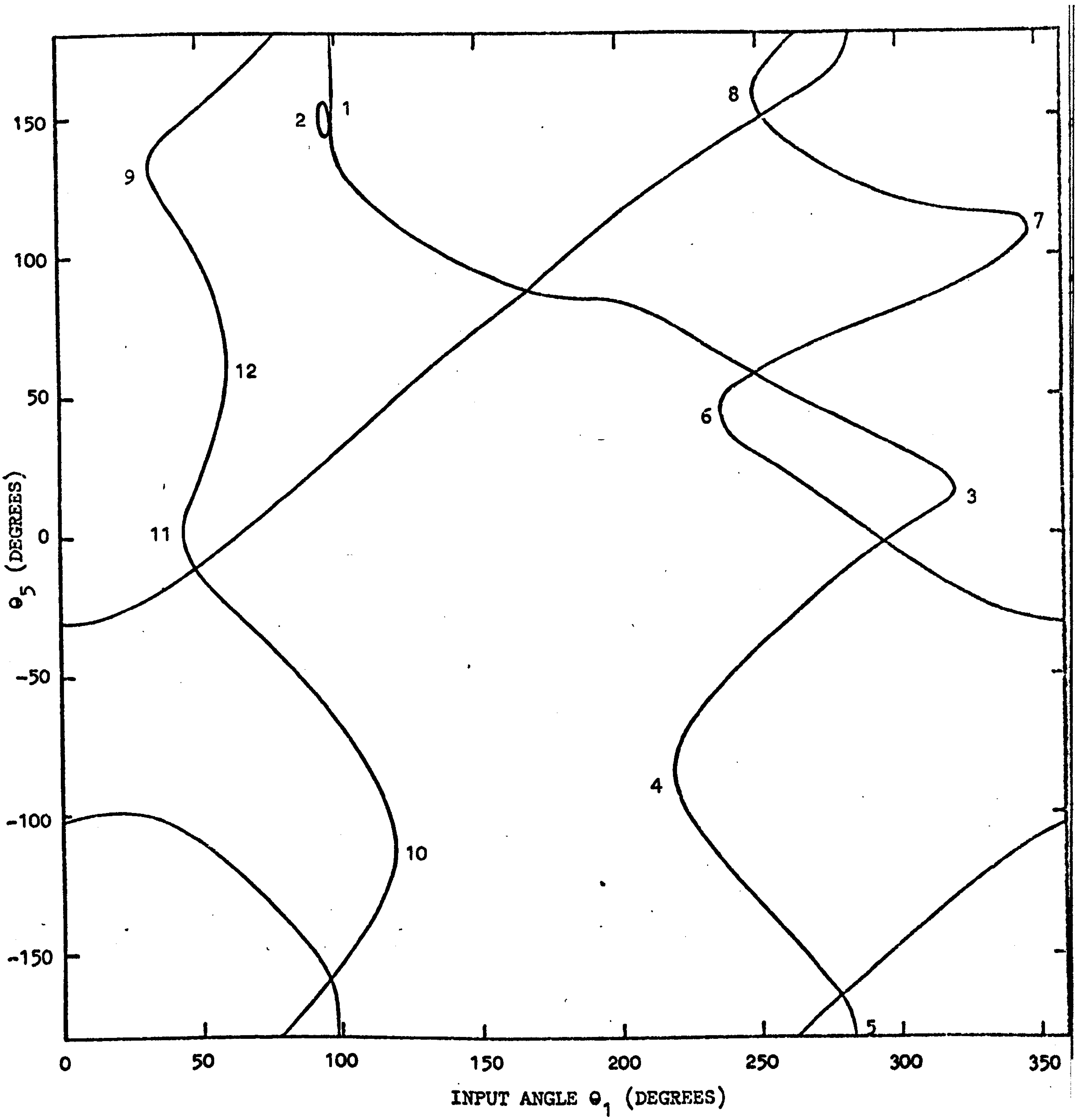


Figure 7.4 Graph of Θ_5 vs Θ_1 for the Six-Link RCRPRR Mechanism.

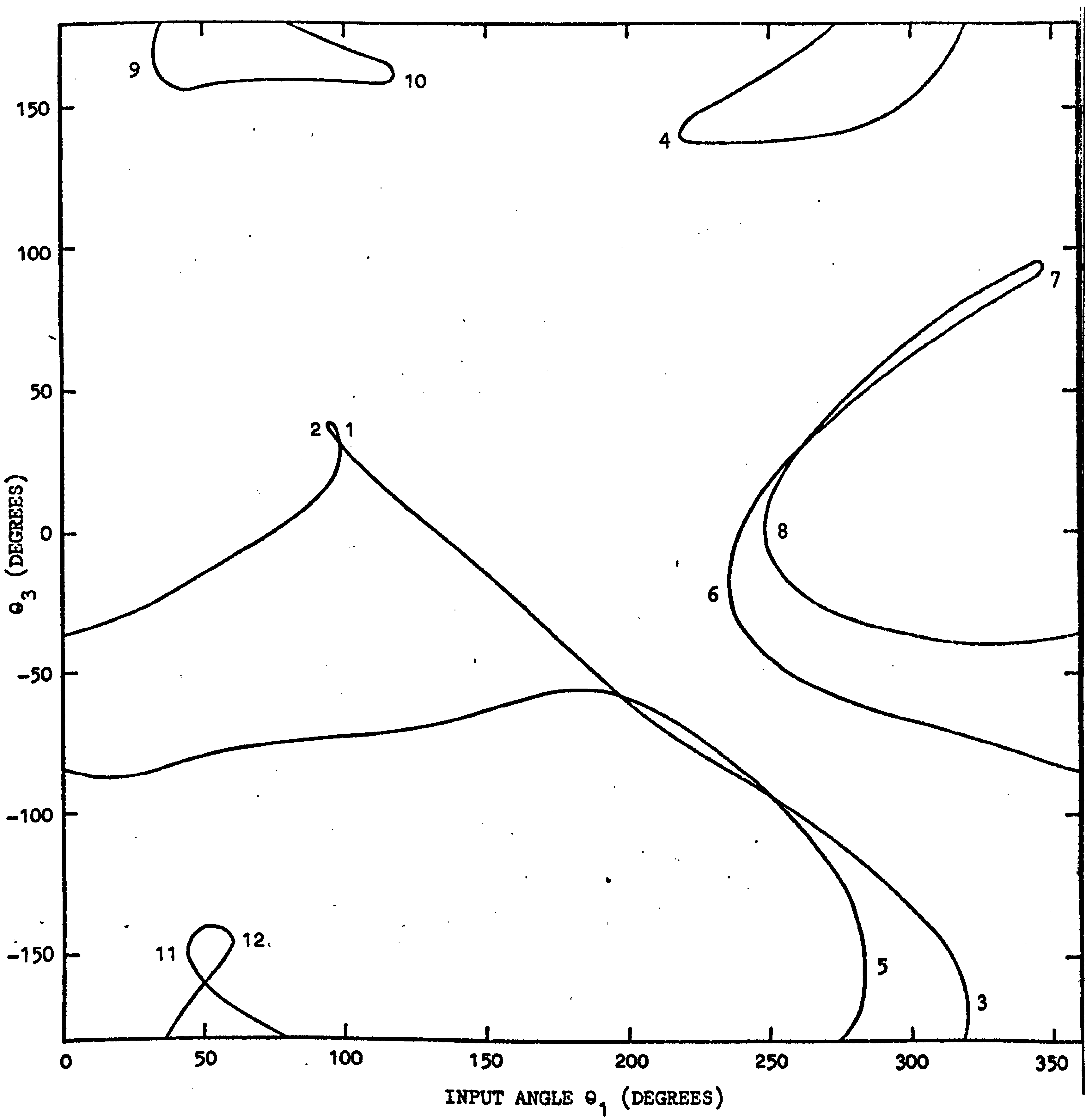


Figure 7.5 Graph of θ_3 vs θ_1 for the Six-Link RCRPRR Mechanism.

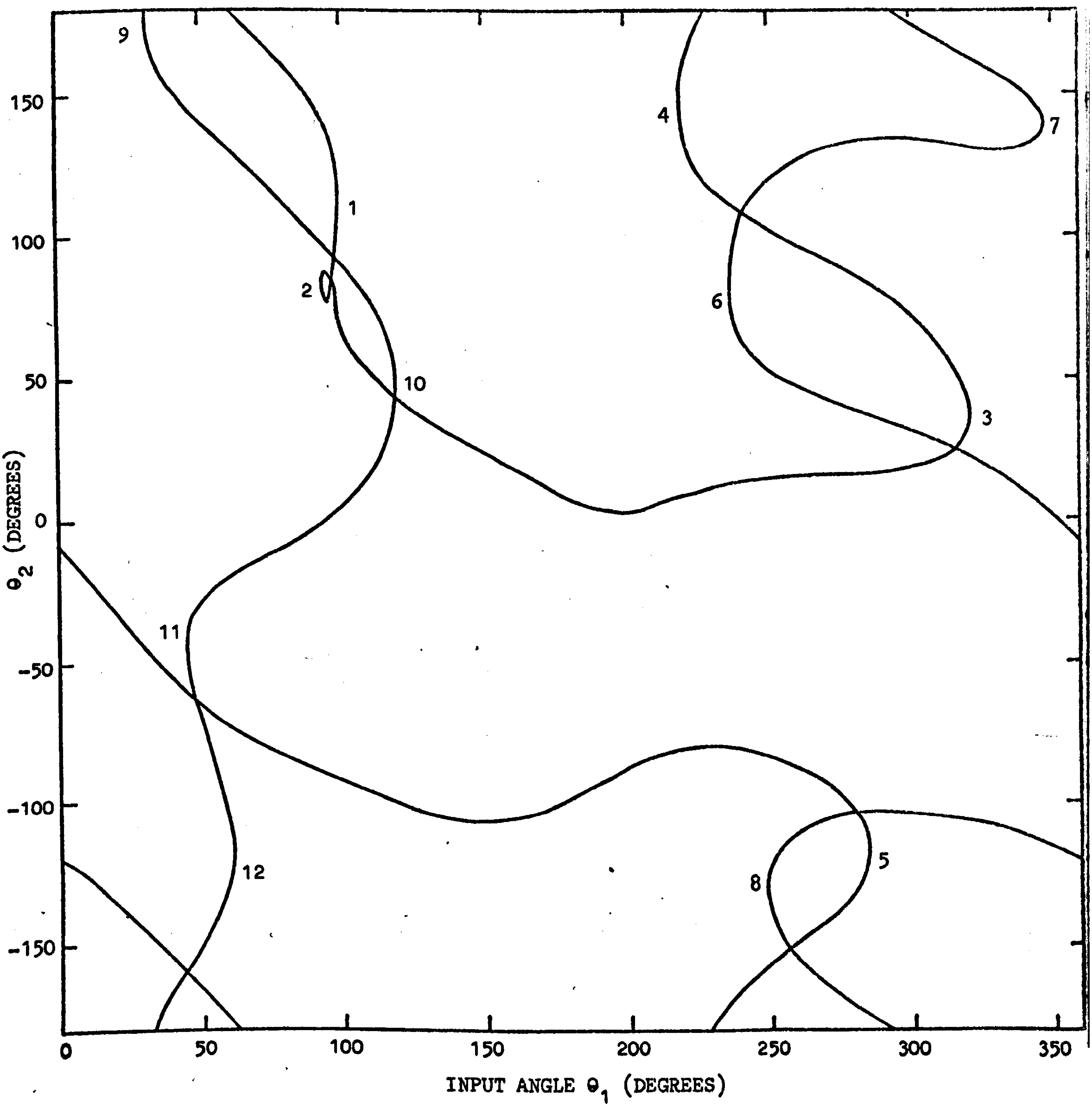


Figure 7.6 Graph of Θ_2 vs Θ_1 for the Six-Link RCRPRR Mechanism.

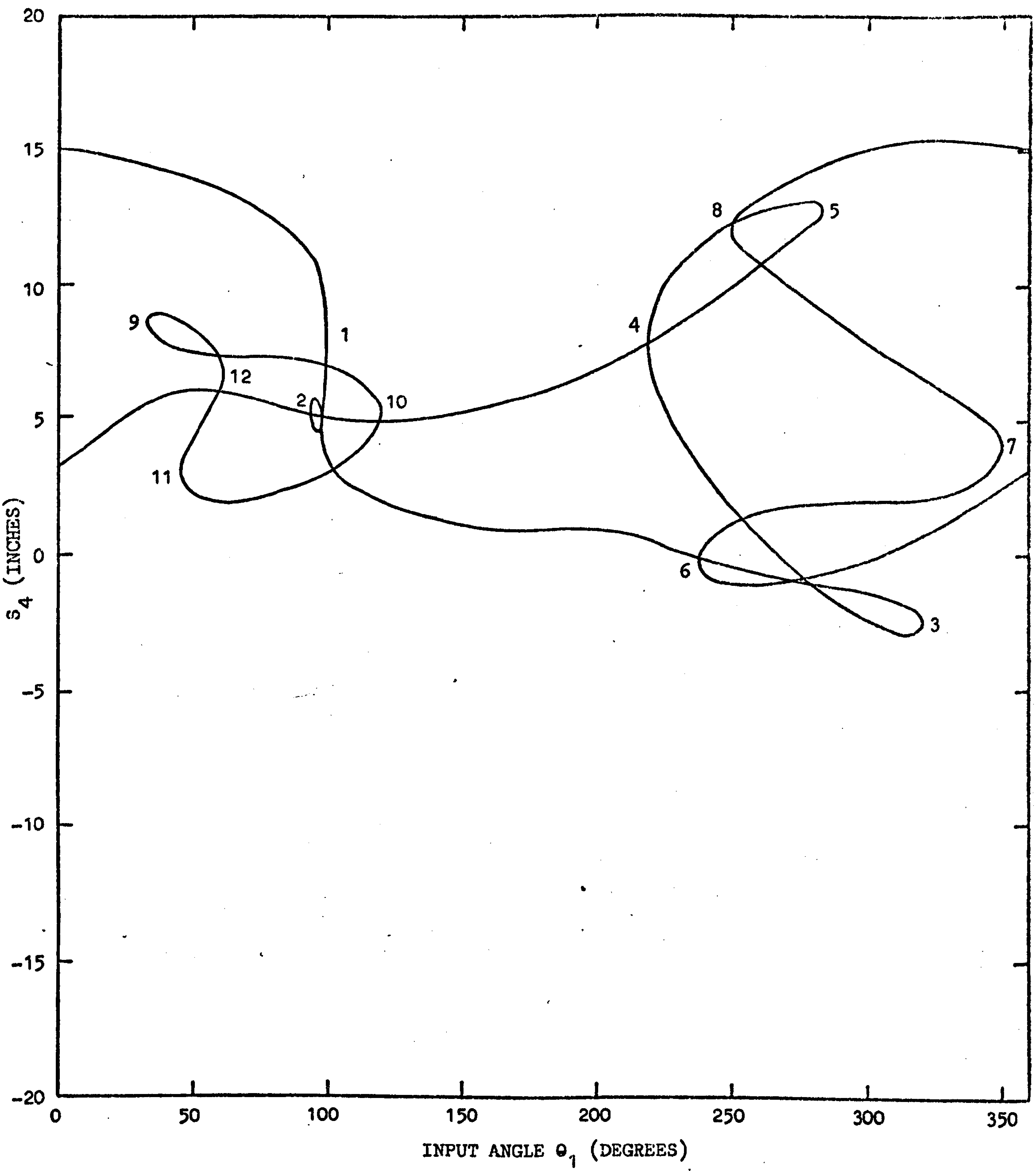


Figure 7.7 Graph of S_4 vs θ_1 for the Six-Link RCRPRR Mechanism.

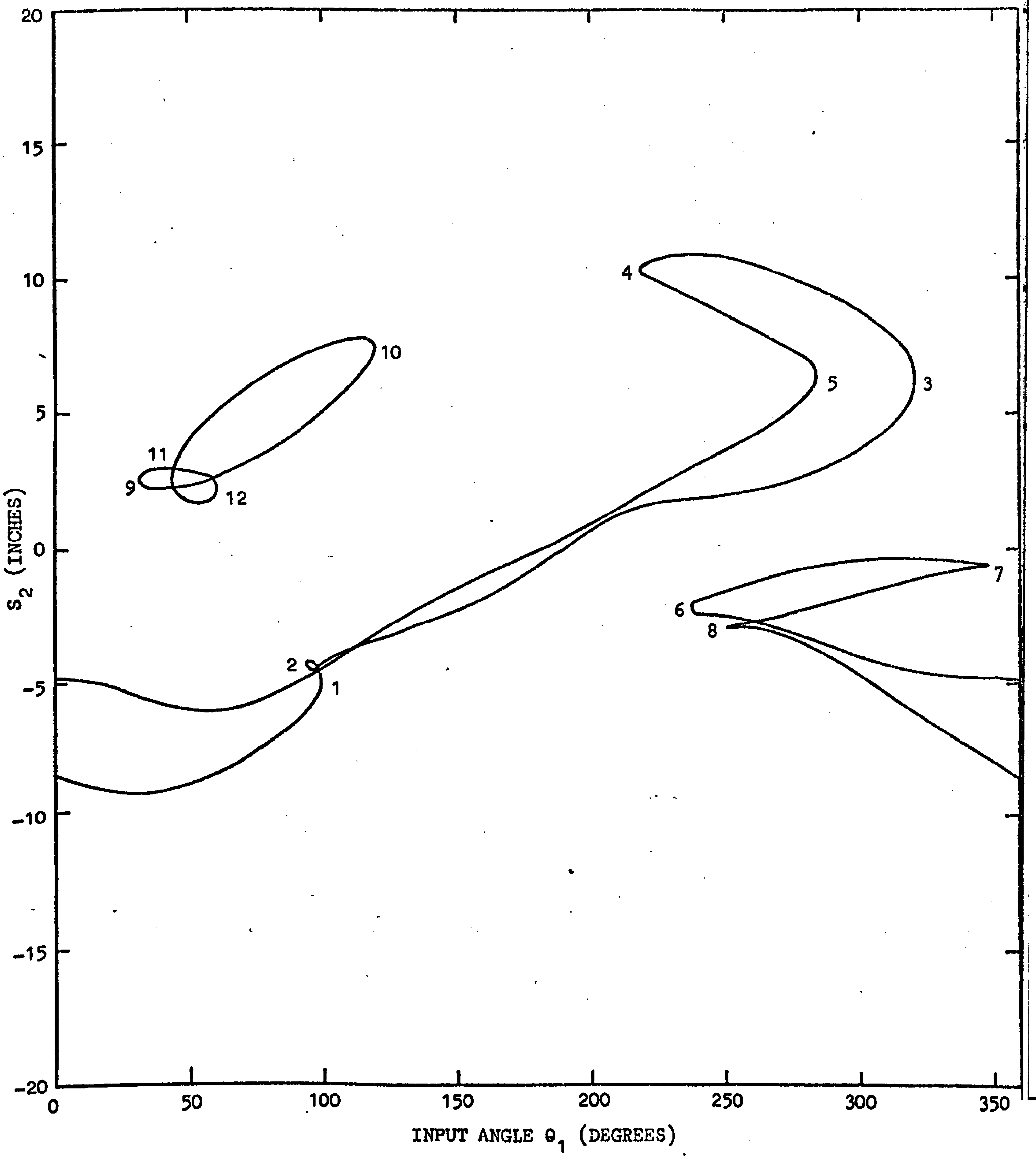


Figure 7.8 Graph of S_2 vs Θ_1 for the Six-Link RCRPRR Mechanism.

CHAPTER 8

A DISPLACEMENT ANALYSIS

OF THE

SPATIAL SIX-LINK RCRRPR MECHANISM

8.1 Introduction.

In this chapter, a novel closed-form input-output displacement equation of degree eight in the half-tangent of the output angular displacement is derived for the spatial six-link RCRRPR mechanism, using the dual number loop equations derived in Chapter 4. This input-output equation in then used to simulate the spatial five-link RCRCR and RCRRC mechanisms by superimposing the fourth revolute pair on the fifth prismatic pair (RCR$\widehat{\text{RP}}$R), and the fifth prismatic pair on the sixth revolute pair (RCRR$\widehat{\text{PR}}$) respectively. Following this procedure, identical input-output relationships to those presented by Yuan [47] and Duffy and Habib-Olahi [11, 13] were obtained for the RCRCR and RCRRC mechanisms.

8.2 Description of the Six-Link RCRRPR Mechanism.

The six-link RCRRPR spatial mechanism is illustrated by Figure 2.26, and is represented mathematically by the following six dual sides and six dual angles:-

$$\begin{aligned}
\hat{\alpha}_{12} &= \alpha_{12} + \epsilon\, a_{12}\\
\hat{\alpha}_{23} &= \alpha_{23} + \epsilon\, a_{23}\\
\hat{\alpha}_{34} &= \alpha_{34} + \epsilon\, a_{34}\\
\hat{\alpha}_{45} &= \alpha_{45} + \epsilon\, a_{45}\\
\hat{\alpha}_{56} &= \alpha_{56} + \epsilon\, a_{56}\\
\hat{\alpha}_{61} &= \alpha_{61} + \epsilon\, a_{61}
\end{aligned} \tag{8.1}$$

$$\begin{aligned}
\hat{\theta}_{1} &= \theta_{1} + \epsilon\, S_{11}\\
\hat{\theta}_{2} &= \theta_{2} + \epsilon\, S_{2}\\
\hat{\theta}_{3} &= \theta_{3} + \epsilon\, S_{33}\\
\hat{\theta}_{4} &= \theta_{4} + \epsilon\, S_{44}\\
\hat{\theta}_{5} &= \theta_{55} + \epsilon\, S_{5}\\
\hat{\theta}_{6} &= \theta_{6} + \epsilon\, S_{66}
\end{aligned} \tag{8.2}$$

where $\epsilon^2 = 0$, and all fixed mechanism proportions have double or repeated suffices. The input and output angular displacements are respectively θ_1 and θ_6, and the frame is the constant dual side, $\hat{\alpha}_{61}$.

A relationship between the input and output angular displacements, of the lowest possible degree in both, is required.

8.3 Derivation of Input-Output Equation for the RCRRPR Mechanism.

For the RCRRPR mechanism one may again write an appropriate primary equation relating the input, output and a single extraneous angular variable. It is then necessary to derive a second equation, from that secondary equation which involves all the fixed parameters, from which to eliminate this extraneous variable and obtain the desired input-output relationship.

8.3.1 First Equation in θ_1, θ_6 and θ_4.

The primary part of the dual cosine law:-

$$\hat{Z}_{1654} = \cos\hat{\alpha}_{23} \tag{8.3}$$

is written as:-

$$Z_{1654} = \cos\alpha_{23} \tag{8.4}$$

where:-

$$Z_{1654} = \sin\alpha_{34}(X_{165}\sin\theta_4 + Y_{165}\cos\theta_4) + \cos\alpha_{34}Z_{165} \tag{8.5}$$

and:-

$$\begin{aligned} X_{165} &= X_{16}\cos\theta_{55} - Y_{16}\sin\theta_{55} \\ Y_{165} &= \cos\alpha_{45}(X_{16}\sin\theta_{55} + Y_{16}\cos\theta_{55}) - \sin\alpha_{45}Z_{16} \\ Z_{165} &= \sin\alpha_{45}(X_{16}\sin\theta_{55} + Y_{16}\cos\theta_{55}) + \cos\alpha_{45}Z_{16} \end{aligned} \tag{8.6}$$

and involves only the input angle (θ_1), output angle (θ_6) and a single extraneous angular variable (θ_4), since the angle, θ_{55}, is a constant mechanism proportion. (In the above equations X_{16}, Y_{16} and Z_{16} are given in Appendix III.). Hence, making the substitutions.

$$\begin{aligned} \sin\theta_4 &\equiv 2x_4/(1 + x_4^2) \\ \cos\theta_4 &\equiv (1 - x_4^2)/(1 + x_4^2) \\ \text{where} \quad x_4 &\equiv \tan(\theta_4/2) \end{aligned} \tag{5.1}$$

in equation (8.4) and rearranging one obtains:-

$$f(x_4) = a_2 x_4^2 + a_1 x_4 + a_0 = 0 \qquad (8.7)$$

where:-

$$a_2 = \cos\alpha_{34} Z_{165} - \sin\alpha_{34} Y_{165} - \cos\alpha_{23}$$
$$a_1 = 2.\sin\alpha_{34} X_{165}$$
$$a_0 = \cos\alpha_{34} Z_{165} + \sin\alpha_{34} Y_{165} - \cos\alpha_{23} \qquad (8.8)$$

Equation (8.7) is now in a suitable form for the elimination of x_4.

8.3.2 Second Equation in θ_1, θ_6 and θ_4.

It is not possible to make use of the secondary equation corresponding to equation (8.4) since this would introduce the variable offset, S_5. Instead, one must use the secondary part of the dual subsidiary cosine law:-

$$\hat{Z}_{16} = \hat{Z}_{34} \qquad (8.9)$$

which is:-

$$Z_{016} = Z_{034} \qquad (8.10)$$

since the above contains all the fixed linkage proportions. Now in equation (8.10) it is convenient to write Z_{016} and Z_{034} in the following forms (see Appendix III.):-

$$\begin{aligned} Z_{016} = \; & a_{56}[\cos\alpha_{56}(\bar{X}_1\sin\theta_6 + \bar{Y}_1\cos\theta_6) - \sin\alpha_{56} Z_1] \\ & + S_{66}\sin\alpha_{56}(\bar{X}_1\cos\theta_6 - \bar{Y}_1\sin\theta_6) \\ & + [\sin\alpha_{56}(\bar{X}_{01}\sin\theta_6 + \bar{Y}_{01}\cos\theta_6) + \cos\alpha_{56}\bar{Z}_{01}] \end{aligned} \qquad (8.11)$$

and:-

$$\begin{aligned} Z_{034} \equiv Z_{043} = \; & a_{23}[\cos\alpha_{23}(\bar{X}_4\sin\theta_3 + \bar{Y}_4\cos\theta_3) - \sin\alpha_{23}\bar{Z}_4] \\ & + S_{33}\sin\alpha_{23}(\bar{X}_4\cos\theta_3 - \bar{Y}_4\sin\theta_3) \\ & + [\sin\alpha_{23}(\bar{X}_{04}\sin\theta_3 + \bar{Y}_{04}\cos\theta_3) + \cos\alpha_{23}\bar{Z}_{04}] \end{aligned} \qquad (8.12)$$

where $\bar{X}_1$, $\bar{Y}_1$, $\bar{Z}_1$ and $\bar{X}_4$, $\bar{Y}_4$, $\bar{Z}_4$ are defined in Appendix III. and where:-

$$\bar{X}_{01} = a_{12}\cos\alpha_{12}\sin\theta_1 + S_{11}\sin\alpha_{12}\cos\theta_1$$
$$\begin{aligned}\bar{Y}_{01} = \; & a_{12}(\sin\alpha_{12}\sin\alpha_{61} - \cos\alpha_{12}\cos\alpha_{61}\cos\theta_1) \\ & - a_{61}(\cos\alpha_{12}\cos\alpha_{61} - \sin\alpha_{12}\sin\alpha_{61}\cos\theta_1) \\ & + S_{11}\cos\alpha_{61}\sin\alpha_{12}\sin\theta_1\end{aligned}$$
$$\begin{aligned}\bar{Z}_{01} = & - a_{12}(\cos\alpha_{61}\sin\alpha_{12} + \sin\alpha_{61}\cos\alpha_{12}\cos\theta_1) \\ & - a_{61}(\sin\alpha_{61}\cos\alpha_{12} + \cos\alpha_{61}\sin\alpha_{12}\cos\theta_1) \\ & + S_{11}\sin\alpha_{61}\sin\alpha_{12}\sin\theta_1\end{aligned} \tag{8.13}$$

$$\bar{X}_{04} = a_{45}\cos\alpha_{45}\sin\theta_4 + S_{44}\sin\alpha_{45}\cos\theta_4$$
$$\begin{aligned}\bar{Y}_{04} = \; & a_{45}(\sin\alpha_{45}\sin\alpha_{34} - \cos\alpha_{45}\cos\alpha_{34}\cos\theta_4) \\ & - a_{34}(\cos\alpha_{45}\cos\alpha_{34} - \sin\alpha_{45}\sin\alpha_{34}\cos\theta_4) \\ & + S_{44}\cos\alpha_{34}\sin\alpha_{45}\sin\theta_4\end{aligned}$$
$$\begin{aligned}\bar{Z}_{04} = & - a_{45}(\cos\alpha_{34}\sin\alpha_{45} + \sin\alpha_{34}\cos\alpha_{45}\cos\theta_4) \\ & - a_{34}(\sin\alpha_{34}\cos\alpha_{45} + \cos\alpha_{34}\sin\alpha_{45}\cos\theta_4) \\ & + S_{44}\sin\alpha_{34}\sin\alpha_{45}\sin\theta_4\end{aligned} \tag{8.14}$$

Thus, with the aid of (8.12), it is possible to rewrite (8.10) as follows, after regrouping terms:-

$$\begin{aligned}& [\sin\alpha_{23}(\bar{X}_{04} - S_{33}\bar{Y}_4) + \cos\alpha_{23}a_{23}\bar{X}_4]\sin\theta_3 \\ & + [\sin\alpha_{23}(\bar{Y}_{04} + S_{33}\bar{X}_4) + \cos\alpha_{23}a_{23}\bar{Y}_4]\cos\theta_3 \\ & + [\cos\alpha_{23}\bar{Z}_{04} - \sin\alpha_{23}a_{23}\bar{Z}_4 - Z_{016}] = 0\end{aligned} \tag{8.15}$$

It is now necessary to replace $\sin\theta_3$ and $\cos\theta_3$ without unnecessarily raising the degree of (8.15). Now $\cos\theta_3$ may be replaced in terms of θ_1, θ_6 and θ_{55} using the subsidiary cosine law:-

$$Z_3 = Z_{165} \tag{8.16}$$

since this may be rearranged in the form:-

$$\cos\theta_3 = (\cos\alpha_{23}\cos\alpha_{34} - Z_{165})\operatorname{cosec}\alpha_{23}\operatorname{cosec}\alpha_{34} \tag{8.17}$$

using the definition of Z_3 (see Appendix III.), and Z_{165} is given by (8.6).

Similarly $\sin\theta_3$ is replaced in terms of θ_1, θ_6, θ_{55} and x_4 by means of the fundamental half-tangent law (see Chapter 5.):-

$$(Y_{165} - Y_3)x_4 - (X_{165} - X_3) = 0 \qquad (8.18)$$

which is a cyclic permutation of (5.36). Thus, using the identity:-

$$\begin{aligned} \sin\alpha_{34} Y_3 &\equiv \cos\alpha_{34} Z_3 - \cos\alpha_{23} \\ &= \cos\alpha_{34} Z_{165} - \cos\alpha_{23} \end{aligned} \qquad (8.19)$$

(see (4.10b)), together with the definition of X_3 (Chapter 4.), one may rewrite (8.18) as:-

$$\begin{aligned} \sin\theta_3 &= \left[\sin\alpha_{34} X_{165} + (\cos\alpha_{34} Z_{165} - \sin\alpha_{34} Y_{165} - \cos\alpha_{23})\, x_4\right]\operatorname{cosec}\alpha_{23}\operatorname{cosec}\alpha_{34} \\ &= \left[\sin\alpha_{34} X_{165} + a_2 x_4\right]\operatorname{cosec}\alpha_{23}\operatorname{cosec}\alpha_{34} \end{aligned} \qquad (8.20)$$

where a_2 is given by (8.8) and X_{165}, Y_{165} and Z_{165} are defined by (8.6)

Hence, it is now possible to substitute into (8.15) for $\sin\theta_3$ and $\cos\theta_3$, using (8.20) and (8.17) giving:-

$$\begin{aligned} &\left[\sin\alpha_{23}(\bar{X}_{04} - S_{33}\bar{Y}_4) + \cos\alpha_{23}\, a_{23}\bar{X}_4\right]\left[\sin\alpha_{34} X_{165} + a_2 x_4\right] \\ &+ \left[\sin\alpha_{23}(\bar{Y}_{04} + S_{33}\bar{X}_4) + \cos\alpha_{23}\, a_{23}\bar{Y}_4\right]\left[\cos\alpha_{23}\cos\alpha_{34} - Z_{165}\right] \\ &+ \left[\cos\alpha_{23}\bar{Z}_{04} - \sin\alpha_{23}\, a_{23}\bar{Z}_4 - Z_{016}\right]\sin\alpha_{23}\sin\alpha_{34} = 0 \end{aligned} \qquad (8.21)$$

Making the substitutions (5.1) for $\sin\theta_4$ and $\cos\theta_4$.

i.e.:-

$$\begin{aligned} \sin\theta_4 &\equiv 2x_4/(1 + x_4^2) \\ \cos\theta_4 &\equiv (1 - x_4^2)/(1 + x_4^2) \end{aligned} \qquad (5.1)$$

where $\qquad x_4 \equiv \tan(\theta_4/2)$

in equation (8.21), and rearranging, one has:-

$$g(x_4) = b_3 x_4^3 + b_2 x_4^2 + b_1 x_4 + b_0 = 0 \qquad (8.22)$$

where:-

$$b_3 = L_2 a_2$$

$$b_2 = \sin\alpha_{34} L_2 X_{165} + L_1 a_2 + M_2(\cos\alpha_{23}\cos\alpha_{34} - Z_{165}) + N_2 - \sin\alpha_{23}\sin\alpha_{34} Z_{016}$$

$$b_1 = \sin\alpha_{34} L_1 X_{165} + L_0 a_2 + M_1(\cos\alpha_{23}\cos\alpha_{34} - Z_{165}) + N_1$$

$$b_0 = \sin\alpha_{34} L_0 X_{165} + M_0(\cos\alpha_{23}\cos\alpha_{34} - Z_{165}) + N_0 - \sin\alpha_{23}\sin\alpha_{34} Z_{016} \quad (8.23)$$

and the nine constants L_2, L_1, L_0, M_2, M_1, M_0, N_2, N_1, N_0 (defined in Appendix VI.) depend only on the fixed mechanism parameters.

8.3.3 Elimination Procedure.

It is now possible to eliminate x_4 between equations (8.7) and (8.22). At, the outset, however, it would seem that the Bézoutian for a cubic and a quadratic is a third order compound determinant and that the resulting eliminant is of the fifth degree in the coefficients a_2, $a_1, \ldots, b_1$, b_0. This would then imply that the input-output equation for the RCRRPR mechanism is of degree ten in input and output. However, this is not the case, since the leading coefficient, b_3, of the cubic is a constant multiple, L_2, of the leading coefficient, a_2, of the quadratic and, wherever this occurs the eliminant is of degree $(m + n - 1)$ (i.e. 4 in this case) in the coefficients instead of $(m + n)$. This is because one may extract a factor, a_2 (from the eliminant) which is non-zero in general. The process is best seen by examining the bigradient, (5.6), rather than the Bézoutian for the system (8.7) and (8.22).

Thus the bigradient of f and g is:-

$$E(f, g) = \begin{vmatrix} a_2 & a_1 & a_0 & 0 & 0 \\ 0 & a_2 & a_1 & a_0 & 0 \\ 0 & 0 & a_2 & a_1 & a_0 \\ 0 & b_3 & b_2 & b_1 & b_0 \\ b_3 & b_2 & b_1 & b_0 & 0 \end{vmatrix} \quad (8.24)$$

But, since $b_3 = L_2 a_2$ (see (8.23)) where L_2 is a constant, one may extract the factor a_2 from the first column of (8.24) and, after performing a series of simple row operations, reduce the above determinant to one of fourth order which is:-

$$E(f, g) = a_2 \cdot \begin{vmatrix} a_2 & a_1 & a_0 & 0 \\ 0 & a_2 & a_1 & a_0 \\ 0 & (b_2 - L_2a_1) & (b_1 - L_2a_0) & b_0 \\ (b_2 - L_2a_1) & (b_1 - L_2a_0) & b_0 & 0 \end{vmatrix} \qquad (8.25)$$

The input-output equation for the RCRRPR mechanism is hence of order four in the coefficients since the factor a_2 is non-zero in general and may be cancelled. Clearly, therefore, it is of degree eight in the input and output variables, and is given by equating the determinant in (8.25) to zero. It may be written in a more concise form by noticing that this determinant is the bigradient of the following system:-

$$\begin{aligned} f(x_4) &= a_2x_4^2 + a_1x_4 + a_0 = 0 \\ g'(x_4) &= (b_2 - L_2a_1)x_4^2 + (b_1 - L_2a_0)x_4 + b_0 = 0 \end{aligned} \qquad (8.26)$$

and subsequently writing the Bézoutian of (8.26), which, when expanded, becomes (see (5.8)):-

$$\begin{aligned} &[a_2(b_1 - L_2a_0) - a_1(b_2 - L_2a_1)]\,[a_1b_0 - a_0(b_1 - L_2a_0)] \\ &- [a_2b_0 - a_0(b_2 - L_2a_1)]^2 \qquad = 0 \end{aligned} \qquad (8.27)$$

Expressing the coefficients a_2, a_1, a_0 and b_2, b_1, b_0 in terms of the half-tangent of the output angular displacement by means of the substitution (5.1), i.e.:-

$$\begin{aligned} \sin\theta_6 &\equiv 2x_6/(1 + x_6^2) \\ \cos\theta_6 &\equiv (1 - x_6^2)/(1 + x_6^2) \qquad (5.1) \\ \text{where} \qquad x_6 &\equiv \tan(\theta_6/2) \end{aligned}$$

one has:-

$$a_2 = p_{22}x_6^2 + p_{12}x_6 + p_{02}$$
$$a_1 = p_{21}x_6^2 + p_{11}x_6 + p_{01}$$
$$a_0 = p_{20}x_6^2 + p_{10}x_6 + p_{00} \qquad (8.28)$$

$$b_2 = q_{22}x_6^2 + q_{12}x_6 + q_{02}$$
$$b_1 = q_{21}x_6^2 + q_{11}x_6 + q_{01}$$
$$b_0 = q_{20}x_6^2 + q_{10}x_6 + q_{00} \qquad (8.29)$$

where the terms p_{ij} and q_{ij} are each a function of the input angular displacement (θ_1) only and are listed in Appendix VI. It is now clear from (8.28) and (8.29) that the input-output equation (8.27) for the RCRRPR mechanism is of degree eight in the output angular displacement. Alternatively, the coefficients a_2, a_1,....,b_0 may be similarly expressed in the half-tangent, x_1, of the input angular displacement and hence (8.27) is also of degree eight in the latter.

These results are in agreement with the predicted degree for the RCRRPR six-link mechanism (see Chapter 2.).

8.4 <u>Displacement Analysis.</u>

Solving the input-output equation (8.27) for x_6, one obtains, in general, eight distinct values for the output angular displacement (i.e. θ_{61}, θ_{62}, θ_{63}, θ_{64}, θ_{65}, θ_{66}, θ_{67}, θ_{68}) for each value of the input angular displacement, θ_1. The resulting eight ordered pairs (θ_1, θ_{61}) (θ_1, θ_{62}),....,(θ_1, θ_{68}) will then each give rise to corresponding values for the remaining linkage variables (S_5, θ_4, θ_3, θ_2, S_2) using procedures outlined below.

Thus, θ_4 may be determined from either of the two expressions for the common root of (8.7) and (8.22) which are derived from the Bézoutian (5.8) of the system (8.26) and are written:-

$$x_4 = -[a_2b_0 - a_0(b_2 - L_2a_1)] / [a_2(b_1 - L_2a_0) - a_1(b_2 - L_2a_1)] \qquad (8.30a)$$

or:-

$$x_4 = -[a_1b_0 - a_0(b_1 - L_2a_0)] / [a_2b_0 - a_0(b_2 - L_2a_1)] \quad (8.30b)$$

where $$x_4 \equiv \tan(\theta_4/2)$$

Having determined corresponding values for θ_1, θ_6 and θ_4 it is then a relatively simple matter to obtain the unique value of θ_3 from either of the two fundamental half-tangent laws (see Appendix IV.):-

$$x_3 = -(Y_{1654} - \sin\alpha_{23})/X_{1654} \quad (8.31a)$$

or:- $$x_3 = X_{1654}/(Y_{1654} + \sin\alpha_{23}) \quad (8.31b)$$

where $$x_3 \equiv \tan(\theta_3/2)$$

which are cyclic permutations of (5.32) and (5.33).

Here:-

$$X_{1654} = X_{165}\cos\theta_4 - Y_{165}\sin\theta_4$$
$$Y_{1654} = \cos\alpha_{34}(X_{165}\sin\theta_4 + Y_{165}\cos\theta_4) - \sin\alpha_{34}Z_{165} \quad (8.32)$$

whilst X_{165}, Y_{165} and Z_{165} are given by (8.6).

In a similar manner one may obtain the value of θ_2 from a cyclic permutation of (8.31a,b) once θ_1, θ_6 and θ_4 are known. Thus:-

$$x_2 = -(Y_{4561} - \sin\alpha_{23})/X_{4561} \quad (8.33a)$$

or:- $$x_2 = X_{4561}/(Y_{4561} + \sin\alpha_{23}) \quad (8.33b)$$

where $$x_2 \equiv \tan(\theta_2/2)$$

and:-

$$X_{4561} = X_{456}\cos\theta_1 - Y_{456}\sin\theta_1$$
$$Y_{4561} = \cos\alpha_{12}(X_{456}\sin\theta_1 + Y_{456}\cos\theta_1) - \sin\alpha_{12}Z_{456} \quad (8.34)$$

The sliding displacement S_5 may be determined from the secondary component of the dual cosine law:-

$$\hat{Z}_{1654} = \cos\hat{\alpha}_{23} \quad (8.35)$$

which is:-

$$Z_{01654} = -a_{23}\sin\alpha_{23} \quad (8.36)$$

where Z_{01654} is defined as follows in its symmetric form (see Appendix III.):-

$$\begin{aligned}
Z_{01654} = \; & a_{34}Y_{1654} \\
& + S_{44}\sin\alpha_{34}X_{1654} \\
& + a_{45}[(X_{16}\sin\theta_5 + Y_{16}\cos\theta_5)Z_4 + Z_{16}Y_4] \\
& - S_5[(X_{16}\sin\theta_5 + Y_{16}\cos\theta_5)X_4 + X_{165}Y_4] \\
& + a_{56}\operatorname{cosec}\alpha_{56}[Z_{16}Z_{45} - \bar{Z}_1Z_4] \\
& - S_{66}[(X_{45}\sin\theta_6 + Y_{45}\cos\theta_6)\bar{X}_1 + X_{456}\bar{Y}_1] \\
& + a_{61}[(X_{45}\sin\theta_6 + Y_{45}\cos\theta_6)\bar{Z}_1 + Z_{45}\bar{Y}_1] \\
& + S_{11}\sin\alpha_{12}X_{4561} \\
& + a_{12}Y_{4561}
\end{aligned} \tag{8.37}$$

The terms X_{1654}, Y_{1654}.....etc., are explained in Chapter 4 and defined in Appendix III. They are each uniquely specified for a given set of θ_1, θ_6, θ_{55} and θ_4.

In a similar manner the sliding displacement S_2 may be calculated from the equation:-

$$Z_{06123} = -a_{45}\sin\alpha_{45} \tag{8.38}$$

which is a cyclic permutation of (8.36) and where:-

$$\begin{aligned}
Z_{06123} = \; & a_{34}Y_{6123} \\
& + S_{33}\sin\alpha_{34}X_{6123} \\
& + a_{23}[(X_{61}\sin\theta_2 + Y_{61}\cos\theta_2)\bar{Z}_3 + Z_{61}\bar{Y}_3] \\
& - S_2[(X_{61}\sin\theta_2 + Y_{61}\cos\theta_2)\bar{X}_3 + X_{612}\bar{Y}_3] \\
& + a_{12}\operatorname{cosec}\alpha_{12}[Z_{61}Z_{32} - Z_6\bar{Z}_3] \\
& - S_{11}[(X_{32}\sin\theta_1 + Y_{32}\cos\theta_1)X_6 + X_{321}Y_6] \\
& + a_{61}[(X_{32}\sin\theta_1 + Y_{32}\cos\theta_1)Z_6 + Z_{32}Y_6] \\
& + S_{66}\sin\alpha_{56}X_{3216} \\
& + a_{56}Y_{3216}
\end{aligned} \tag{8.39}$$

Again, the terms X_{6123}, Y_{6123},....,etc., are defined in Appendix III.

8.5 Numerical Results.

The input-output equation (8.27) for the RCRRPR mechanism was solved numerically for a given set of mechanism proportions, and graphs of the output angular variable, Θ_6, and remaining variables, Θ_4, Θ_3, Θ_2, S_5 and S_2 against the input, Θ_1, were plotted (see Figures 8.3, 8.4, 8.5, 8.6, 8.7 and 8.8. respectively).

In addition, since a combination of revolute and prismatic pairs may be used to simulate a cylindric pair, the input-output equation (8.27) for the RCRRPR mechanism may be used to generate input-output relationships for five-link RCRCR or RCRRC mechanisms. Figures 8.1, and 8.2 show plots for these two mechanisms.

The following sets of data for the mechanism proportions were chosen in each case:-

8.5.1 RCRCR Mechanism.

a_{12} = 2.5 ins.	α_{12} = 60 deg.	S_{11} = 3.0 ins.
a_{23} = 3.0 ins.	α_{23} = 45 deg.	S_{33} = 2.5 ins.
a_{34} = 4.0 ins.	α_{34} = 35 deg.	S_{44} = 0.0 ins.
a_{45} = 0.0 ins.	α_{45} = 0 deg.	S_{66} = 0.0 ins.
a_{56} = 1.0 ins.	α_{56} = 30 deg.	
a_{61} = 3.2 ins.	α_{61} = 10 deg.	Θ_{55} = 0 deg. (8.40)

Here the fourth revolute pair has been superimposed on the fifth sliding pair by selecting the proportions, $a_{45} = \alpha_{45} = S_{44} = \Theta_{55} = 0$. The remaining proportions were selected to give the same RCRCR mechanism as that previously analysed in [11]. Figure 8.1 is identical to the input-output relationship presented in [11].

8.5.2 RCRRC Mechanism.

a_{12} = 3.0 ins.	α_{12} = 45 deg.	S_{11} = 2.5 ins.
a_{23} = 2.5 ins	α_{23} = 60 deg.	S_{33} = 3.0 ins.
a_{34} = 3.2 ins.	α_{34} = 10 deg.	S_{44} = 0.0 ins.
a_{45} = 1.0 ins.	α_{45} = 30 deg.	S_{66} = 0.0 ins.
a_{56} = 0.0 ins.	α_{56} = 0 deg.	
a_{61} = 4.0 ins.	α_{61} = 35 deg.	θ_{55} = 0 deg. (8.41)

Here the fifth sliding pair has been superimposed on the sixth revolute pair by selecting the proportions, $a_{56} = \alpha_{56} = S_{66} = \theta_{55} = 0$. The remaining proportions were selected to give the same RCRRC mechanism as that previously analysed by Duffy and Habib-Olahi [13]. Figure 8.2 is identical to the input-output relationship presented in [13].

8.5.3 RCRRPR Mechanism.

a_{12} = 0.3 ins.	α_{12} = 312 deg.	S_{11} = -4.0 ins.
a_{23} = 1.4 ins.	α_{23} = 126 deg.	S_{33} = 0.5 ins.
a_{34} = 3.2 ins.	α_{34} = 282 deg.	S_{44} = 0.1 ins.
a_{45} = 0.3 ins.	α_{45} = 78 deg.	S_{66} = -2.3 ins.
a_{56} = 0.5 ins.	α_{56} = 196 deg.	
a_{61} = 1.1 ins.	α_{61} = 236 deg.	θ_{55} = 332 deg. (8.42)

These proportions were chosen to yield eight real closures for various ranges of the input angular displacement, and the results have been plotted in Figures 8.3-8.8 inclusive. On these graphs, the turning points, which occur at twelve distinct values of the input variable, θ_1, are labelled 1-12 in order to identify easily the different closures.

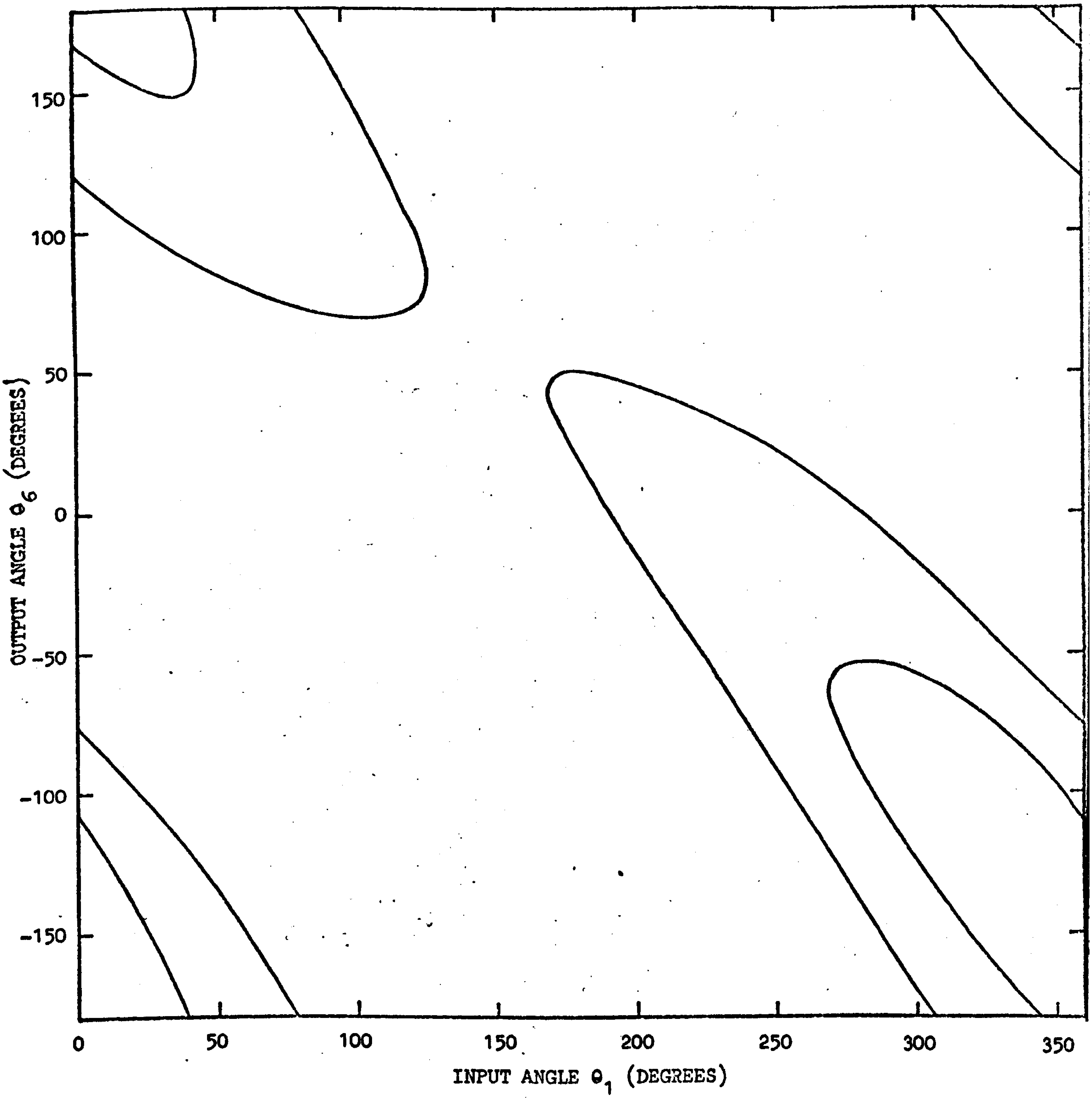


Figure 8.1 Graph of Input-Output Relationship (i.e. Θ_6 vs Θ_1) for the Six-Link RCRRPR Mechanism with Proportions chosen to Reduce the Latter to the Five-Link RCRCR Mechanism.

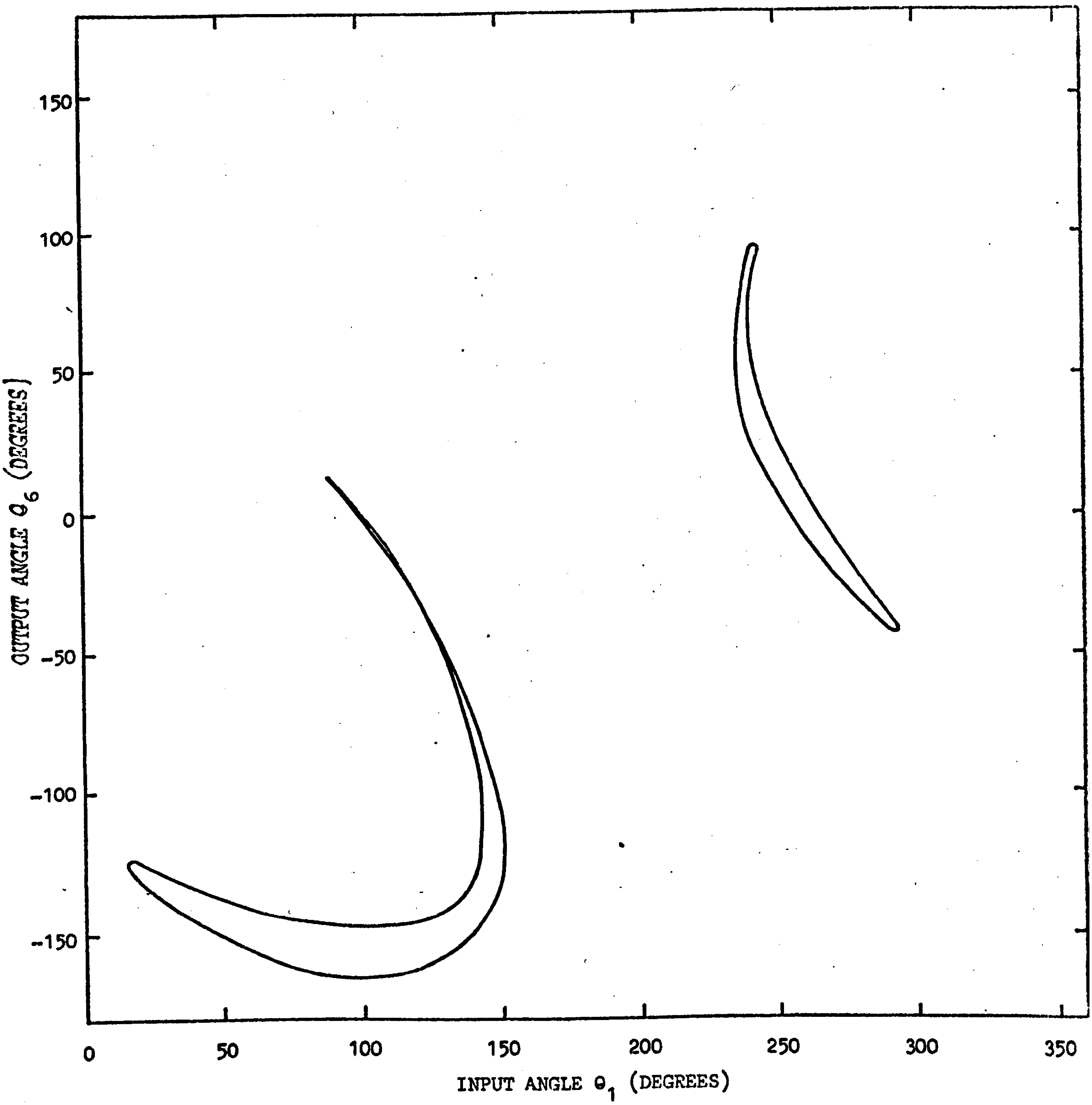


Figure 8.2 Graph of Input-Output Relationship (i.e. Θ_6 vs Θ_1) for the Six-Link RCRRPR Mechanism with Proportions chosen to Reduce the Latter to the Five-Link RCRRC Mechanism.

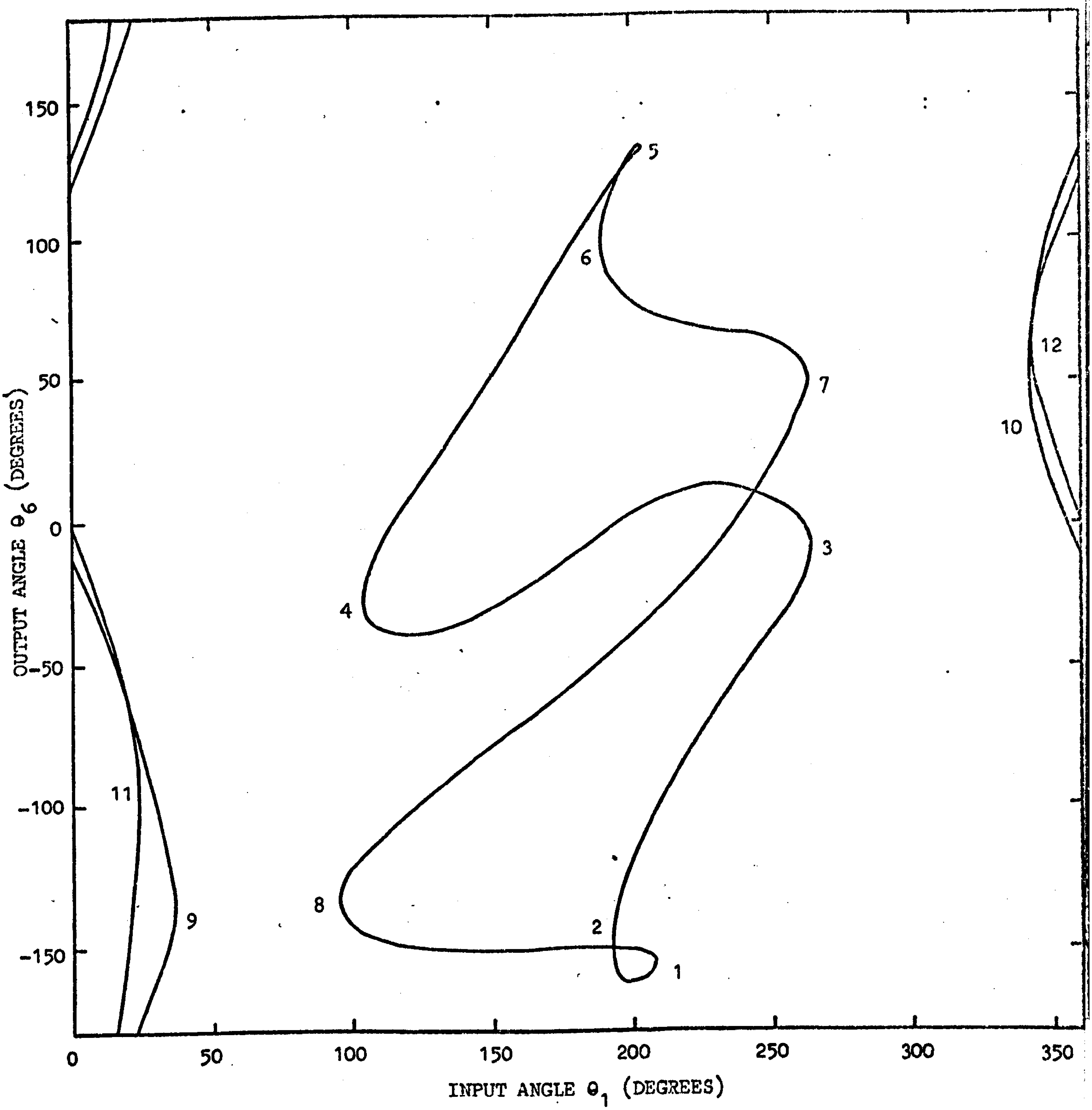


Figure 8.3 Graph of Input-Output Relationship (i.e. θ_6 vs θ_1) for the Six-Link RCRRPR Mechanism.

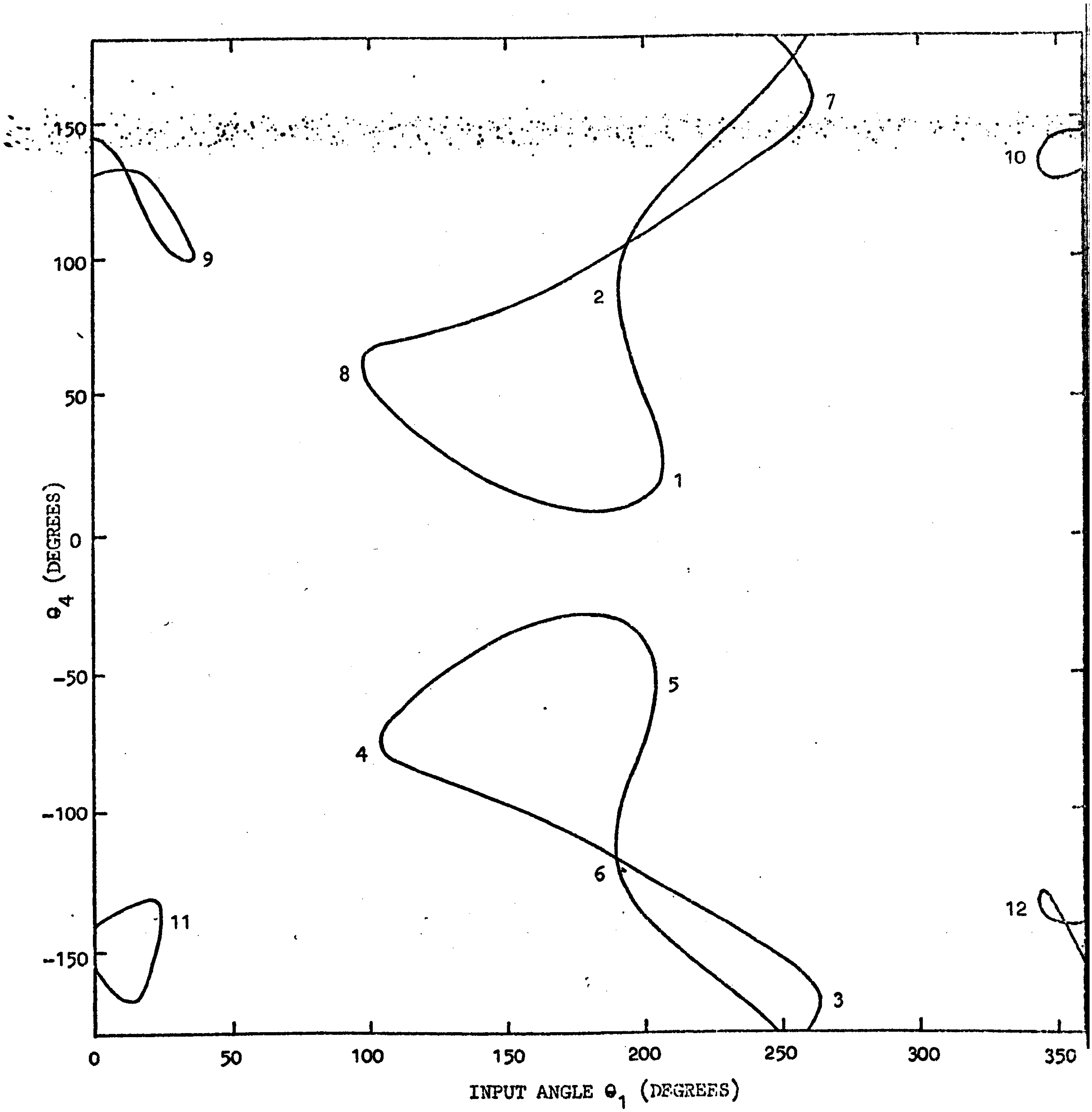


Figure 8.4 Graph of Θ_4 vs Θ_1 for the Six-Link RCRRPR Mechanism.

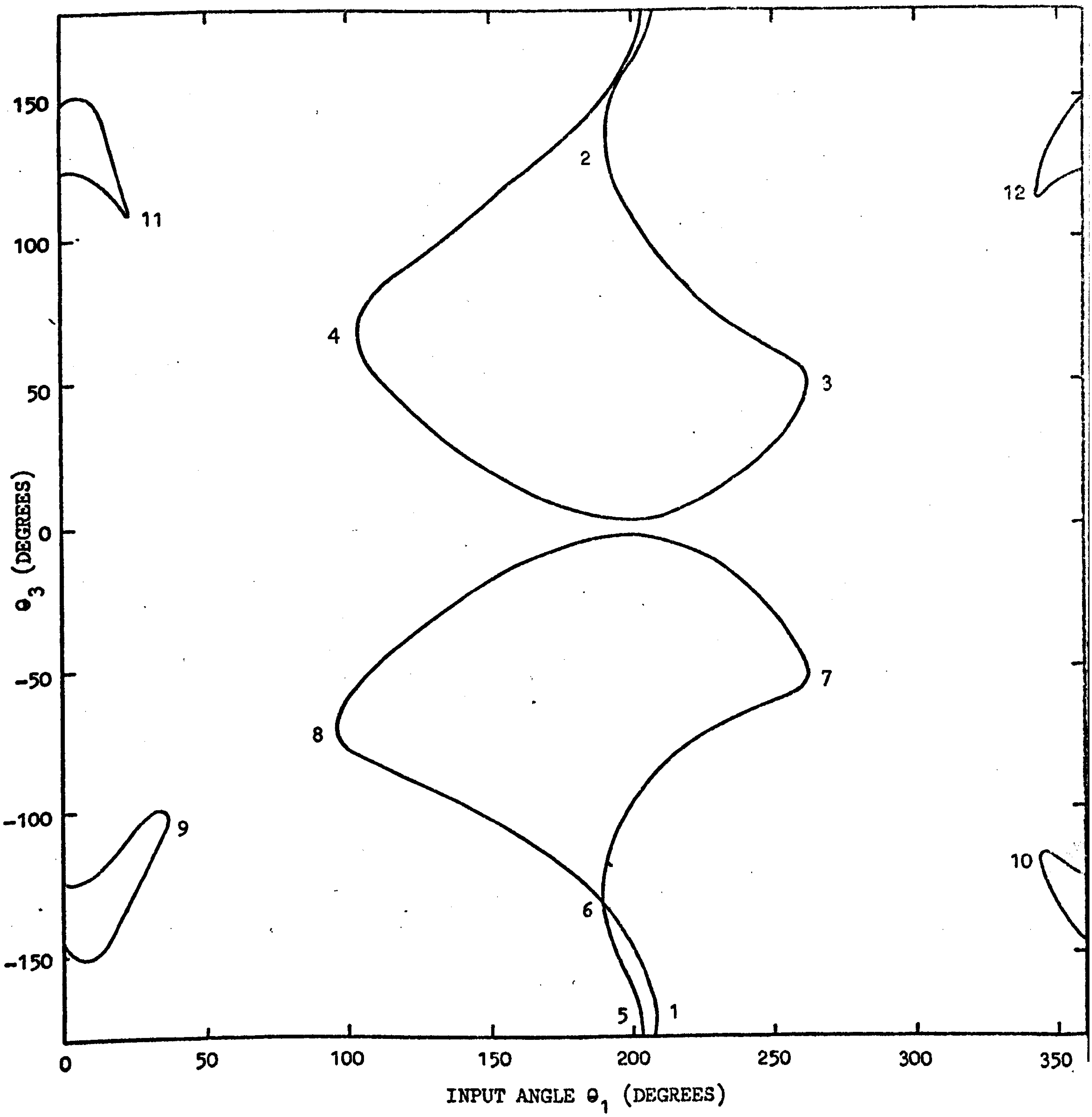


Figure 8.5 Graph of Θ_3 vs Θ_1 for the Six-Link RCRRPR Mechanism.

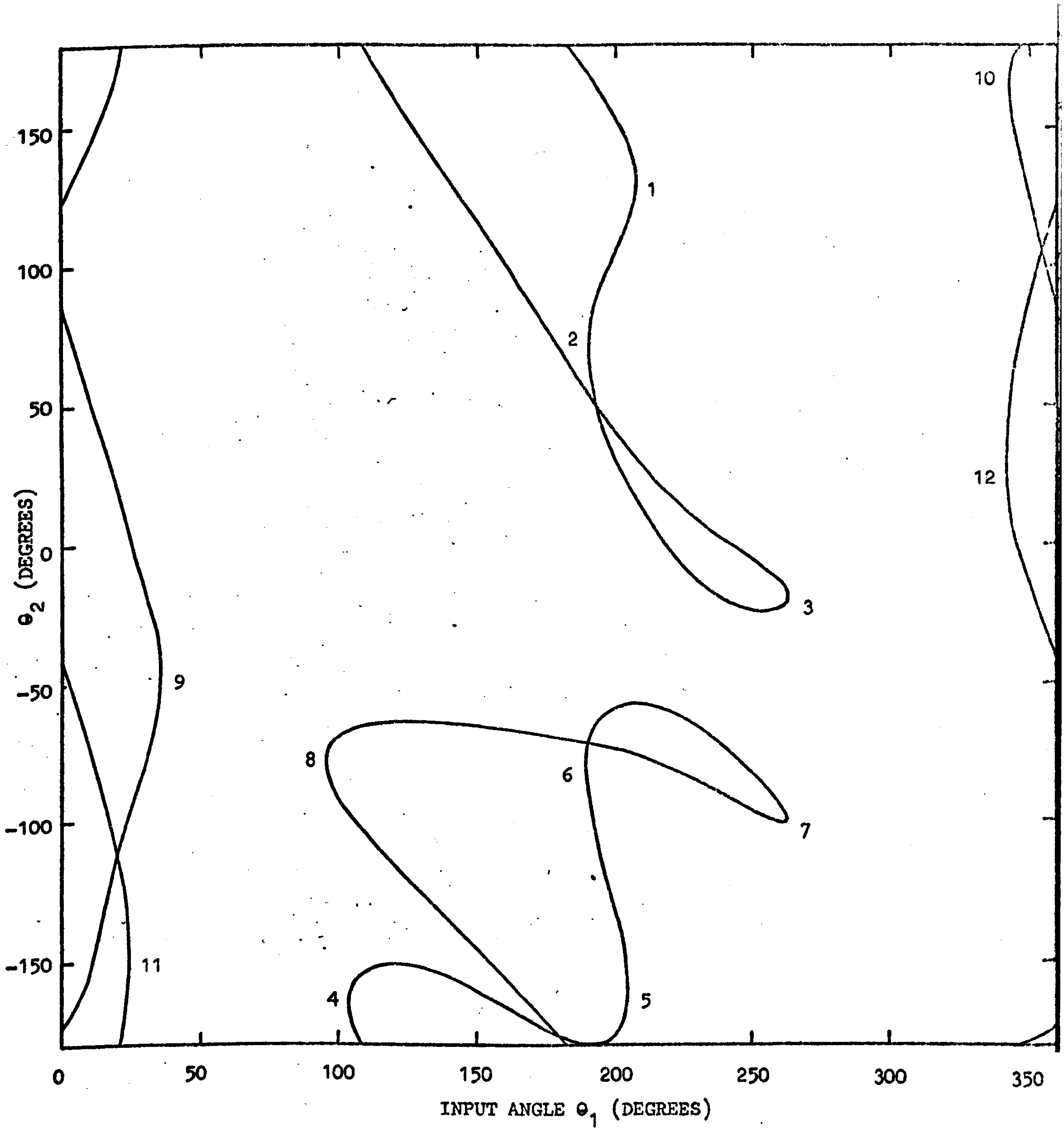


Figure 8.6 Graph of θ_2 vs θ_1 for the Six-Link RCRRPR Mechanism.

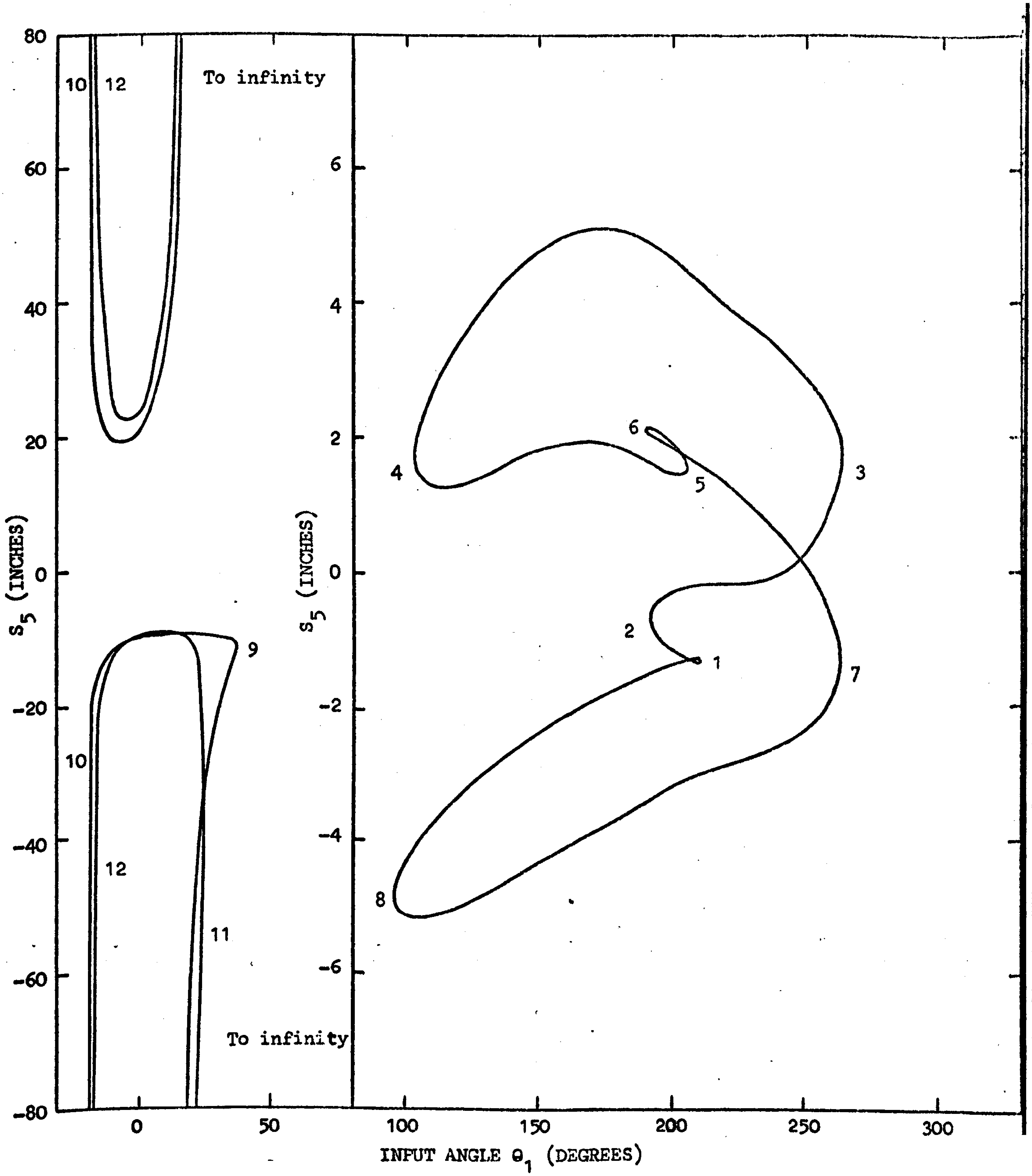


Figure 8.7 Graph of S_5 vs Θ_1 for the Six-Link RCRRPR Mechanism.

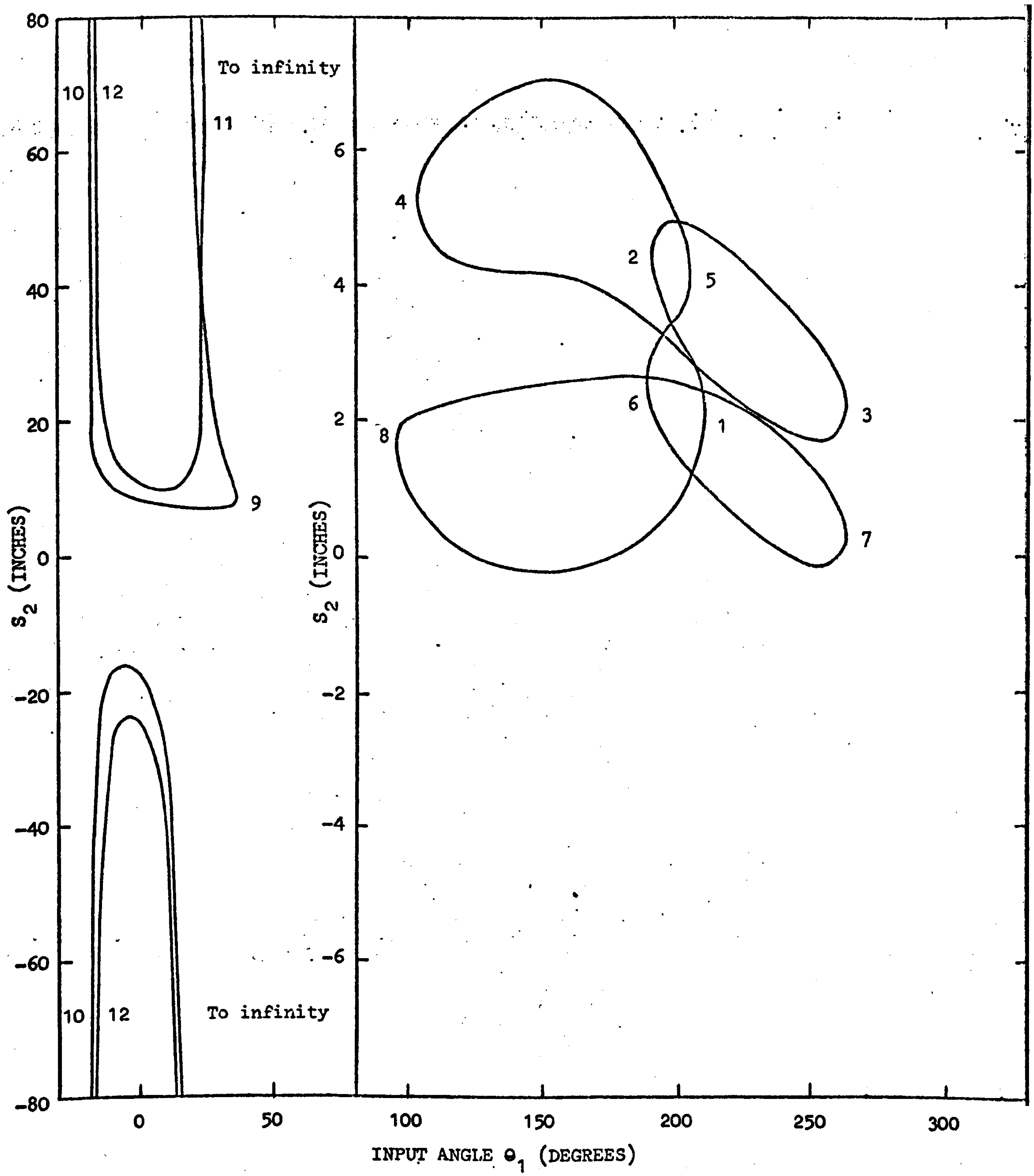


Figure 8.8 Graph of S_2 vs Θ_1 for the Six-Link RCRRPR Mechanism.

CHAPTER 9

A DISPLACEMENT ANALYSIS

OF THE

SPATIAL SIX-LINK RRRPCR MECHANISM

9.1 Introduction.

In this chapter, a novel closed-form input-output displacement equation of degree eight in the half-tangent of the output angular displacement is derived for the spatial six-link RRRPCR mechanism using the dual number loop equations derived in Chapter 4. This input-output equation is then used to simulate the spatial five-link RRCCR mechanism by superimposing the third revolute pair on the fourth prismatic pair ($RR\widehat{RP}CR$). Following this procedure, an identical input-output relationship to that presented by Yuan [48] for the RRCCR mechanism was obtained.

9.2 Description of the Six-Link RRRPCR Mechanism.

The six-link RRRPCR spatial mechanism is illustrated by Figure 2.27 and is represented mathematically by the following six dual sides and six dual angles:-

$$\begin{aligned}
\hat{\alpha}_{12} &= \alpha_{12} + \varepsilon\, a_{12}\\
\hat{\alpha}_{23} &= \alpha_{23} + \varepsilon\, a_{23}\\
\hat{\alpha}_{34} &= \alpha_{34} + \varepsilon\, a_{34}\\
\hat{\alpha}_{45} &= \alpha_{45} + \varepsilon\, a_{45}\\
\hat{\alpha}_{56} &= \alpha_{56} + \varepsilon\, a_{56}\\
\hat{\alpha}_{61} &= \alpha_{61} + \varepsilon\, a_{61}
\end{aligned} \qquad (9.1)$$

$$\begin{aligned}
\hat{\theta}_1 &= \theta_1 + \varepsilon\, S_{11}\\
\hat{\theta}_2 &= \theta_2 + \varepsilon\, S_{22}\\
\hat{\theta}_3 &= \theta_3 + \varepsilon\, S_{33}\\
\hat{\theta}_4 &= \theta_{44} + \varepsilon\, S_4\\
\hat{\theta}_5 &= \theta_5 + \varepsilon\, S_5\\
\hat{\theta}_6 &= \theta_6 + \varepsilon\, S_{66}
\end{aligned} \qquad (9.2)$$

where $\varepsilon^2 = 0$, and all fixed mechanism proportions have double or repeated suffices. The input and output angular displacements are respectively θ_1 and θ_6, and the frame is the constant dual side, $\hat{\alpha}_{61}$.

A relationship between the input and output angular displacements, of the lowest possible degree in both, is required.

9.3 Derivation of Input-Output Equation for the RRRPCR Mechanism.

As with the previous two 4R-P-C mechanisms, dealt with in Chapters 7 and 8, one may write an appropriate primary equation relating the input, output and a single extraneous angular variable, for the RRRPCR mechanism. It is then necessary to derive a second such equation from that secondary equation which involves all the fixed mechanism parameters. The extraneous variable may then be eliminated from the two equations and the desired input-output relationship obtained.

9.3.1 First Equation in θ_1, θ_6 and θ_5.

For the RRRPCR mechanism one may write the following primary equation which is the primary part of a dual cosine law for a spatial hexagon, and which relates the variables θ_1 (input), θ_6 (output), θ_5 and the constant angular displacement, θ_{44}:-

$$Z_{1654} = \cos\alpha_{23} \tag{9.3}$$

where:-

$$Z_{1654} = \sin\alpha_{34}(X_{165}\sin\theta_{44} + Y_{165}\cos\theta_{44}) + \cos\alpha_{34}Z_{165} \tag{9.4}$$

and:-

$$X_{165} = X_{16}\cos\theta_5 - Y_{16}\sin\theta_5$$
$$Y_{165} = \cos\alpha_{45}(X_{16}\sin\theta_5 + Y_{16}\cos\theta_5) - \sin\alpha_{45}Z_{16}$$
$$Z_{165} = \sin\alpha_{45}(X_{16}\sin\theta_5 + Y_{16}\cos\theta_5) + \cos\alpha_{45}Z_{16} \tag{9.5}$$

In (9.5) the terms X_{16}, Y_{16} and Z_{16} are defined in Appendix III. and are each functions of θ_1 and θ_6 only (see Chapter 4.).

Clearly one may rewrite (9.3) in the form:-

$$L_{416}\sin\theta_5 + M_{416}\cos\theta_5 + N_{416} = \cos\alpha_{23} \tag{9.6}$$

where:-

$$L_{416} = -(Y_4X_{16} + X_4Y_{16})$$
$$M_{416} = (X_4X_{16} - Y_4Y_{16})$$
$$N_{416} = Z_4Z_{16} \tag{9.7}$$

and X_4, Y_4, Z_4 are given in Appendix III.(see also equation (7.8)).

Now, making the substitution (5.1) in (9.6) for $\sin\theta_5$ and $\cos\theta_5$ i.e.:-

$$\sin\theta_5 \equiv 2x_5/(1 + x_5^2)$$
$$\cos\theta_5 \equiv (1 - x_5^2)/(1 + x_5^2) \qquad (5.1)$$
where $$x_5 \equiv \tan(\theta_5/2)$$

and rearranging, one has:-

$$f(x_5) = a_2x_5^2 + a_1x_5 + a_0 = 0 \qquad (9.8)$$

where:-

$$a_2 = N_{416} - M_{416} - \cos\alpha_{23}$$
$$a_1 = 2.L_{416}$$
$$a_0 = N_{416} + M_{416} - \cos\alpha_{23} \qquad (9.9)$$

Equation (9.8) is the required first equation in a suitable form for the elimination of x_5.

9.3.2 <u>Second Equation in θ_1, θ_6 and θ_5.</u>

It is not possible to make use of the secondary equation corresponding to equation (9.3) directly, since this would involve (as extra variables) the offsets S_4 and S_5. Consequently one must use the secondary part of the dual cosine law:-

$$\hat{Z}_{6123} = \cos\hat{\alpha}_{45} \qquad (9.10)$$

which is:-

$$Z_{06123} = -a_{45}\sin\alpha_{45} \qquad (9.11)$$

since this involves all of the fixed mechanism proportions and may be transformed into the required form.

In equation (9.11) the term Z_{06123} is most conveniently written in its symmetric form (see Appendix III.) as follows:-

$$
\begin{aligned}
Z_{06123} = \; & a_{34}Y_{6123} \\
& + S_{33}\sin\alpha_{34}X_{6123} \\
& + a_{23}[(X_{61}\sin\theta_2 + Y_{61}\cos\theta_2)\bar{Z}_3 + Z_{61}\bar{Y}_3] \\
& - S_{22}[(X_{61}\sin\theta_2 + Y_{61}\cos\theta_2)\bar{X}_3 + X_{612}\bar{Y}_3] \\
& + a_{12}\operatorname{cosec}\alpha_{12}[Z_{61}Z_{32} - Z_6\bar{Z}_3] \\
& - S_{11}[(X_{32}\sin\theta_1 + Y_{32}\cos\theta_1)X_6 + X_{321}Y_6] \\
& + a_{61}[(X_{32}\sin\theta_1 + Y_{32}\cos\theta_1)Z_6 + Z_{32}Y_6] \\
& + S_{66}\sin\alpha_{56}X_{3216} \\
& + a_{56}Y_{3216}
\end{aligned} \qquad (9.12)
$$

where the terms X_{6123}, Y_{6123},.....etc., are also defined in Appendix III.

In order to transform (9.12) into an equation in θ_1, θ_6, θ_5 and θ_{44} only, it is natural to examine the coefficients of each term, a_{34},.....,a_{56} separately and to reduce these to the correct form in turn. In order to achieve this end onerequires the following sine, sine-cosine and cosine laws for a spherical hexagon (see Chapter 4):-

$$
\begin{aligned}
X_{6123} &= \sin\alpha_{45}\sin\theta_{44} \\
Y_{6123} &= \sin\alpha_{45}\cos\theta_{44}
\end{aligned} \qquad (9.13)
$$

$$
\begin{aligned}
X_{61}\cos\theta_2 - Y_{61}\sin\theta_2 &= X_{43} \\
X_{61}\sin\theta_2 + Y_{61}\cos\theta_2 &= -Y_{43} \\
Z_{61} &= Z_{43}
\end{aligned} \qquad (9.14)
$$

$$
\begin{aligned}
X_{32}\cos\theta_1 - Y_{32}\sin\theta_1 &= X_{56} \\
X_{32}\sin\theta_1 + Y_{32}\cos\theta_1 &= -Y_{56} \\
Z_{32} &= Z_{56}
\end{aligned} \qquad (9.15)
$$

$$
\begin{aligned}
X_{3216} &= \sin\alpha_{45}\sin\theta_5 \\
Y_{3216} &= \sin\alpha_{45}\cos\theta_5
\end{aligned} \qquad (9.16)
$$

where the terms X_{6123}, Y_{6123},.....,etc., are defined in Chapter 4 and Appendix III.

Thus the coefficients of a_{34} and S_{33} in (9.12) can be immediately replaced using (9.13) whilst those of a_{56} and S_{66} can be similarly dealt with using (9.16). The remaining five terms (i.e. a_{23}, S_{22}, a_{12}, S_{11} and a_{61}) must now be examined in detail.

If one labels these five coefficients as follows:-

$$\begin{aligned} k_1 &= \text{coefficient of } a_{23} \\ k_2 &= \text{coefficient of } S_{22} \\ k_3 &= \text{coefficient of } a_{12} \\ k_4 &= \text{coefficient of } S_{11} \\ k_5 &= \text{coefficient of } a_{61} \end{aligned} \qquad (9.17)$$

then this will facilitate the manipulations.

Hence using (9.14) one may write:-

$$\begin{aligned} k_1 &= (X_{61}\sin\theta_2 + Y_{61}\cos\theta_2)\bar{Z}_3 + Z_{61}\bar{Y}_3 \\ &= -Y_{43}\bar{Z}_3 + Z_{43}\bar{Y}_3 \end{aligned} \qquad (9.18)$$

Applying the definitions of Y_{43}, Z_{43}, $\bar{Y}_3$, $\bar{Z}_3$ from Chapter 4 and rearranging, this reduces to:-

$$\begin{aligned} k_1 = &-(\sin\alpha_{34}\bar{Z}_4 + \cos\alpha_{34}\bar{Y}_4)\cos\theta_3 \\ &- \cos\alpha_{34}\sin\alpha_{45}\sin\theta_4\sin\theta_3 \end{aligned} \qquad (9.19)$$

But from the identity (4.11a) this may be rewritten as:-

$$\begin{aligned} k_1 &= \sin\alpha_{45}(\cos\theta_3\cos\theta_4 - \sin\theta_3\sin\theta_4\cos\alpha_{34}) \\ &= \sin\alpha_{45}\left[(\sin\alpha_{23}\cos\theta_3)\cos\theta_4 - (\sin\alpha_{23}\sin\theta_3)\sin\theta_4\cos\alpha_{34}\right] / \sin\alpha_{23} \end{aligned} \qquad (9.20)$$

and, using the following sine and sine-cosine laws:-

$$\begin{aligned} \sin\alpha_{23}\sin\theta_3 &= X_{1654} \\ \sin\alpha_{23}\cos\theta_3 &= Y_{1654} \end{aligned} \qquad (9.21)$$

which are cyclic permutations of (9.13) or (9.16), equation (9.20) becomes:-

$$k_1 = \sin\alpha_{45}\operatorname{cosec}\alpha_{23}(Y_{1654}\cos\theta_4 - X_{1654}\sin\theta_4\cos\alpha_{34}) \qquad (9.22)$$

which finally simplifies to the following form using the definitions of X_{1654} and Y_{1654} from Chapter 4:-

$$k_1 = \sin\alpha_{45}\operatorname{cosec}\alpha_{23}\left[(X_{16}\sin\theta_5 + Y_{16}\cos\theta_5)Z_4 + Z_{16}Y_4\right] \tag{9.23}$$

Thus the coefficient, k_1, is now in the required form.

Clearly, exactly analogous, but equally lengthy, procedures may be applied to the four remaining coefficients (i.e. k_2, k_3, k_4 and k_5) and the results may be listed as follows:-

Thus, from (9.14), k_2 may be written as:-

$$\begin{aligned} k_2 &= -(X_{61}\sin\theta_2 + Y_{61}\cos\theta_2)\bar{X}_3 - X_{612}\bar{Y}_3 \\ &= Y_{43}\bar{X}_3 - X_{43}\bar{Y}_3 \end{aligned} \tag{9.24}$$

After expanding and rearranging (9.24) one has:-

$$\begin{aligned} k_2 &= \sin\alpha_{45}(X_3\cos\theta_4 - Y_3\sin\theta_4) \\ &= \sin\alpha_{45}X_{165} \end{aligned} \tag{9.25}$$

from the subsidiary sine law.

Similarly one may rearrange k_3 as follows, using the definitions of Z_{61} and Z_{32} (see Chapter 4.):-

$$\begin{aligned} k_3 &= \operatorname{cosec}\alpha_{12}(Z_{61}Z_{32} - Z_6\bar{Z}_3) \\ &= (X_6\sin\theta_1 + Y_6\cos\theta_1)Z_{32} + Z_6Y_{32} \end{aligned} \tag{9.26}$$

and from the subsidiary sine-cosine and cosine laws:-

$$\begin{aligned} -Y_{32} &= (X_{56}\sin\theta_1 + Y_{56}\cos\theta_1) \\ Z_{32} &= Z_{56} \end{aligned} \tag{9.27}$$

one may rewrite (9.26) in the following form, after regrouping terms:-

$$\begin{aligned} k_3 = {} & (Z_{56}X_6 - X_{56}Z_6)\sin\theta_1 \\ & + (Z_{56}Y_6 - Y_{56}Z_6)\cos\theta_1 \end{aligned} \tag{9.28}$$

Finally, expanding and regrouping (9.28) using the identity (4.10a) one obtains:-

$$\begin{aligned} k_3 &= \sin\alpha_{45}\left[\sin\theta_5(U_{16}\sin\alpha_{56} + V_{16}\cos\alpha_{56}) + \cos\theta_5W_{16}\right] \\ &= \sin\alpha_{45}W_{165} \end{aligned} \tag{9.29}$$

where U_{16}, V_{16} and W_{16} are expressions for the polar spherical hexagon (see Chapter 4) and are defined as follows:-

$$U_{16} = \sin\theta_1 \sin\alpha_{61}$$

$$V_{16} = -(\cos\theta_1 \sin\theta_6 + \sin\theta_1 \cos\theta_6 \cos\alpha_{61})$$

$$W_{16} = (\cos\theta_1 \cos\theta_6 - \sin\theta_1 \sin\theta_6 \cos\alpha_{61}) \qquad (9.30)$$

Now the remaining two coefficients of (9.12), (i.e. k_4 and k_5) may clearly be written as follows, with reference to (9.23) and (9.25), since Z_{06123} is symmetric:-

$$k_4 = -(X_{32}\sin\theta_1 + Y_{32}\cos\theta_1)X_6 - X_{321}Y_6$$
$$= \sin\alpha_{45} X_{234} \qquad (9.31)$$

and:-

$$k_5 = (X_{32}\sin\theta_1 + Y_{32}\cos\theta_1)Z_6 + Z_{32}Y_6$$
$$= \sin\alpha_{45} \operatorname{cosec}\alpha_{61} [(X_{23}\sin\theta_{44} + Y_{23}\cos\theta_{44})\bar{Z}_5 + Z_{23}\bar{Y}_5] \qquad (9.32)$$

However, using the three subsidiary laws:-

$$X_{234} \equiv (X_{23}\cos\theta_{44} - Y_{23}\sin\theta_{44}) = X_{65}$$
$$(X_{23}\sin\theta_{44} + Y_{23}\cos\theta_{44}) = -Y_{65}$$
$$Z_{23} = Z_{65} \qquad (9.33)$$

equations (9.31) and (9.32) are transformed into the following:-

$$k_4 = \sin\alpha_{45} X_{65} \qquad (9.34)$$

and:-

$$k_5 = \sin\alpha_{45}(\cos\theta_6\cos\theta_5 - \sin\theta_6\sin\theta_5\cos\alpha_{56})$$
$$= \sin\alpha_{45} W_{65} \qquad (9.35)$$

It is now possible to rewrite the whole of (9.12) in a form which involves only θ_1, θ_6, θ_5 and θ_{44}. Thus, from (9.23), (9.25), (9.29), (9.34) and (9.35), together with (9.13) and (9.16), equation (9.11) becomes:-

$$
\begin{aligned}
-a_{45}\sin\alpha_{45} = \; & a_{34}\sin\alpha_{45}\cos\theta_{44} \\
& + S_{33}\sin\alpha_{45}\sin\alpha_{34}\sin\theta_{44} \\
& + a_{23}\sin\alpha_{45}\,\mathrm{cosec}\,\alpha_{23}\left[(X_{16}\sin\theta_5 + Y_{16}\cos\theta_5)Z_4 + Z_{16}Y_4\right] \\
& + S_{22}\sin\alpha_{45}X_{165} \\
& + a_{12}\sin\alpha_{45}W_{165} \\
& + S_{11}\sin\alpha_{45}X_{65} \\
& + a_{61}\sin\alpha_{45}W_{65} \\
& + S_{66}\sin\alpha_{45}\sin\alpha_{56}\sin\theta_5 \\
& + a_{56}\sin\alpha_{45}\cos\theta_5 \; .
\end{aligned}
\qquad (9.36)
$$

After dividing throughout by $\sin\alpha_{45}$, equation (9.36) may be arranged in the form:-

$$P_{416}\sin\theta_5 + Q_{416}\cos\theta_5 + R_{416} = -a_{45} \qquad (9.37)$$

where:-

$$
\begin{aligned}
P_{416} = \; & a_{23}\,\mathrm{cosec}\,\alpha_{23}Z_4X_{16} - S_{22}Y_{16} \\
& + a_{12}(U_{16}\sin\alpha_{56} + V_{16}\cos\alpha_{56}) - S_{11}Y_6 \\
& - a_{61}\cos\alpha_{56}\sin\theta_6 + S_{66}\sin\alpha_{56}
\end{aligned}
$$

$$
\begin{aligned}
Q_{416} = \; & a_{23}\,\mathrm{cosec}\,\alpha_{23}Z_4Y_{16} + S_{22}X_{16} \\
& + a_{12}W_{16} + S_{11}X_6 \\
& + a_{61}\cos\theta_6 + a_{56}
\end{aligned}
$$

$$
\begin{aligned}
R_{416} = \; & a_{34}\cos\theta_{44} + S_{33}\sin\alpha_{34}\sin\theta_{44} \\
& + a_{23}\,\mathrm{cosec}\,\alpha_{23}Y_4Z_{16}
\end{aligned}
\qquad (9.38)
$$

Hence making the substitutions (5.1) in (9.37) for $\sin\theta_5$, and $\cos\theta_5$ one obtains, after rearrangement:-

$$g(x_5) = b_2x_5^2 + b_1x_5 + b_0 = 0 \qquad (9.39)$$

where:-

$$
\begin{aligned}
b_2 &= R_{416} - Q_{416} + a_{45} \\
b_1 &= 2.P_{416} \\
b_0 &= R_{416} + Q_{416} + a_{45}
\end{aligned}
\qquad (9.40)
$$

and clearly (9.39) is in the desired form for the elimination of $x_5(\equiv \tan(\theta_5/2))$.

9.3.3 Elimination Procedure.

The elimination of x_5 between equations (9.8) and (9.39) may now easily be carried out using the Bézoutian, (5.8), for two quadratics and the input-output equation for the RRRPCR mechanism is thus of the form:-

$$(a_2b_0 - a_0b_2)^2 - (a_2b_1 - a_1b_2)(a_1b_0 - a_0b_1) = 0 \qquad (9.41)$$

where a_2, a_1, a_0 and b_2, b_1, b_0 are defined by (9.9) and (9.40) respectively. Since each of these terms is a quadratic expression in x_6 it is clear from (9.41) that the input-output equation is of degree eight in the half-tangent of the output angular displacement. One may express these coefficients in terms of $x_6(\equiv \tan(\theta_6/2))$ by means of the substitution (5.1),

i.e.:-

$$\begin{aligned} \sin\theta_6 &\equiv 2x_6/(1 + x_6^2) \\ \cos\theta_6 &\equiv (1 - x_6^2)/(1 + x_6^2) \end{aligned} \qquad (5.1)$$

as follows:-

$$\begin{aligned} a_2 &= p_{22}x_6^2 + p_{12}x_6 + p_{02} \\ a_1 &= p_{21}x_6^2 + p_{11}x_6 + p_{01} \\ a_0 &= p_{20}x_6^2 + p_{10}x_6 + p_{00} \end{aligned} \qquad (9.42)$$

and:-

$$\begin{aligned} b_2 &= q_{22}x_6^2 + q_{12}x_6 + q_{02} \\ b_1 &= q_{21}x_6^2 + q_{11}x_6 + q_{01} \\ b_0 &= q_{20}x_6^2 + q_{10}x_6 + q_{00} \end{aligned} \qquad (9.43)$$

where the terms p_{ij} and q_{ij} are each a function of the input angle (θ_1) only, and are listed in Appendix VII.

Furthermore, it is also possible to express a_2, a_1,...,b_0 as quadratics in $x_1(\equiv \tan(\theta_1/2))$ and again, from (9.41), it is clear that the input-output equation is of degree eight in the input variable. These results are in agreement with the predicted degree for the RRRPCR mechanism (see Chapter 2.).

9.4 Displacement Analysis.

Solving the input-output equation (9.41) for x_6, one obtains, in general, eight distinct real values for the output angular displacement (i.e. θ_{61}, θ_{62}...., θ_{68}), for each value of the input angular displacement, θ_1. The eight ordered pairs (θ_1, θ_{61}), (θ_1, θ_{62}),....,(θ_1, θ_{68}) thus produced will then each give rise to corresponding values for the remaining linkage variables (θ_5, S_5, S_4, θ_3, θ_2) using procedures outlined below.

Thus θ_5 may be determined from either of the two expressions for the common root of (9.8) and (9.39), which are derived from the Bézoutian (5.8). (see Chapter 5). These expressions are:-

$$\begin{aligned} x_5 &= -(a_2b_0)/(a_2b_1) \\ &= -(a_2b_0 - a_0b_2)/(a_2b_1 - a_1b_2) \end{aligned} \tag{9.44a}$$

or:-

$$\begin{aligned} x_5 &= -(a_1b_0)/(a_2b_0) \\ &= -(a_1b_0 - a_0b_1)/(a_2b_0 - a_0b_2) \end{aligned} \tag{9.44b}$$

where:- $x_5 \equiv \tan(\theta_5/2)$

Having determined θ_1, θ_6 and θ_5 it is a simple matter to obtain the unique value of θ_3 from either of the two fundamental half-tangent laws:-

$$x_3 = -(Y_{1654} - \sin\alpha_{23})/X_{1654} \tag{9.45a}$$

or:-

$$x_3 = X_{1654}/(Y_{1654} + \sin\alpha_{23}) \tag{9.45b}$$

where:-

$$x_3 \equiv \tan(\theta_3/2)$$

(see equations (5.32) and (5.33)).

Here:-

$$\begin{aligned} X_{1654} &= X_{165}\cos\theta_{44} - Y_{16}\sin\theta_{44} \\ Y_{1654} &= \cos\alpha_{34}(X_{165}\sin\theta_{44} + Y_{165}\cos\theta_{44}) - \sin\alpha_{34}Z_{165} \end{aligned} \tag{9.46}$$

In a similar manner one may obtain θ_2 from a cyclic permutation of (9.45a,b) for known values of θ_1, θ_6 and θ_5. Thus:-

$$x_2 = -(Y_{4561} - \sin\alpha_{23})/X_{4561} \tag{9.47a}$$

or:-

$$x_2 = X_{4561}/(Y_{4561} + \sin\alpha_{23}) \tag{9.47b}$$

where:-

$$x_2 \equiv \tan(\theta_2/2)$$

and:-

$$\begin{aligned} X_{4561} &= X_{456}\cos\theta_1 - Y_{456}\sin\theta_1 \\ Y_{4561} &= \cos\alpha_{12}(X_{456}\sin\theta_1 + Y_{456}\cos\theta_1) - \sin\alpha_{12}Z_{456} \end{aligned} \tag{9.48}$$

The sliding displacement S_4 may be determined from the secondary component of the dual subsidiary cosine law:-

$$\hat{Z}_{16} = \hat{Z}_{34} \tag{9.49}$$

which is:-

$$Z_{016} = Z_{034} \tag{9.50}$$

where:-

$$\begin{aligned} Z_{016} = \; & a_{56}Y_{16} \\ & + S_{66}\sin\alpha_{56}X_{16} \\ & + a_{61}\operatorname{cosec}\alpha_{61}(\bar{Z}_1 Z_6 - \cos\alpha_{12}\cos\alpha_{56}) \\ & + S_{11}\sin\alpha_{12}X_{61} \\ & + a_{12}Y_{61} \end{aligned} \tag{9.51}$$

and:-

$$\begin{aligned} Z_{034} = \; & a_{45}Y_{34} \\ & + S_4\sin\alpha_{45}X_{34} \\ & + a_{34}\operatorname{cosec}\alpha_{34}(Z_3\bar{Z}_4 - \cos\alpha_{23}\cos\alpha_{45}) \\ & + S_{33}\sin\alpha_{23}X_{43} \\ & + a_{23}Y_{43} \end{aligned} \tag{9.52}$$

In a similar manner the sliding displacement S_5 may be calculated from the secondary component of the dual cosine law:-

$$\hat{Z}_{2165} = \cos\hat{\alpha}_{34} \tag{9.53}$$

which is:-

$$Z_{02165} = -a_{34}\sin\alpha_{34} \tag{9.54}$$

where:-

$$\begin{aligned} Z_{02165} = \; & a_{45}Y_{2165} \\ & + S_5\sin\alpha_{45}X_{2165} \\ & + a_{56}[(X_{21}\sin\theta_6 + Y_{21}\cos\theta_6)Z_5 + Z_{21}Y_5] \\ & - S_{66}[(X_{21}\sin\theta_6 + Y_{21}\cos\theta_6)X_5 + X_{216}Y_5] \\ & + a_{61}\mathrm{cosec}\alpha_{61}[Z_{21}Z_{56} - \bar{Z}_2Z_5] \\ & - S_{11}[(X_{56}\sin\theta_1 + Y_{56}\cos\theta_1)\bar{X}_2 + X_{561}\bar{Y}_2] \\ & + a_{12}[(X_{56}\sin\theta_1 + Y_{56}\cos\theta_1)\bar{Z}_2 + Z_{56}\bar{Y}_2] \\ & + S_{22}\sin\alpha_{23}X_{5612} \\ & + a_{23}Y_{5612} \end{aligned} \tag{9.55}$$

and the terms X_{2165}, Y_{2165}....,etc., are defined in Appendix III.

This completes the displacement analysis for the RRRPCR six-link mechanism.

9.5 Numerical Results.

The input-output equation (9.41) for the RRRPCR mechanism was solved numerically for a given set of mechanism proportions, and graphs of the output angular variable, θ_6, and remaining variables θ_5, θ_3, θ_2, S_4 and S_5 against the input, θ_1, were plotted (see Figures 9.2, 9.3, 9.4, 9.5, 9.6, and 9.7 respectively).

In addition since a combination of revolute and prismatic pairs may be used to simulate a cylindric pair, equation (9.41) may be used to generate the input-output relationship for the five-link RRCCR mechanism. Figure 9.1 shows a plot of θ_6 vs θ_1 for this mechanism.

The following sets of data for the mechanism proportions were chosen in each case:-

9.5.1 RRCCR Mechanism.

$$\begin{array}{lll} a_{12} = 1.0 \text{ ins.} & \alpha_{12} = 30 \text{ deg.} & S_{11} = 1.0 \text{ ins.} \\ a_{23} = 3.5 \text{ ins.} & \alpha_{23} = 45 \text{ deg.} & S_{22} = -6.0 \text{ ins.} \\ a_{34} = 0.0 \text{ ins.} & \alpha_{34} = 0 \text{ deg.} & S_{33} = 0.0 \text{ ins.} \\ a_{45} = 2.5 \text{ ins.} & \alpha_{45} = 45 \text{ deg.} & S_{66} = 3.0 \text{ ins.} \\ a_{56} = 2.0 \text{ ins.} & \alpha_{56} = 60 \text{ deg.} & \\ a_{61} = 3.0 \text{ ins.} & \alpha_{61} = 20 \text{ deg.} & \theta_{44} = 0 \text{ deg.} \end{array} \qquad (9.56)$$

Here the third revolute pair has been superimposed on the fourth sliding pair by selecting the proportions, $a_{34} = \alpha_{34} = S_{33} = \theta_{44} = 0$. The remaining proportions were selected to give the same RRCCR mechanism as that previously analysed by Yuan [48]. Figure 9.1 is identical to the input-output relationship presented in [48].

9.5.2 RRRPCR Mechanism.

$$\begin{array}{lll} a_{12} = 2.0 \text{ ins} & \alpha_{12} = 90 \text{ deg.} & S_{11} = 8.0 \text{ ins.} \\ a_{23} = 2.0 \text{ ins.} & \alpha_{23} = 80 \text{ deg.} & S_{22} = -4.0 \text{ ins.} \\ a_{34} = 1.0 \text{ ins.} & \alpha_{34} = 10 \text{ deg.} & S_{33} = 0.0 \text{ ins.} \\ a_{45} = 2.0 \text{ ins.} & \alpha_{45} = 90 \text{ deg.} & S_{66} = -2.0 \text{ ins.} \\ a_{56} = 2.0 \text{ ins.} & \alpha_{56} = 90 \text{ deg.} & \\ a_{61} = 4.0 \text{ ins.} & \alpha_{61} = 90 \text{ deg.} & \theta_{44} = 0 \text{ deg.} \end{array} \qquad (9.57)$$

These proportions were chosen to yield eight real closures for various ranges of the input angular displacement, and the results have been plotted in Figures 9.2-9.7 inclusive. On these graphs, the turning points (which occur at sixteen distinct values of the input variable, θ_1) are labelled 1-16 in order to identify easily the different closures.

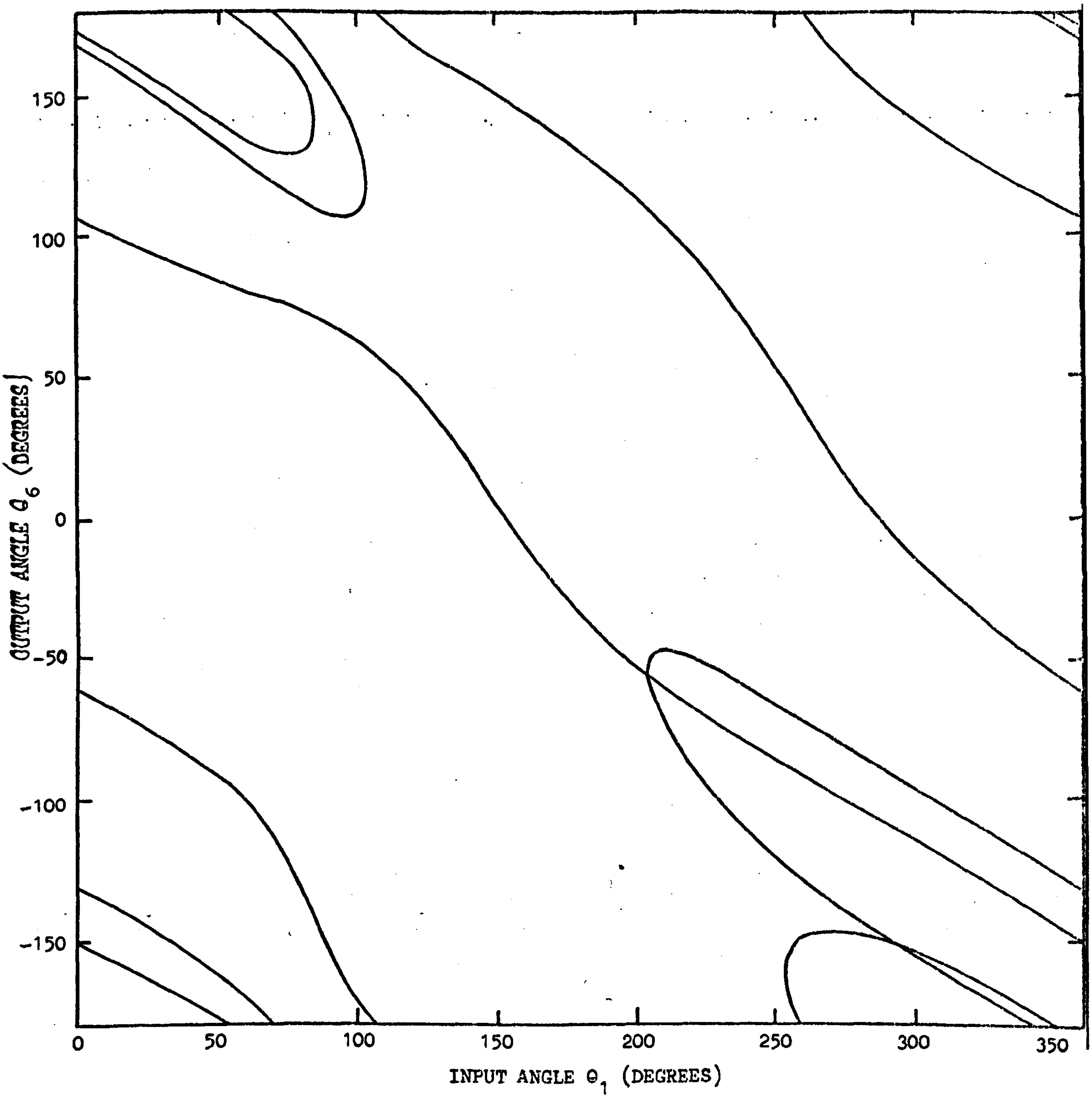


Figure 9.1 Graph of Input-Output Relationship (i.e. θ_6 vs θ_1) for the Six-Link RRRPCR Mechanism with Proportions chosen to Reduce the Latter to the Five-Link RRCCR Mechanism.

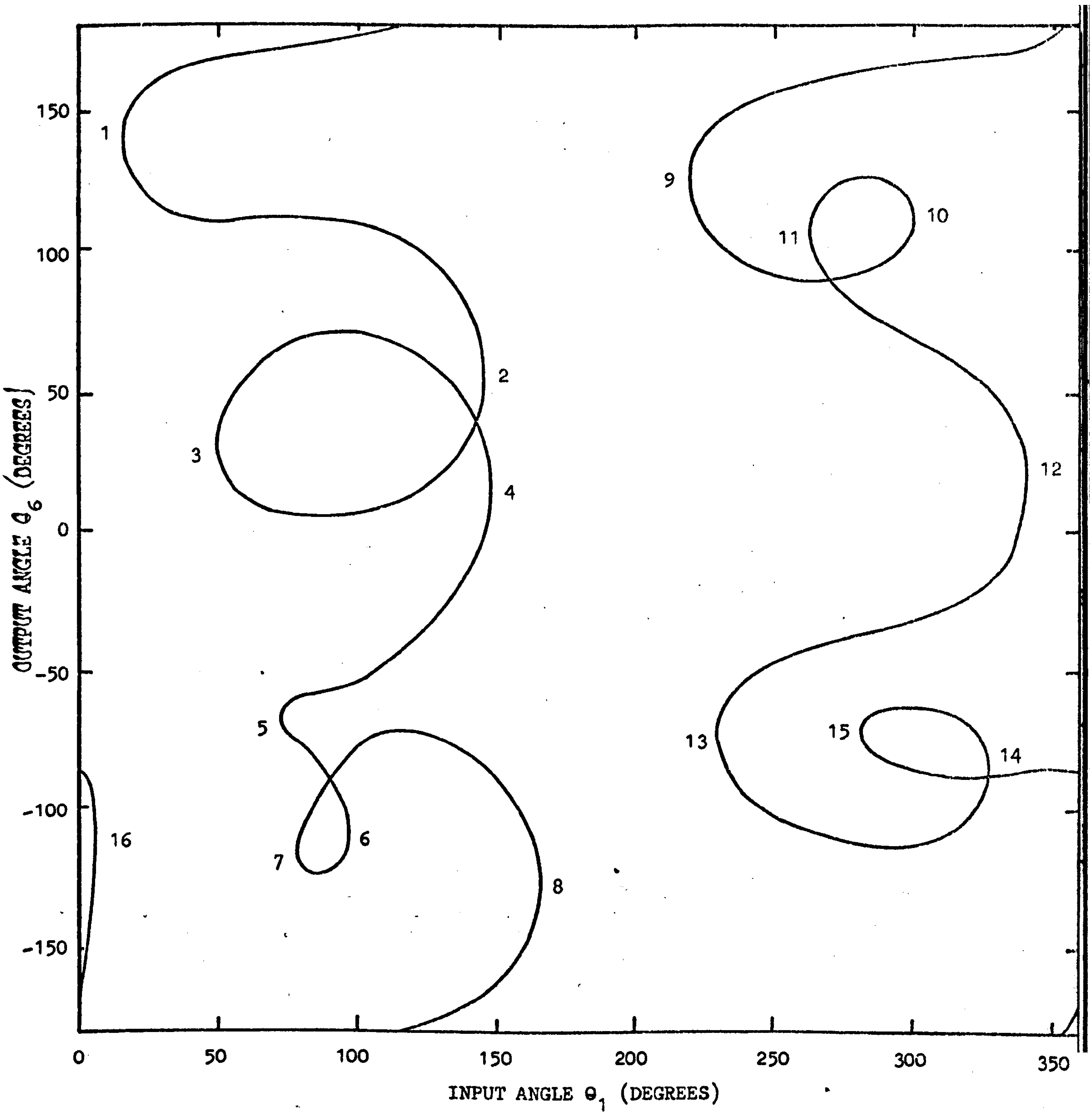


Figure 9.2 Graph of Input-Output Relationship (i.e. θ_6 vs θ_1) for the Six-Link RRRPCR Mechanism.

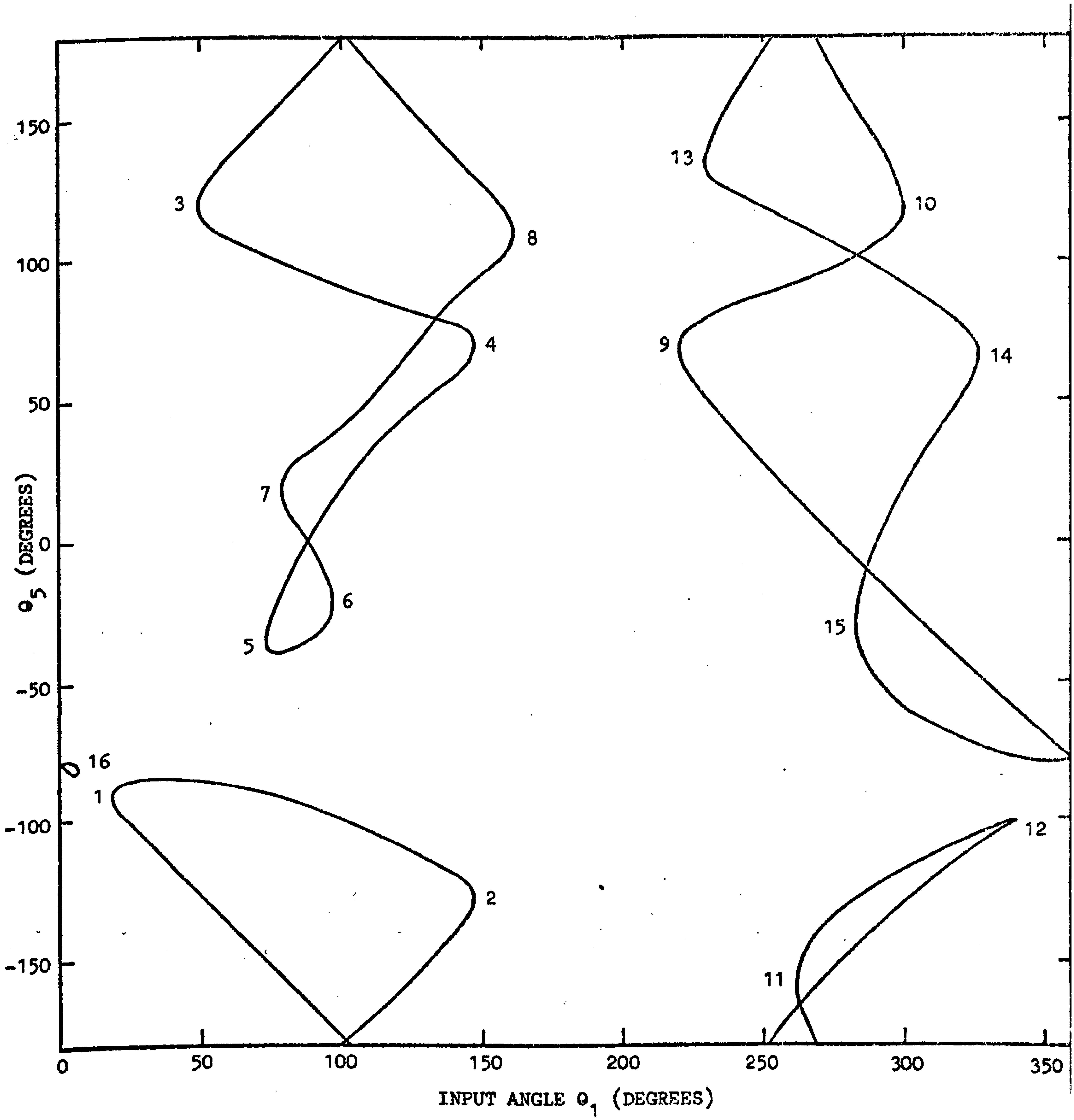


Figure 9.3 Graph of Θ_5 vs Θ_1 for the Six-Link RRRPCR Mechanism.

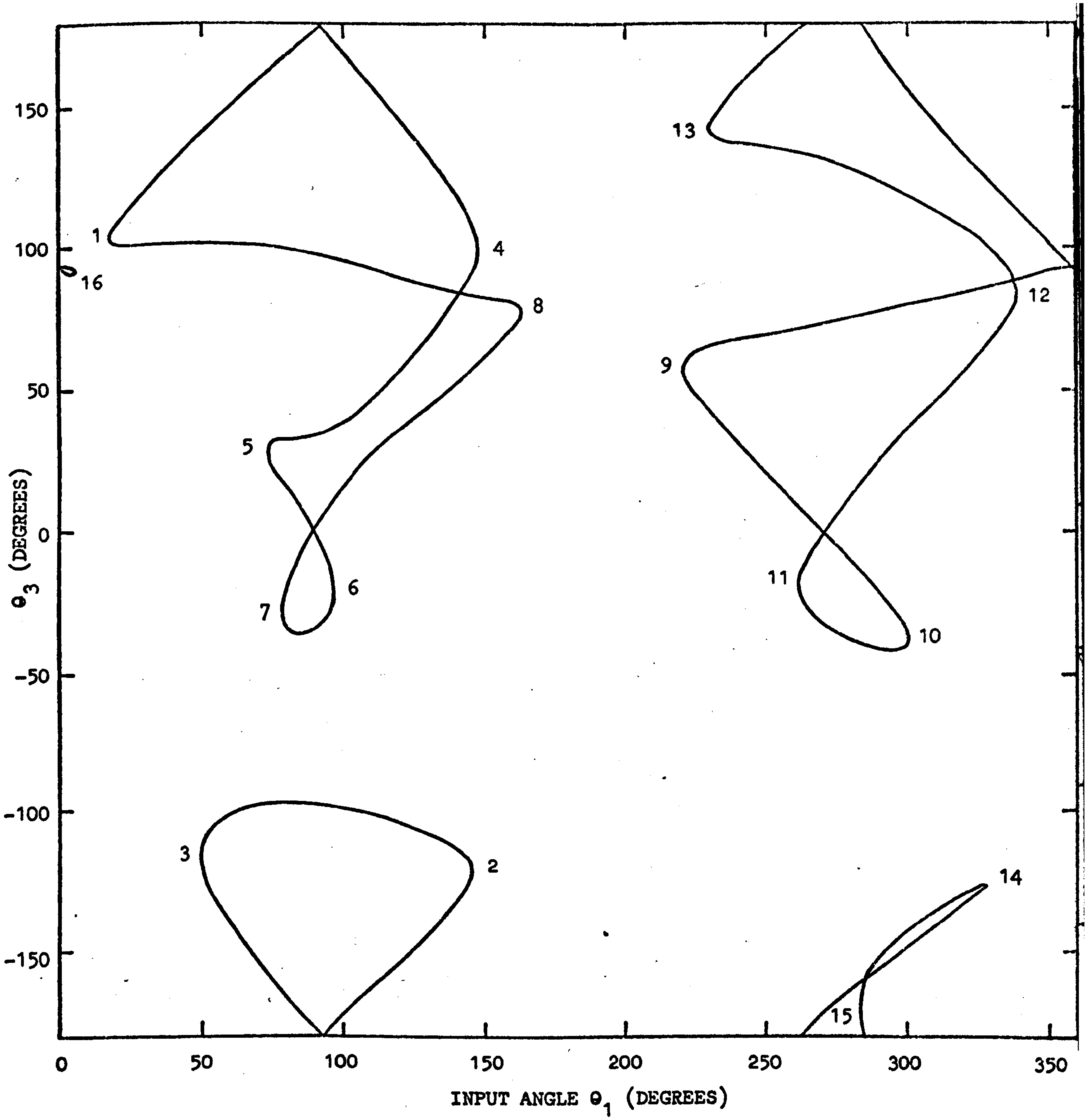


Figure 9.4 Graph of θ_3 vs θ_1 for the Six-Link RRRPCR Mechanism.

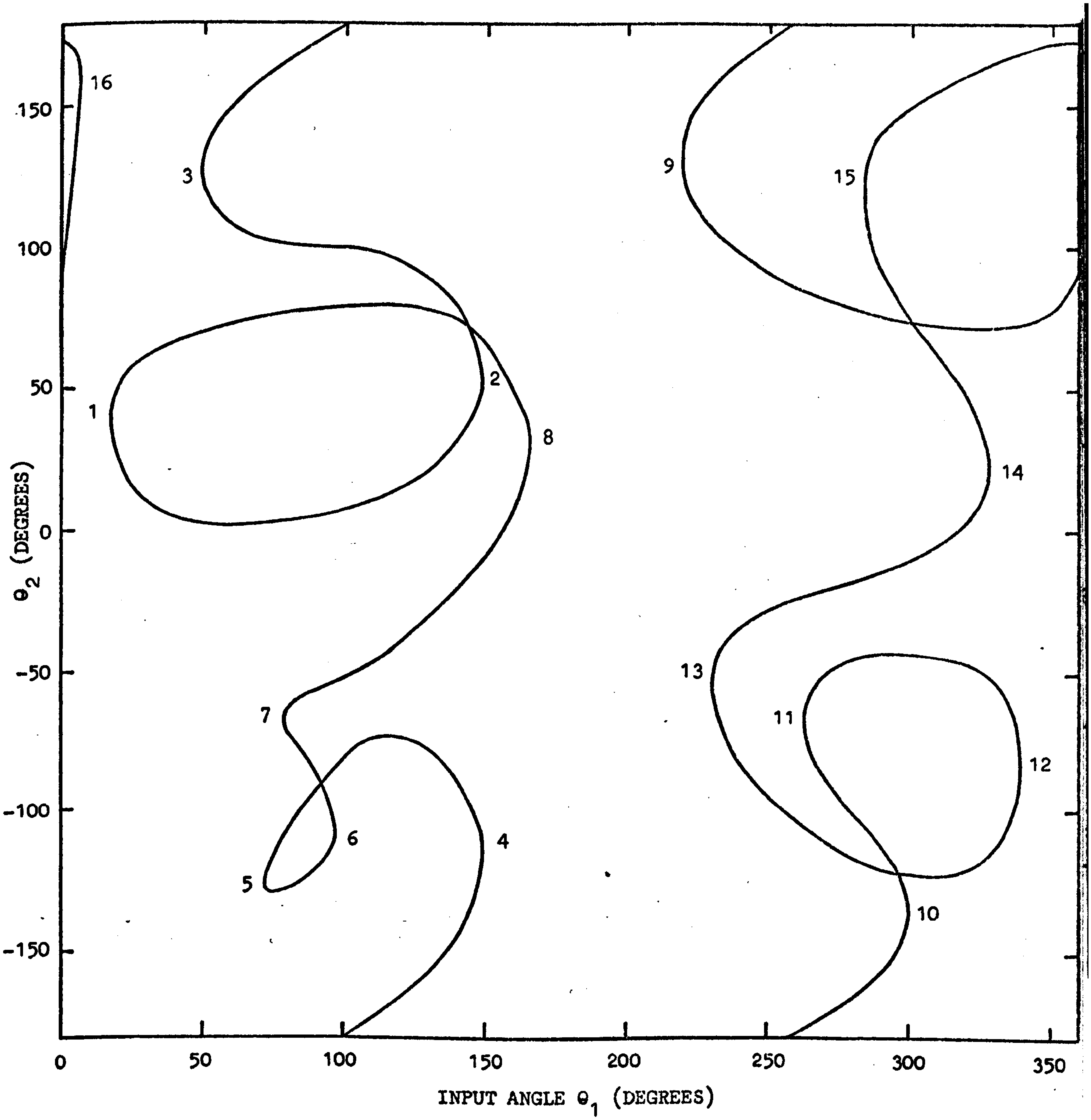


Figure 9.5 Graph of θ_2 vs θ_1 for the Six-Link RRRPCR Mechanism.

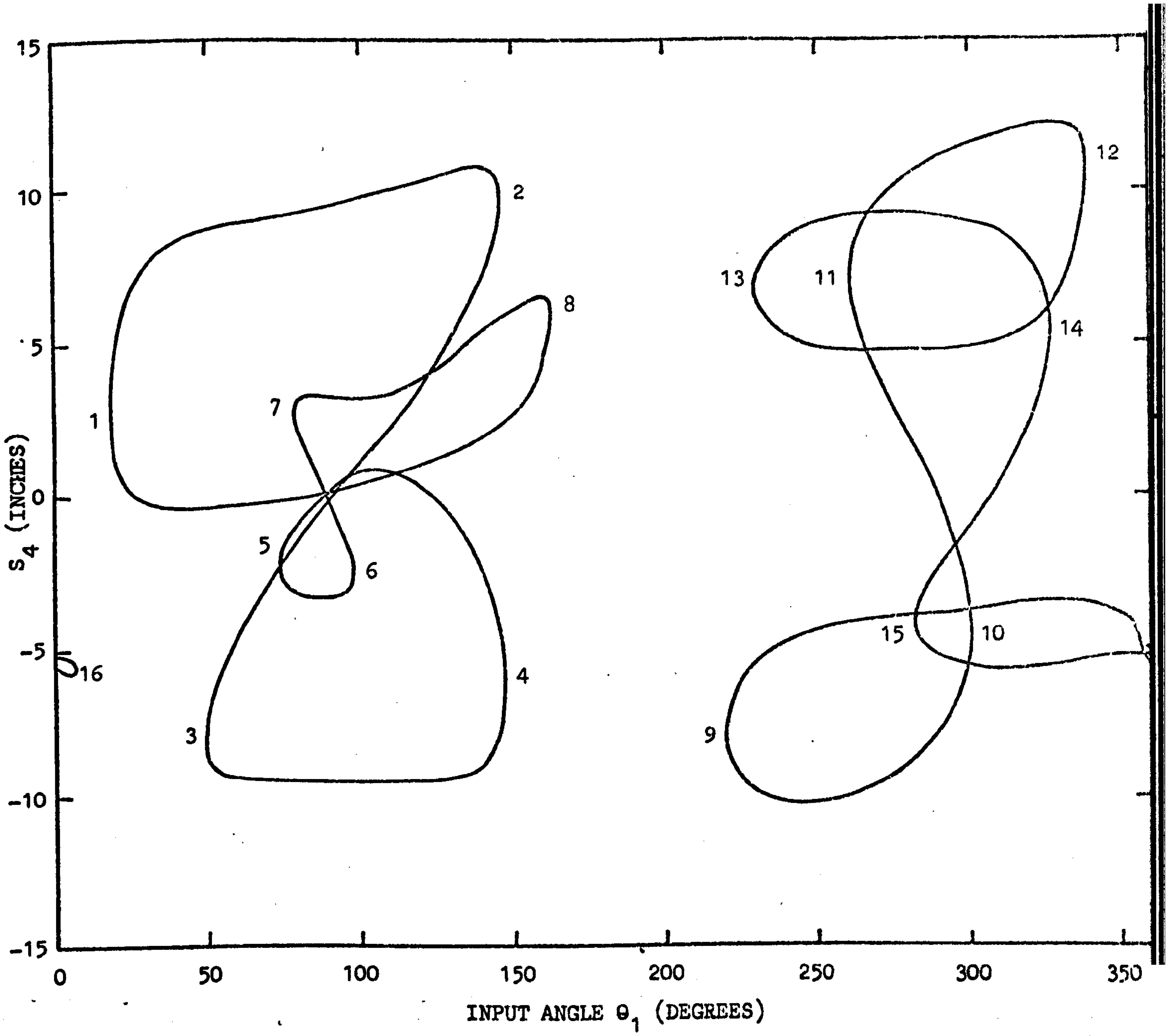


Figure 9.6 Graph of S_4 vs Θ_1 for the Six-Link RRRPCR Mechanism.

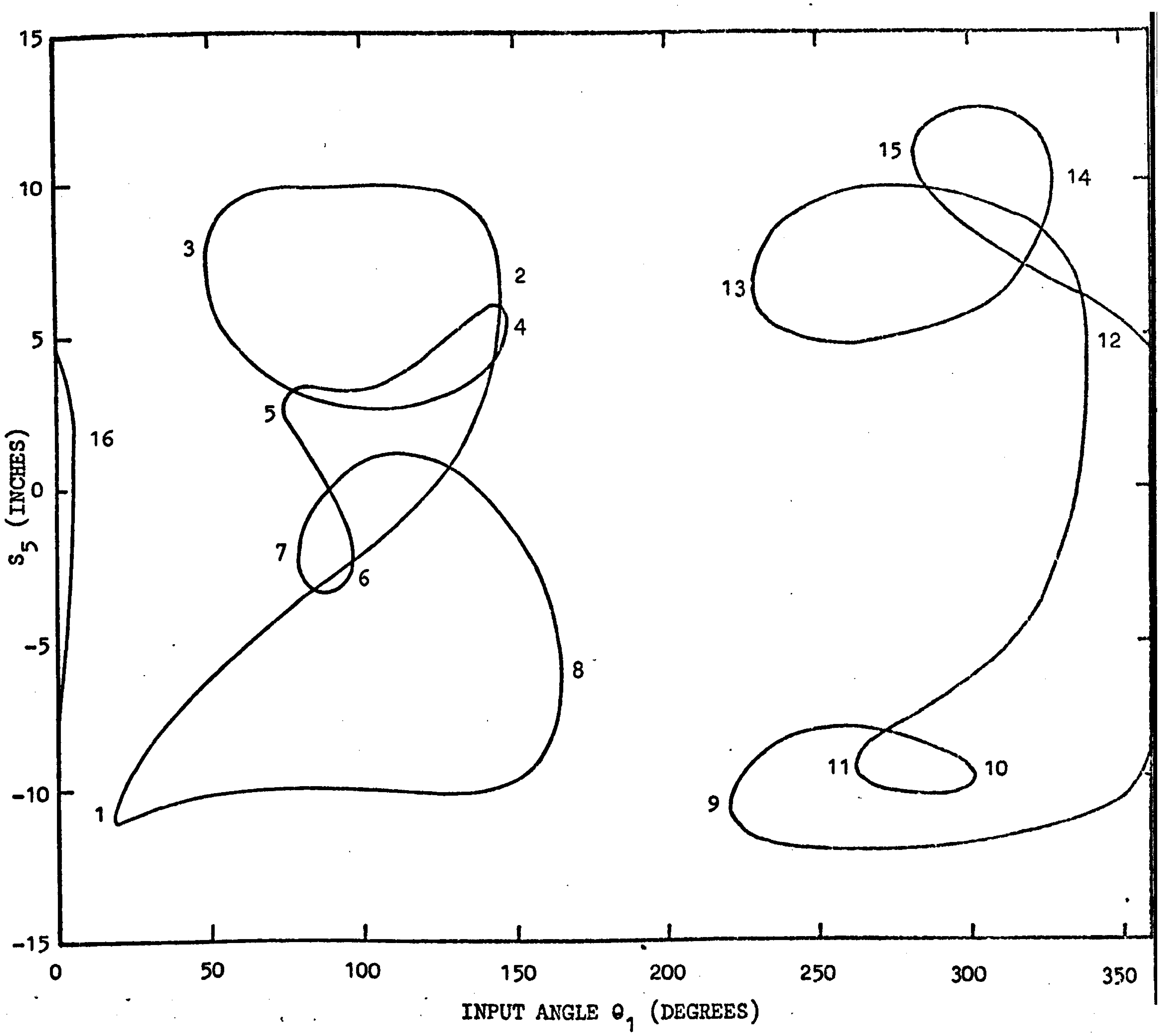


Figure 9.7 Graph of S_5 vs θ_1 for the Six-Link RRRPCR Mechanism.

CHAPTER 10

A DISPLACEMENT ANALYSIS

OF

SPATIAL SIX-LINK 5R-C MECHANISMS

10.1 Introduction.

For all spatial five-link 3R-2C (Chapter 6) and six-link 4R-P-C (Chapters 7, 8 and 9) mechanisms, the input-output equation must be derived by eliminating a single angular displacement in one operation from two simultaneous equations (see Chapter 5). However, in the case of spatial six-link 5R-C mechanisms the problem formulation is far more difficult. It is no longer possible to write equations involving the input, output and a single extraneous angular displacement, and one must eliminate two extraneous angular displacements. Furthermore, it is now well established that this elimination must be performed in one operation to avoid the introduction of extraneous roots into the final eliminant.

Now for two quadratics in one variable (see equations (5.3) and (5.4)) for example, it is possible to derive the following two conditions on their coefficients:-

(i) The condition that the two polynomials should possess a common zero or root.

(ii) The condition that they should possess a common linear factor.

The former condition is known as the eliminant whilst the latter is termed the resultant (see Bôcher [2].). However, in this case the two conditions are identical and the terms 'eliminant' and 'resultant' are used synonymously.

For polynomials in more than one variable, however, the above two conditions are no longer synonymous since if two or more such equations possess, for example, a single common linear factor they clearly possess an infinite number of zeroes or roots.

e.g.:-

$$f(x, y) = (x + y - 2)(x^2 - 2x + 4) = 0 \tag{10.1}$$

$$g(x, y) = (x + y - 2)(x^3 + 4x^2 - 1) = 0 \tag{10.2}$$

Thus (10.1) and (10.2) have a common linear factor but are satisfied by an infinite number of ordered pairs, (x, y).

Now, when considering the 5R-C mechanisms, one requires the eliminant (rather than the resultant) of a system of polynomials in two extraneous variables in order to obtain the input-output relationships, and since the two conditions are no longer synonymous, one cannot use expressions corresponding to the bigradient or Bézoutian. This is because the latter are essentially resultants (see Bôcher [2]).

Nevertheless, it is still possible to obtain the required eliminant for the 5R-C mechanisms in determinantal form (see also Salmon [31]). At the outset one would expect to derive three equations from which to eliminate the two extraneous variables, although this has not proved to be the case. In this chapter a set of four non-linear equations in two unknown angular displacements is derived, from which the input-output equations for both the RRRRCR and the RRCRRR mechanisms may be obtained.

The set is derived from the fundamental half-tangent laws for the spatial hexagon and each equation is a function of the input, output and two extraneous angular displacements. This new problem formulation is a significant result since it simplifies the elimination process and provides solutions (for the spatial six-link RRRRCR and RRCRRR mechanisms) of a much lower degree than any quoted elsewhere (see [16, 17]). In addition, the input-output equations contain, as special cases, eighth-degree polynomials for spatial six-link 4R-P-C slider-crank mechanisms. These results are, again, in agreement with the predictions of Chapter 2.

10.2 Description of the Six-Link RRRRCR and RRCRRR Mechanisms.

The six-link RRRRCR and RRCRRR (an inversion of the RRRRCR) spatial mechanisms are illustrated by Figure 2.28 and are represented mathematically by the following six dual sides and six dual angles:-

$$\hat{\alpha}_{12} = \alpha_{12} + \varepsilon\, a_{12}$$
$$\hat{\alpha}_{23} = \alpha_{23} + \varepsilon\, a_{23}$$
$$\hat{\alpha}_{34} = \alpha_{34} + \varepsilon\, a_{34}$$
$$\hat{\alpha}_{45} = \alpha_{45} + \varepsilon\, a_{45}$$
$$\hat{\alpha}_{56} = \alpha_{56} + \varepsilon\, a_{56}$$
$$\hat{\alpha}_{61} = \alpha_{61} + \varepsilon\, a_{61} \qquad (10.3)$$

$$\hat{\theta}_1 = \theta_1 + \varepsilon\, S_{11}$$
$$\hat{\theta}_2 = \theta_2 + \varepsilon\, S_{22}$$
$$\hat{\theta}_3 = \theta_3 + \varepsilon\, S_{33}$$
$$\hat{\theta}_4 = \theta_4 + \varepsilon\, S_{44}$$
$$\hat{\theta}_5 = \theta_5 + \varepsilon\, S_{5}$$
$$\hat{\theta}_6 = \theta_6 + \varepsilon\, S_{66} \qquad (10.4)$$

where $\varepsilon^2 = 0$, and all fixed mechanism proportions have double or repeated suffices. For the RRRRCR mechanism the input and output angular displacements are respectively θ_1 and θ_6, and the frame is the constant dual side, $\hat{\alpha}_{61}$, whilst for the RRCRRR the input, output and frame are respectively θ_1, θ_2 and $\hat{\alpha}_{12}$.

10.3 Derivation of Input-Output Equations for the RRRRCR and RRCRRR Mechanisms.

The input-output relationships for the RRRRCR and RRCRRR inversions may be obtained from the same set of four equations, which is derived below.

10.3.1 Derivation of Four Equations in θ_6, θ_1, θ_2 and x_3.

For the spherical hexagon it is possible to write the two fundamental half-tangent laws relating θ_6, θ_1, θ_2, θ_4 and x_3, expressed by equations (5.35) and (5.36) and hence, after introducing the dual symbol, one has the following two dual laws for a spatial hexagon:-

$$(\hat{X}_{612} + \hat{\bar{X}}_4)\hat{x}_3 + (\hat{Y}_{612} + \hat{\bar{Y}}_4) = \hat{0} \qquad (10.5)$$
$$(\hat{Y}_{612} - \hat{\bar{Y}}_4)\hat{x}_3 - (\hat{X}_{612} - \hat{\bar{X}}_4) = \hat{0} \qquad (10.6)$$

The primary parts of (10.5) and (10.6) are equations (5.35) and (5.36) respectively i.e.:-

$$(X_{612} + \bar{X}_4)x_3 + (Y_{612} + \bar{Y}_4) = 0 \qquad (5.35)$$

$$(Y_{612} - \bar{Y}_4)x_3 - (X_{612} - \bar{X}_4) = 0 \qquad (5.36)$$

whilst the secondary equations corresponding to these are given, after regrouping terms, by (5.62) and (5.63) (see Chapter 5). i.e.:-

$$S_{33}[(X_{612} + \bar{X}_4)x_3^2 + (X_{612} + \bar{X}_4)]/2 + (X_{612} + \bar{X}_4)_0 x_3 + (Y_{612} + \bar{Y}_4)_0 = 0 \qquad (5.62)$$

$$S_{33}[(Y_{612} - \bar{Y}_4)x_3^2 + (Y_{612} - \bar{Y}_4)]/2 + (Y_{612} - \bar{Y}_4)_0 x_3 - (X_{612} - \bar{X}_4)_0 = 0 \qquad (5.63)$$

Equations (5.62) and (5.63) are quadratic in x_3 at present but it is possible to reduce them to linear form. This may be achieved by writing the subsidiary sine and sine-cosine laws (4.70 a,b) in terms of $x_3(\equiv \tan(\theta_3/2))$ as follows:-

Thus $$(X_{612}\cos\theta_3 - Y_{612}\sin\theta_3) = \bar{X}_4 \qquad (4.70a)$$

$$(X_{612}\sin\theta_3 + Y_{612}\cos\theta_3) = -\bar{Y}_4 \qquad (4.70b)$$

and, making the substitutions (5.1) for $\sin\theta_3$ and $\cos\theta_3$, one has, after regrouping terms:-

$$(X_{612} + \bar{X}_4)x_3^2 = -2.Y_{612}x_3 + (X_{612} - \bar{X}_4) \qquad (10.7)$$

and:- $$(Y_{612} - \bar{Y}_4)x_3^2 = 2.X_{612}x_3 + (Y_{612} + \bar{Y}_4) \qquad (10.8)$$

Clearly (10.7) may be used with (5.62) to give:-

$$[(X_{612} + \bar{X}_4)_0 - S_{33}Y_{612}]x_3 + [S_{33}X_{612} + (Y_{612} + \bar{Y}_4)_0] = 0 \qquad (10.9)$$

whilst from (5.63) and (10.8) one obtains:-

$$[(Y_{612} - \bar{Y}_4)_0 + S_{33}X_{612}]x_3 + [S_{33}Y_{612} - (X_{612} - \bar{X}_4)_0] = 0 \qquad (10.10)$$

Thus, at this stage one has four equations ((5.35), (5.36), (10.9) and (10.10)) which involve θ_6, θ_1, θ_2, x_3 and θ_4. It is now necessary to remove the terms $\sin\theta_4$ and $\cos\theta_4$ from these, without unduly increasing their degree.

Now from the identities (4.10b) and (4.11b) one may write:-

$$Y_4 \equiv \cot\alpha_{45} Z_4 - \cos\alpha_{34} \operatorname{cosec}\alpha_{45} \tag{10.11}$$

$$\bar{Y}_4 \equiv \cot\alpha_{34} \bar{Z}_4 - \cos\alpha_{45} \operatorname{cosec}\alpha_{34} \tag{10.12}$$

In addition one has the primary and secondary parts of the dual subsidiary cosine law:-

$$\hat{Z}_{612} = \hat{\bar{Z}}_4 \tag{10.13}$$

which are:-

$$Z_{612} = \bar{Z}_4 \ (\equiv Z_4) \tag{10.14}$$

and:-

$$Z_{0612} = \bar{Z}_{04} \tag{10.15}$$

where Z_{0612} and $\bar{Z}_{04}$ are given in their symmetric forms by:-

$$\begin{aligned} Z_{0612} = \ & a_{23}Y_{612} \\ & + S_{22}\sin\alpha_{23} X_{612} \\ & + a_{12}[(X_6\sin\theta_1 + Y_6\cos\theta_1)\bar{Z}_2 + Z_6\bar{Y}_2] \\ & + S_{11}[(Y_6\bar{Y}_2 - X_6\bar{X}_2)\sin\theta_1 - (Y_6\bar{X}_2 + X_6\bar{Y}_2)\cos\theta_1] \\ & + a_{61}[(\bar{X}_2\sin\theta_1 + \bar{Y}_2\cos\theta_1)Z_6 + \bar{Z}_2Y_6] \\ & + S_{66}\sin\alpha_{56} X_{216} \\ & + a_{56}Y_{216} \end{aligned} \tag{10.16}$$

and:-

$$\begin{aligned} \bar{Z}_{04} = \ & a_{34}\bar{Y}_4 \\ & + S_{44}\sin\alpha_{34} \sin\alpha_{45} \sin\theta_4 \\ & + a_{45}Y_4 \end{aligned} \tag{10.17}$$

Thus from, (10.11) and (10.12) using (10.14) one may rewrite (10.15) as:-

$$\begin{aligned} & Z_{0612} - (a_{34}\cot\alpha_{34} + a_{45}\cot\alpha_{45}) Z_{612} \\ & + (a_{34}\cos\alpha_{45} \operatorname{cosec}\alpha_{34} + a_{45}\cos\alpha_{34} \operatorname{cosec}\alpha_{45}) = S_{44}\sin\alpha_{34} \sin\alpha_{45} \sin\theta_4 \\ & \qquad = S_{44}\sin\alpha_{34} \bar{X}_4 \end{aligned} \tag{10.18}$$

Finally, from the definition of $\bar{Z}_4$ (see Chapter 4), equation (10.14) can be written as:-

$$\cos\alpha_{34}\cos\alpha_{45} - Z_{612} = \sin\alpha_{34}\sin\alpha_{45}\cos\theta_4 \tag{10.19}$$

It is now possible to remove the $\bar{X}_4$ terms from (5.35) and (5.36) using (10.18), and the $\bar{Y}_4$ terms using the identity (10.12) together with (10.14). Thus, after multiplying throughout by $S_{44}.\sin\alpha_{34}$, equations (5.35) and (5.36) become respectively:-

$$\begin{aligned}&[S_{44}\sin\alpha_{34}X_{612} + Z_{0612} - (a_{34}\cot\alpha_{34} + a_{45}\cot\alpha_{45})Z_{612}\\ &\quad + (a_{34}\cos\alpha_{45}\operatorname{cosec}\alpha_{34} + a_{45}\cos\alpha_{34}\operatorname{cosec}\alpha_{45})]x_3\\ &+ S_{44}(\sin\alpha_{34}Y_{612} + \cos\alpha_{34}Z_{612} - \cos\alpha_{45}) = 0\end{aligned} \tag{10.20}$$

and:-

$$\begin{aligned}&S_{44}(\sin\alpha_{34}Y_{612} - \cos\alpha_{34}Z_{612} + \cos\alpha_{45})\,x_3\\ &- [S_{44}\sin\alpha_{34}X_{612} - Z_{0612} + (a_{34}\cot\alpha_{34} + a_{45}\cot\alpha_{45})Z_{612}\\ &\quad - (a_{34}\cos\alpha_{45}\operatorname{cosec}\alpha_{34} + a_{45}\cos\alpha_{34}\operatorname{cosec}\alpha_{45})] = 0\end{aligned} \tag{10.21}$$

Equations (10.20) and (10.21) are two equations of the desired form in θ_6, θ_1, θ_2 and x_3, and it now remains to obtain a further two such equations by removing terms in θ_4 from (10. 9) and (10.10).

From the definitions of $\bar{X}_{04}$ and $\bar{Y}_{04}$ (see Appendix III.) equation (10. 9) becomes:-

$$\begin{aligned}&(X_{0612} + a_{45}\cos\alpha_{45}\sin\theta_4 + S_{44}\sin\alpha_{45}\cos\theta_4 - S_{33}Y_{612})x_3\\ &+ [S_{33}X_{612} + Y_{0612} + a_{45}(\sin\alpha_{45}\sin\alpha_{34} - \cos\alpha_{45}\cos\alpha_{34}\cos\theta_4)\\ &\quad - a_{34}(\cos\alpha_{45}\cos\alpha_{34} - \sin\alpha_{45}\sin\alpha_{34}\cos\theta_4)\\ &\quad + S_{44}\sin\alpha_{45}\cos\alpha_{34}\sin\theta_4] = 0\end{aligned} \tag{10.22}$$

whilst (10.10) becomes:-

$$\Big[Y_{0612} - a_{45}(\sin\alpha_{45}\sin\alpha_{34} - \cos\alpha_{45}\cos\alpha_{34}\cos\theta_4)$$
$$+ a_{34}(\cos\alpha_{45}\cos\alpha_{34} - \sin\alpha_{45}\sin\alpha_{34}\cos\theta_4)$$
$$- S_{44}\sin\alpha_{45}\cos\alpha_{34}\sin\theta_4 + S_{33}X_{612}\Big]x_3$$
$$+ (S_{33}Y_{612} - X_{0612} + a_{45}\cos\alpha_{45}\sin\theta_4 + S_{44}\sin\alpha_{45}\cos\theta_4) = 0 \qquad (10.23)$$

Now, using (10.18) and (10.19) to replace some of the terms involving $\sin\theta_4$ and $\cos\theta_4$, (10.22) gives, after multiplying throughout by $\sin\alpha_{34}$:-

$$\Big[\sin\alpha_{34}X_{0612} + S_{44}(\cos\alpha_{34}\cos\alpha_{45} - Z_{612}) - S_{33}\sin\alpha_{34}Y_{612}$$
$$+ a_{45}\sin\alpha_{34}\cos\alpha_{45}\sin\theta_4\Big]x_3$$
$$+ \Big[S_{33}\sin\alpha_{34}X_{612} + \sin\alpha_{34}Y_{0612} - a_{34}\sin\alpha_{34}Z_{612}$$
$$+ a_{45}\sin\alpha_{34}[\sin\alpha_{45}\sin\alpha_{34} - \cot\alpha_{45}\cot\alpha_{34}(\cos\alpha_{34}\cos\alpha_{45} - Z_{612})]$$
$$+ \cos\alpha_{34}[Z_{0612} - (a_{34}\cot\alpha_{34} + a_{45}\cot\alpha_{45})Z_{612}$$
$$+ (a_{34}\cos\alpha_{45}\operatorname{cosec}\alpha_{34} + a_{45}\cos\alpha_{34}\operatorname{cosec}\alpha_{45})]\Big] = 0 \qquad (10.24)$$

In a similar manner (10.23) becomes:-

$$\Big[S_{33}\sin\alpha_{34}X_{612} + \sin\alpha_{34}Y_{0612} + a_{34}\sin\alpha_{34}Z_{612}$$
$$- a_{45}\sin\alpha_{34}[\sin\alpha_{45}\sin\alpha_{34} - \cot\alpha_{45}\cot\alpha_{34}(\cos\alpha_{34}\cos\alpha_{45} - Z_{612})]$$
$$- \cos\alpha_{34}[Z_{0612} - (a_{34}\cot\alpha_{34} + a_{45}\cot\alpha_{45})Z_{612}$$
$$+ (a_{34}\cos\alpha_{45}\operatorname{cosec}\alpha_{34} + a_{45}\cos\alpha_{34}\operatorname{cosec}\alpha_{45})]\Big]x_3$$
$$- \Big[\sin\alpha_{34}X_{0612} - S_{44}(\cos\alpha_{34}\cos\alpha_{45} - Z_{612}) - S_{33}\sin\alpha_{34}Y_{612}$$
$$- a_{45}\sin\alpha_{34}\cos\alpha_{45}\sin\theta_4\Big] = 0 \qquad (10.25)$$

Finally the $x_3\sin\theta_4$ term in (10.24) and the $\sin\theta_4$ term in (10.25) may be replaced using equations (5.35) and (5.36) respectively, since the latter can be written (with the aid of identity (10.12)) in the form:-

$$\sin\alpha_{34}\sin\alpha_{45}\sin\theta_4 x_3 = -\sin\alpha_{34}X_{612}x_3$$
$$- (\sin\alpha_{34}Y_{612} + \cos\alpha_{34}Z_{612} - \cos\alpha_{45}) \qquad (10.26)$$

and:-

$$\sin\alpha_{34}\sin\alpha_{45}\sin\theta_4 = -(\sin\alpha_{34}Y_{612} - \cos\alpha_{34}Z_{612} + \cos\alpha_{45})x_3$$
$$+ \sin\alpha_{34}X_{612} \qquad (10.27)$$

Thus from (10.24) and (10.26) one obtains:-

$$\Big[\sin\alpha_{34} X_{0612} + S_{44}(\cos\alpha_{34}\cos\alpha_{45} - Z_{612}) - S_{33}\sin\alpha_{34} Y_{612} - a_{45}\sin\alpha_{34}\cot\alpha_{45} X_{612}\Big]x_3$$
$$+ \Big[(\sin\alpha_{34} Y_{0612} + \cos\alpha_{34} Z_{0612}) + S_{33}\sin\alpha_{34} X_{612} + a_{34}\mathrm{cosec}\,\alpha_{34}(\cos\alpha_{34}\cos\alpha_{45} - Z_{612}) + a_{45}[\mathrm{cosec}\,\alpha_{45} - \cot\alpha_{45}(\sin\alpha_{34} Y_{612} + \cos\alpha_{34} Z_{612})]\Big] = 0 \qquad (10.28)$$

and from (10.25) and (10.27) one has:-

$$\Big[(\sin\alpha_{34} Y_{0612} - \cos\alpha_{34} Z_{0612}) + S_{33}\sin\alpha_{34} X_{612} - a_{34}\mathrm{cosec}\,\alpha_{34}(\cos\alpha_{34}\cos\alpha_{45} - Z_{612}) - a_{45}[\mathrm{cosec}\,\alpha_{45} + \cot\alpha_{45}(\sin\alpha_{34} Y_{612} - \cos\alpha_{34} Z_{612})]\Big]x_3$$
$$- \Big[\sin\alpha_{34} X_{0612} - S_{44}(\cos\alpha_{34}\cos\alpha_{45} - Z_{612}) - S_{33}\sin\alpha_{34} Y_{612} - a_{45}\sin\alpha_{34}\cot\alpha_{45} X_{612}\Big] = 0 \qquad (10.29)$$

Equations (10.20), (10.21), (10.28) and (10.29) are now the desired four equations in θ_6, θ_1, θ_2 and x_3.

10.3.2 Elimination Procedure for the RRRRCR Mechanism.

By making the substitutions:-

$$\sin\theta_2 = 2x_2/(1 + x_2^2)$$
$$\cos\theta_2 = (1 - x_2^2)/(1 + x_2^2) \qquad (5.1)$$

where:- $x_2 = \tan(\theta_2/2)$

in equations (10.20), (10.21), (10.28) and (10.29), one obtains the following set of four equations which are respectively:-

$$F_1(x_2, x_3) = (a_1x_2^2 + b_1x_2 + c_1)x_3 + (d_1x_2^2 + e_1x_2 + f_1) = 0$$
$$F_2(x_2, x_3) = (a_2x_2^2 + b_2x_2 + c_2)x_3 + (d_2x_2^2 + e_2x_2 + f_2) = 0$$
$$F_3(x_2, x_3) = (a_3x_2^2 + b_3x_2 + c_1)x_3 + (d_3x_2^2 + e_3x_2 + f_3) = 0$$
$$F_4(x_2, x_3) = (a_4x_2^2 + b_4x_2 + c_4)x_3 + (d_4x_2^2 + e_4x_2 + f_4) = 0 \qquad (10.30)$$

where the coefficients a_i, b_i, c_i, d_i, e_i, f_i (i = 1, 2, 3, 4) are given in Appendix VIII and are each of degree two in the half-tangents of both

θ_1 (input) and θ_6 (output). They also collectively involve all of the mechanism proportions.

The two unknowns x_2 and x_3 can be eliminated from equations (10.30) in a single operation by firstly multiplying the latter system throughout by x_2. This produces a further four equations giving, together with (10.30), a total of eight equations. By treating these as a system of eight non-homogeneous linear equations in the seven unknowns $x_2^3x_3$, x_2^3, $x_2^2x_3$, x_2x_3, x_3, x_2^2 and x_2 it is possible to eliminate the latter and obtain as the eliminant an (8 x 8)-determinant (see also Salmon [31]). This eliminant, equated to zero, is then the required input-output equation for the RRRRCR six-link spatial mechanism, and it may be written as follows:-

$$E(F_1, F_2, F_3, F_4) = \begin{vmatrix} 0 & 0 & a_1 & b_1 & c_1 & d_1 & e_1 & f_1 \\ 0 & 0 & a_2 & b_2 & c_2 & d_2 & e_2 & f_2 \\ 0 & 0 & a_3 & b_3 & c_3 & d_3 & e_3 & f_3 \\ 0 & 0 & a_4 & b_4 & c_4 & d_4 & e_4 & f_4 \\ a_1 & d_1 & b_1 & c_1 & 0 & e_1 & f_1 & 0 \\ a_2 & d_2 & b_2 & c_2 & 0 & e_2 & f_2 & 0 \\ a_3 & d_3 & b_3 & c_3 & 0 & e_3 & f_3 & 0 \\ a_4 & d_4 & b_4 & c_4 & 0 & e_4 & f_4 & 0 \end{vmatrix} = 0 \qquad (10.31)$$

Clearly, (10.31) is of order 8 in the coefficients $a_i, \ldots\ldots, f_i$ and is therefore of degree <u>sixteen</u> in both the input and the output half-tangents, x_1 and x_6. This novel result agrees with the predicted degree for the RRRRCR input-output equation given in Chapter 2.

10.3.3 Elimination Procedure for the RRCRRR Mechanism.

In a similar manner, by making the substitutions:-

$$\sin\theta_6 \equiv 2x_6/(1 + x_6^2)$$
$$\cos\theta_6 \equiv (1 - x_6^2)/(1 + x_6^2) \qquad (5.1)$$

where $$x_6 \equiv \tan(\theta_6/2)$$

and regrouping terms, equations (10.20), (10.21), (10.28) and (10.29) can be written as a set of four equations in the following form:-

$$F'_1(x_6, x_3) = (a'_1x_6^2 + b'_1x_6 + c'_1)x_3 + (d'_1x_6^2 + e'_1x_6 + f'_1) = 0$$
$$F'_2(x_6, x_3) = (a'_2x_6^2 + b'_2x_6 + c'_2)x_3 + (d'_2x_6^2 + e'_2x_6 + f'_2) = 0$$
$$F'_3(x_6, x_3) = (a'_3x_6^2 + b'_3x_6 + c'_3)x_3 + (d'_3x_6^2 + e'_3x_6 + f'_3) = 0$$
$$F'_4(X_6, x_3) = (a'_4x_6^2 + b'_4x_6 + c'_4)x_3 + (d'_4x_6^2 + e'_4x_6 + f'_4) = 0 \qquad (10.32)$$

where the coefficients a'_i, b'_i, c'_i, d'_i, e'_i, f'_i ($i = 1, 2, 3, 4$) are given in Appendix VIII and are of degree two in the half-tangents of both θ_1 (input) and θ_2 (output). They also collectively involve all the mechanism proportions.

Now, in analogy with the previous inversion (the RRRRCR), it is possible to eliminate the unknowns x_6 and x_3 from (10.32) in a single operation by, firstly, multiplying throughout by x_6; then eliminating the seven unknowns $x_6^3x_3$, x_6^3, $x_6^2x_3$, x_6x_3, x_3, x_6^2 and x_6 from the resulting eight equations. In this manner one obtains the following eliminant:-

$$E(F'_1, F'_2, F'_3, F'_4) = \begin{vmatrix} 0 & 0 & a'_1 & b'_1 & c'_1 & d'_1 & e'_1 & f'_1 \\ 0 & 0 & a'_2 & b'_2 & c'_2 & d'_2 & e'_2 & f'_2 \\ 0 & 0 & a'_3 & b'_3 & c'_3 & d'_3 & e'_3 & f'_3 \\ 0 & 0 & a'_4 & b'_4 & c'_4 & d'_4 & e'_4 & f'_4 \\ a'_1 & d'_1 & b'_1 & c'_1 & 0 & e'_1 & f'_1 & 0 \\ a'_2 & d'_2 & b'_2 & c'_2 & 0 & e'_2 & f'_2 & 0 \\ a'_3 & d'_3 & b'_3 & c'_3 & 0 & e'_3 & f'_3 & 0 \\ a'_4 & d'_4 & b'_4 & c'_4 & 0 & e'_4 & f'_4 & 0 \end{vmatrix} = 0 \qquad (10.33)$$

Equation (10.33) is the required input-output equation for the RRCRRR six-link spatial mechanism and it is clearly of order 8 in the coefficients $a'_i, \ldots\ldots, f'_i$. It is, therefore, of degree <u>sixteen</u> in both the input and the output half-tangents, x_1 and x_2. This result agrees with the predicted degree for the RRCRRR input-output equation given in Chapter 2.

10.4 Displacement Analyses for 5R-C Mechanisms.

For spatial five-link 3R-2C, six-link 4R-P-C and seven-link 5R-2P mechanisms it is a relatively simple matter to devise procedures for determining uniquely the remaining variables once the input-output equation has been solved (see Chapters 6, 7, 8 and 9). However, for the 5R-C mechanisms, analysed here, the problem is of a more complex nature since one cannot write down immediately expressions for the common root obtained from the Bézoutian. Indeed the 'common root' of either of the systems (10.30) or (10.32) is no longer a single value but rather an ordered pair $((x_2, x_3)$ or $(x_6, x_3))$ of values.

Nevertheless, explicit expressions for x_2 and x_3 (x_6 and x_3) in terms of the coefficients $a_i, \ldots, f_i$ ($a_i', \ldots, f_i'$) may be determined in the case of the RRRRCR (RRCRRR) mechanism, using what is, essentially, an extension of Bézout's method for dealing with equations in a single unknown.

10.4.1 Displacement Analysis of the RRRRCR Mechanism.

Thus, for the RRRRCR mechanism, one may write the system (10.30) in the following form:-

$$\begin{aligned}
a_1x_2^2x_3 + b_1x_2x_3 + d_1x_2^2 + (e_1x_2 + c_1x_3 + f_1) &= 0\\
a_2x_2^2x_3 + b_2x_2x_3 + d_2x_2^2 + (e_2x_2 + c_2x_3 + f_2) &= 0\\
a_3x_2^2x_3 + b_3x_2x_3 + d_3x_2^2 + (e_3x_2 + c_3x_3 + f_3) &= 0\\
a_4x_2^2x_3 + b_4x_2x_3 + d_4x_2^2 + (e_4x_2 + c_4x_3 + f_4) &= 0
\end{aligned} \qquad (10.34)$$

Treating (10.34) as a set of four non-homogeneous linear equations in the three variables $x_2^2x_3$, x_2x_3 and x_2^2, and eliminating the latter, one obtains, after expansion of determinants, the following linear equation in x_2 and x_3:-

$$A_1x_2 + B_1x_3 + C_1 = 0 \qquad (10.35)$$

where:-

$$A_1 = \begin{vmatrix} a_1 & b_1 & d_1 & e_1 \\ a_2 & b_2 & d_2 & e_2 \\ a_3 & b_3 & d_3 & e_3 \\ a_4 & b_4 & d_4 & e_4 \end{vmatrix} \qquad (10.36a)$$

$$B_1 = \begin{vmatrix} a_1 & b_1 & d_1 & c_1 \\ a_2 & b_2 & d_2 & c_2 \\ a_3 & b_3 & d_3 & c_3 \\ a_4 & b_4 & d_4 & c_4 \end{vmatrix} \qquad (10.36b)$$

and:-

$$C_1 = \begin{vmatrix} a_1 & b_1 & d_1 & f_1 \\ a_2 & b_2 & d_2 & f_2 \\ a_3 & b_3 & d_3 & f_3 \\ a_4 & b_4 & d_4 & f_4 \end{vmatrix} \qquad (10.36c)$$

Alternatively, it is possible to write the system (10.30) in the form:-

$$\begin{aligned} a_1x_2^2x_3 + c_1x_3 + (d_1x_2 + b_1x_3 + e_1)x_2 + f_1 = 0 \\ a_2x_2^2x_3 + c_2x_3 + (d_2x_2 + b_2x_3 + e_2)x_2 + f_2 = 0 \\ a_3x_2^2x_3 + c_3x_3 + (d_3x_2 + b_3x_3 + e_3)x_2 + f_3 = 0 \\ a_4x_2^2x_3 + c_4x_3 + (d_4x_2 + b_4x_3 + e_4)x_2 + f_4 = 0 \end{aligned} \qquad (10.37)$$

and, teating (10.37) as a set of four non-homogeneous linear equationsin the three variables $x_2^2x_3$, x_3 and x_2, one obtains, after eliminating the latter, the following linear equation in x_2 and x_3:-

$$A_2x_2 + B_2x_3 + C_2 = 0 \qquad (10.38)$$

where:-

$$A_2 = \begin{vmatrix} a_1 & c_1 & d_1 & f_1 \\ a_2 & c_2 & d_2 & f_2 \\ a_3 & c_3 & d_3 & f_3 \\ a_4 & c_4 & d_4 & f_4 \end{vmatrix} \qquad (10.39a)$$

$$B_2 = \begin{vmatrix} a_1 & c_1 & b_1 & f_1 \\ a_2 & c_2 & b_2 & f_2 \\ a_3 & c_3 & b_3 & f_3 \\ a_4 & c_4 & b_4 & f_4 \end{vmatrix} \qquad (10.39b)$$

and:-

$$C_2 = \begin{vmatrix} a_1 & c_1 & e_1 & f_1 \\ a_2 & c_2 & e_2 & f_2 \\ a_3 & c_3 & e_3 & f_3 \\ a_4 & c_4 & e_4 & f_4 \end{vmatrix} \qquad (10.39c)$$

It is now possible to obtain explicit expressions for x_2 and x_3 by solving the two linear equations (10.35) and (10.38) simultaneously. The required expressions for the common root (or zero) of (10.30) are therefore:-

$$x_2 = (B_1C_2 - B_2C_1)/(A_1B_2 - A_2B_1) \qquad (10.40)$$

and:-

$$x_3 = -(A_1C_2 - A_2C_1)/(A_1B_2 - A_2B_1) \qquad (10.41)$$

The determination of the remaining variables is now straightforward since θ_1, θ_6, θ_2 and θ_3 are now known. A complete displacement analysis for the RRRRCR mechanism can be outlined and summarised as follows:-

(i) The degree sixteen polynomial obtained from (10.31) gives, in general, sixteen real values of θ_6 (i.e. $\theta_{61}, \theta_{62}, \ldots, \theta_{616}$) for each specified value of the input angular displacement θ_1.

(ii) For each of the sixteen ordered pairs (θ_1, θ_{61}), $(\theta_1, \theta_{62}), \ldots\ldots, (\theta_1, \theta_{616})$, the corresponding numerical values of the coefficients $a_i, \ldots\ldots, f_i$ (see Appendix VIII) may be calculated and hence the unique values for x_2 and x_3 can be obtained from (10.40) and (10.41).

(iii) The corresponding values for x_4 ($\equiv \tan(\theta_4/2)$) and x_5 ($\equiv \tan(\theta_5/2)$) may now be easily calculated from either form of the relevant fundamental half-tangent laws (see equations (5.32) and (5.33)). Thus:-

$$x_4 = (\sin\alpha_{45} - Y_{6123})/X_{6123} \tag{10.42a}$$

or:-

$$x_4 = X_{6123}/(\sin\alpha_{45} + Y_{6123}) \tag{10.42b}$$

and:-

$$x_5 = (\sin\alpha_{45} - Y_{3216})/X_{3216} \tag{10.43a}$$

or:-

$$x_5 = X_{3216}/(\sin\alpha_{45} + Y_{3216}) \tag{10.43b}$$

(iv) Finally, the sliding displacement, S_5, is calculated from the secondary part of the dual subsidiary cosine law:-

$$\hat{Z}_{123} = \hat{\bar{Z}}_5 \tag{10.44}$$

which is:-

$$Z_{0123} = \bar{Z}_{05} \tag{10.45}$$

where Z_{0123} and $\bar{Z}_{05}$ are written in their symmetric forms as:-

$$\begin{aligned} Z_{0123} = \; & a_{34}Y_{123} \\ & + S_{33}\sin\alpha_{34}X_{123} \\ & + a_{23}[(X_1\sin\theta_2 + Y_1\cos\theta_2)\bar{Z}_3 + Z_1\bar{Y}_3] \\ & + S_{22}[(Y_1\bar{Y}_3 - X_1\bar{X}_3)\sin\theta_2 - (Y_1\bar{X}_3 + X_1\bar{Y}_3)\cos\theta_2] \\ & + a_{12}[(\bar{X}_3\sin\theta_2 + \bar{Y}_3\cos\theta_2)Z_1 + \bar{Z}_3Y_1] \\ & + S_{11}\sin\alpha_{61}X_{321} \\ & + a_{61}Y_{321} \end{aligned} \tag{10.46}$$

and:-

$$\bar{Z}_{05} = a_{45}\bar{Y}_5 + S_5 \sin\alpha_{45}\bar{X}_5 + a_{56}Y_5 \qquad (10.47)$$

10.4.2 Displacement Analysis of the RRCRRR Mechanism.

For the RRCRRR six-link spatial mechanism, the remaining variables may be calculated using a procedure exactly analogous to that used above for the RRRRCR inversion. Thus, by writing the system (10.32) in the following two alternative forms:-

$$\begin{aligned}
a_1'x_6^2x_3 + b_1'x_6x_3 + d_1'x_6^2 + (e_1'x_6 + c_1'x_3 + f_1') &= 0\\
a_2'x_6^2x_3 + b_2'x_6x_3 + d_2'x_6^2 + (e_2'x_6 + c_2'x_3 + f_2') &= 0\\
a_3'x_6^2x_3 + b_3'x_6x_3 + d_3'x_6^2 + (e_3'x_6 + c_3'x_3 + f_3') &= 0\\
a_4'x_6^2x_3 + b_4'x_6x_3 + d_4'x_6^2 + (e_4'x_6 + c_4'x_3 + f_4') &= 0
\end{aligned} \qquad (10.48)$$

or:-

$$\begin{aligned}
a_1'x_6^2x_3 + c_1'x_3 + (d_1'x_6 + b_1'x_3 + e_1')x_6 + f_1' &= 0\\
a_2'x_6^2x_3 + c_2'x_3 + (d_2'x_6 + b_2'x_3 + e_2')x_6 + f_2' &= 0\\
a_3'x_6^2x_3 + c_3'x_3 + (d_3'x_6 + b_3'x_3 + e_3')x_6 + f_3' &= 0\\
a_4'x_6^2x_3 + c_4'x_3 + (d_4'x_6 + b_4'x_3 + e_4')x_6 + f_4' &= 0
\end{aligned} \qquad (10.49)$$

and eliminating the three variables $x_6^2x_3$, x_6x_3 and x_6^2 from (10.48) one obtains the following linear equation in x_6 and x_3:-

$$A_1'x_6 + B_1'x_3 + C_1' = 0 \qquad (10.50)$$

where:-

$$A_1' = \begin{vmatrix} a_1' & b_1' & d_1' & e_1' \\ a_2' & b_2' & d_2' & e_2' \\ a_3' & b_3' & d_3' & e_3' \\ a_4' & b_4' & d_4' & e_4' \end{vmatrix} \qquad (10.51a)$$

$$B_1' = \begin{vmatrix} a_1' & b_1' & d_1' & c_1' \\ a_2' & b_2' & d_2' & c_2' \\ a_3' & b_3' & d_3' & c_3' \\ a_4' & b_4' & d_4' & c_4' \end{vmatrix} \qquad (10.51b)$$

and:-

$$C_1' = \begin{vmatrix} a_1' & b_1' & d_1' & f_1' \\ a_2' & b_2' & d_2' & f_2' \\ a_3' & b_3' & d_3' & f_3' \\ a_4' & b_4' & d_4' & f_4' \end{vmatrix} \qquad (10.51c)$$

Alternatively, by eliminating $x_6^2x_3$, x_3 and x_6 from (10.49), a further linear equation in x_6 and x_3 is produced and this is:-

$$A_2'x_6 + B_2'x_3 + C_2' = 0 \qquad (10.52)$$

where:-

$$A_2' = \begin{vmatrix} a_1' & c_1' & d_1' & f_1' \\ a_2' & c_2' & d_2' & f_2' \\ a_3' & c_3' & d_3' & f_3' \\ a_4' & c_4' & d_4' & f_4' \end{vmatrix} \qquad (10.53a)$$

$$B_2' = \begin{vmatrix} a_1' & c_1' & b_1' & f_1' \\ a_2' & c_2' & b_2' & f_2' \\ a_3' & c_3' & b_3' & f_3' \\ a_4' & c_4' & b_4' & f_4' \end{vmatrix} \qquad (10.53b)$$

$$C_2' = \begin{vmatrix} a_1' & c_1' & e_1' & f_1' \\ a_2' & c_2' & e_2' & f_2' \\ a_3' & c_3' & e_3' & f_3' \\ a_4' & c_4' & e_4' & f_4' \end{vmatrix} \qquad (10.53c)$$

Explicit expressions for x_6 and x_3 may now be written by solving (10.50) and (10.52) simultaneously for x_6 and x_3. Thus:-

$$x_6 = (B_1'C_2' - B_2'C_1')/(A_1'B_2' - A_2'B_1') \tag{10.54}$$

and:-

$$x_3 = -(A_1'C_2' - A_2'C_1')/(A_1'B_2' - A_2'B_1') \tag{10.55}$$

A complete displacement analysis for the RRCRRR mechanism can be summarised as follows:-

(i) The degree sixteen polynomial obtained from (10.33) gives, in general, sixteen real values of θ_2 (i.e. $\theta_{21}, \theta_{22}, \ldots \ldots, \theta_{216}$) for each specified value of the input angular displacement, θ_1.

(ii) For each of the sixteen ordered pairs $(\theta_1, \theta_{21}), (\theta_1, \theta_{22}), \ldots\ldots, (\theta_1, \theta_{216})$, the corresponding numerical values of the coefficients $a_i', \ldots\ldots, f_i'$ (see Appendix VIII) may be calculated and hence the unique values for x_6 and x_3 can be obtained from (10.54) and (10.55).

(iii) The corresponding values for x_4 ($\equiv \tan(\theta_4/2)$) and x_5 ($\equiv \tan(\theta_5/2)$) may now be easily calculated from the fundamental half-tangent laws, (10.42 a or b) and (10.43 a or b).

(iv) Finally the sliding displacement, S_5, is calculated from equations (10.45), (10.46) and (10.47).

This completes the analysis procedures for both of the 5R-C mechanisms dealt with here.

10.5 Numerical Results.

The input-output equations (10.31) and (10.33) for the RRRRCR and RRCRRR mechanisms were solved numerically for given sets of mechanism proportions, and graphs of the output angular variables (θ_6 for the RRRRCR and θ_2 for the RRCRRR) and remaining variables θ_3, θ_4, θ_5 and S_5, against the input angle, θ_1, were plotted (see Figures 10.1, 10.2, 10.3, 10.4, 10.5 and 10.6 respectively). The mechanism parameters were selected from a consideration of the intersecting right circular torii used in Chapter 2 (to determine the number of closures of these mechanisms) and, as a consequence of this, the input-output equations each possess sixteen real roots for various ranges of values of the input angle, θ_1 (see Figures 10.1 and 10.2). In addition, the parameters were chosen so that both inversions would produce identical sets of graphs, and thus Figure 10.2 is both the input-output graph for the RRCRRR and a plot of θ_2 vs θ_1 for the RRRRCR.

The following sets of data were selected in each case:-

10.5.1 RRRRCR Mechanism.

$$
\begin{array}{lll}
a_{12} = 2.0 \text{ ins.} & \alpha_{12} = 90 \text{ deg.} & S_{11} = 8.0 \text{ ins.} \\
a_{23} = 0.0 \text{ ins.} & \alpha_{23} = 90 \text{ deg.} & S_{22} = 3.0 \text{ ins.} \\
a_{34} = 8.0 \text{ ins.} & \alpha_{34} = 90 \text{ deg.} & S_{33} = 0.0 \text{ ins.} \\
a_{45} = 2.0 \text{ ins.} & \alpha_{45} = 90 \text{ deg.} & S_{44} = 0.0 \text{ ins.} \\
a_{56} = 2.0 \text{ ins.} & \alpha_{56} = 90 \text{ deg.} & S_{66} = 2.0 \text{ ins.} \\
a_{61} = 3.0 \text{ ins.} & \alpha_{61} = 90 \text{ deg.} &
\end{array}
\qquad (10.56)
$$

For this inversion the input-output relationship is plotted in Figure 10.1.

10.5.2 RRCRRR Mechanism.

$$
\begin{array}{lll}
a_{12} = 2.0 \text{ ins.} & \alpha_{12} = 90 \text{ deg.} & S_{11} = -8.0 \text{ ins.} \\
a_{23} = 0.0 \text{ ins.} & \alpha_{23} = 90 \text{ deg.} & S_{22} = -3.0 \text{ ins.} \\
a_{34} = 8.0 \text{ ins.} & \alpha_{34} = 90 \text{ deg.} & S_{33} = 0.0 \text{ ins.} \\
a_{45} = 2.0 \text{ ins.} & \alpha_{45} = 90 \text{ deg.} & S_{44} = 0.0 \text{ ins.} \\
a_{56} = 2.0 \text{ ins.} & \alpha_{56} = 90 \text{ deg.} & S_{66} = -2.0 \text{ ins.} \\
a_{61} = 3.0 \text{ ins.} & \alpha_{61} = 90 \text{ deg.} &
\end{array}
\qquad (10.57)
$$

For this inversion the input-output relationship is plotted in Figure 10.2.

10.5.3 Note on the Selection of Data.

The two sets of data above differ only in the signs of the offset distances $S_{11},\ldots,S_{66}$ (those for the RRCRRR being the negatives of those for the RRRRCR). The reason for this choice is that, although the two kinematic loops are identical, Figure 2.28 clearly shows that the RRRRCR loop is described in the opposite sense to the RRCRRR loop, and hence, in order to obtain identical sets of graphs for the two inversions, one must consider the line vectors, $\underline{S}_i$ (i = 1, 2, 3, 4, 5, 6), to be in opposite directions for the two loops.

10.6 Discussion of Results.

The results illustrated in Figures 10.1-10.6 are in agreement with the predictions of Chapter 2. In addition the numerical values of the remaining variables for the RRRRCR mechanism (i.e. θ_2, θ_3, θ_4, θ_5 and S_5) were in exact agreement with those obtained for the RRCRRR mechanism (i.e. θ_6, θ_3, θ_4, θ_5 and S_5). Thus, for example the results presented in Figure 10.2 (θ_2 vs θ_1) were obtained in two independent ways, firstly by solving (10.33), and secondly by sequential solution of (10.31) and (10.40). This agreement provides verification of the analysis procedures for the two mechanisms.

The drawing of Figures 10.1-10.6 proved to be rather difficult owing to the complexity of the situation, and, in order to simplify matters, the corresponding turning points have been numbered on each graph. There are essentially six complete "circuits" on each Figure, and the following sequence of numbers has been used to identify them:-

		No. of Turning or Limit Points
1st Circuit	1 to 11	8
2nd Circuit	12 to 14	2
3rd Circuit	15 to 18	4
4th Circuit	19 to 22	4
5th Circuit	23 to 26	4
6th Circuit	(not labelled)	0

The 1st circuit is the most complex and contains four double points. The 2nd and 6th circuits each contain a single double point and traverse the width of the graph. The 3rd, 4th and 5th circuits on the other hand are three small loops, but they also each contain a single double point.

10.7 Special Cases.

The input-output equation, (10.31), for the RRRRCR mechanism contains as special cases closed form input-output displacement equations for the two spatial six-link, PRRRCR and RRRRCP slider-crank mechanisms. The reduction to the PRRRCR (RRRRCP) mechanism is achieved by transforming the input (output) revolute pair θ_1, S_{11} (θ_6, S_{66}) of the RRRRCR mechanism into a sliding pair θ_{11}, S_1 (θ_{66}, S_6). Following this, equation (10.31) can be arranged as an eighth degree polynomial in either the sliding displacement, S_1 (PRRRCR mechanism), or the sliding displacement, S_6 (RRRRCP mechanism), since the coefficients $a_i, \ldots, f_i$, are linear in the latter.

In an exactly analogous manner one can derive an eighth degree polynomial in either S_1 (PRCRRR mechanism) or S_2 (RRCRRP mechanism) from the input-output equation, (10.33), for the RRCRRR mechanism, by transforming θ_1, S_{11} (θ_2, S_{22}) into θ_{11}, S_1 (θ_{22}, S_2).

These results, together with those presented in Chapters 7, 8 and 9 complete the analysis of the spatial six-link 4R-P-C mechanisms.

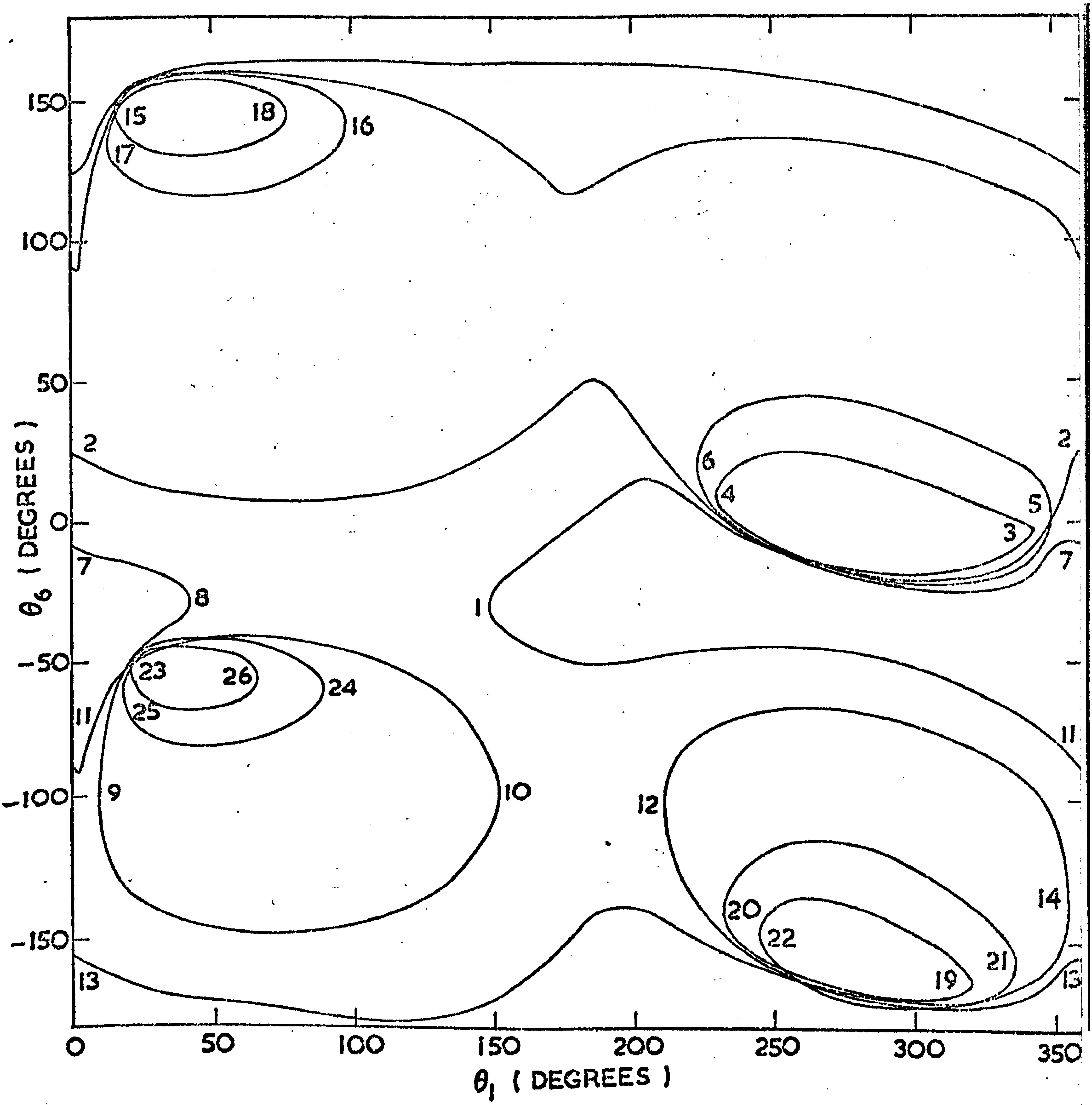


Figure 10.1 Graph of Input-Output Relationship (i.e. θ_6 vs θ_1) for the Six-Link RRRRCR Mechanism.

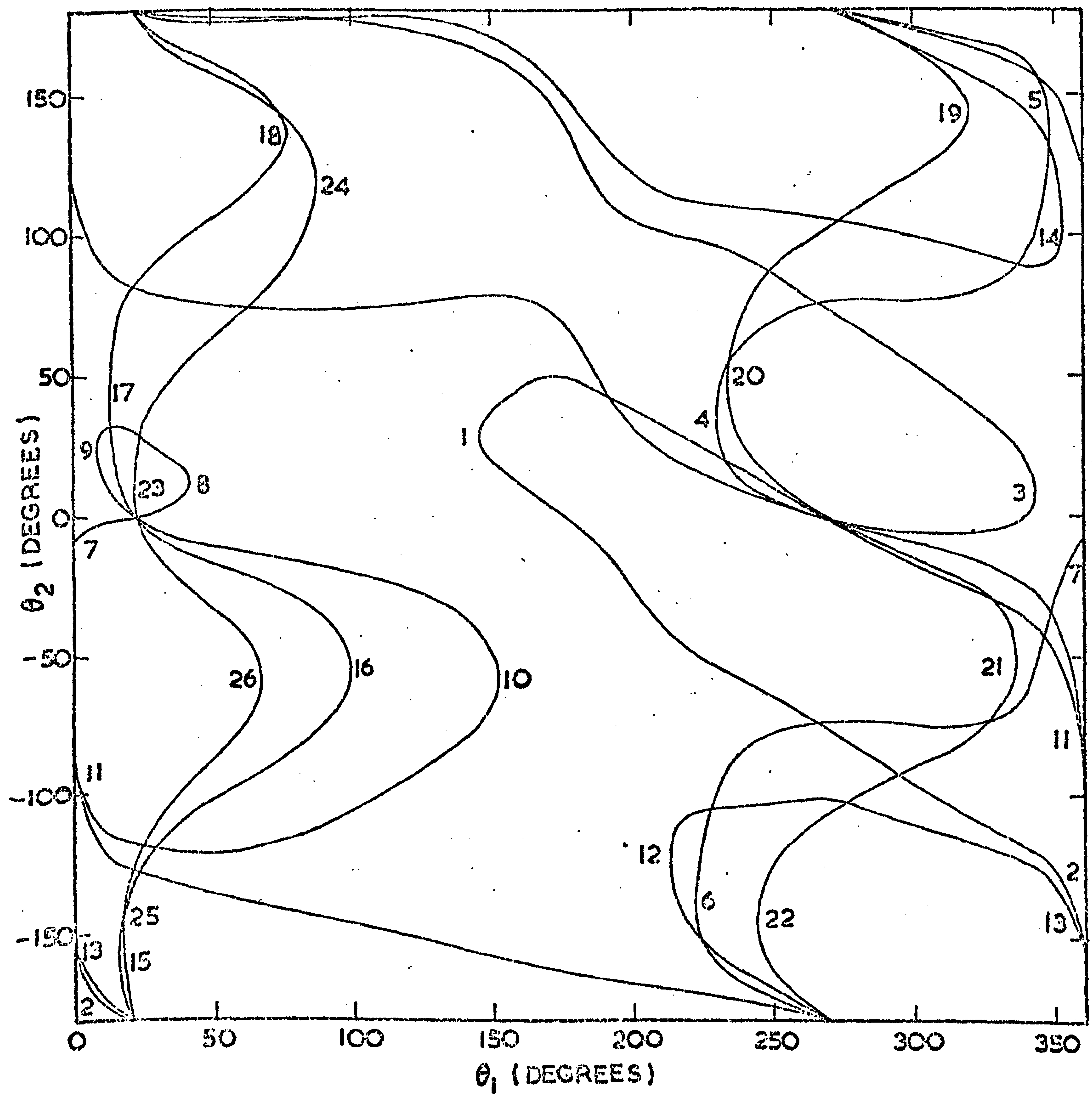


Figure 10.2 Graph of Input-Output Relationship (i.e. θ_2 vs θ_1) for the Six-Link RRCRRR Mechanism.

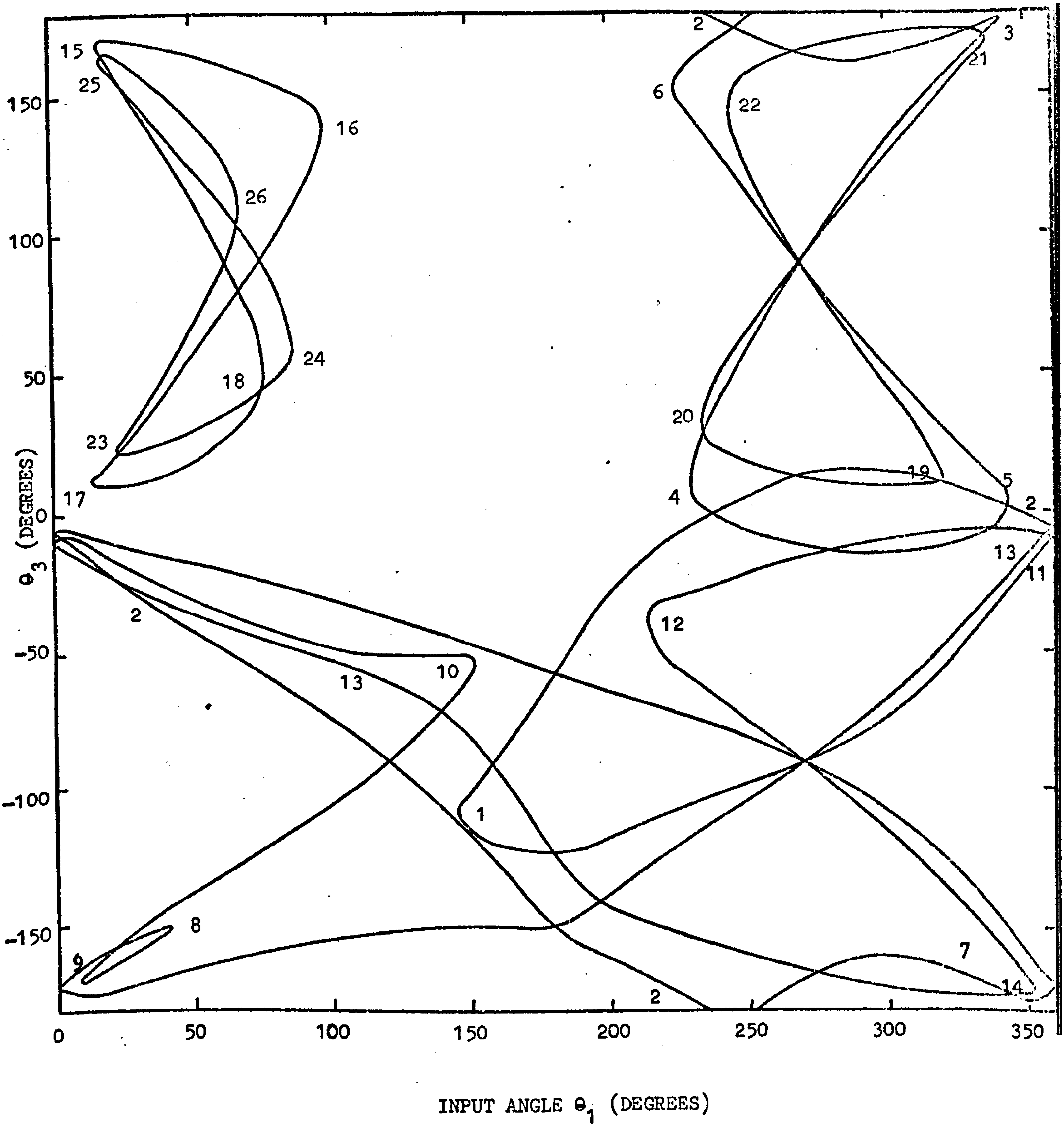


Figure 10.3 Graph of θ_3 vs θ_1 for the Six-Link 5R-C Mechanisms.

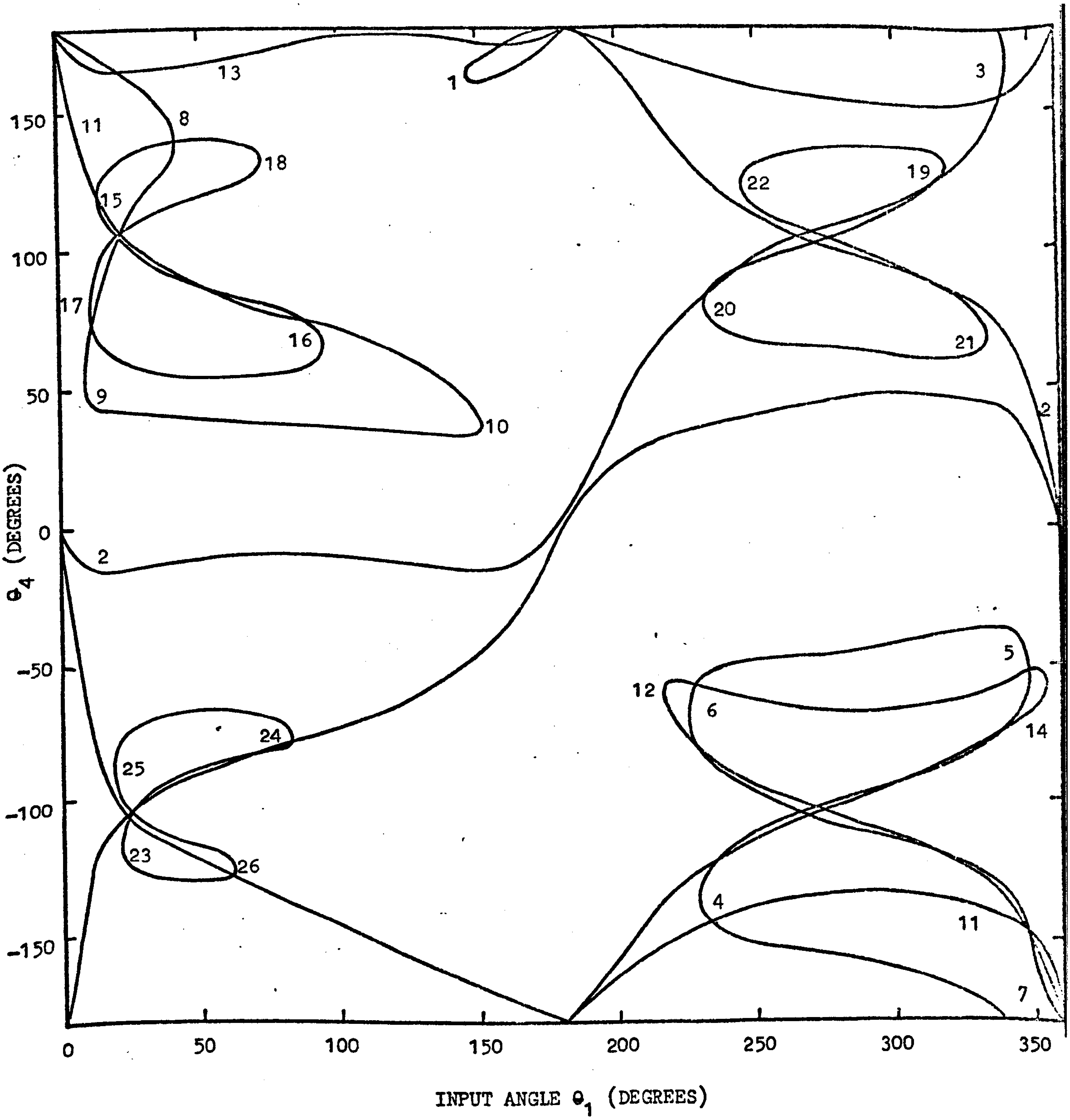


Figure 10.4 Graph of Θ_4 vs Θ_1 for the Six-Link 5R-C Mechanisms.

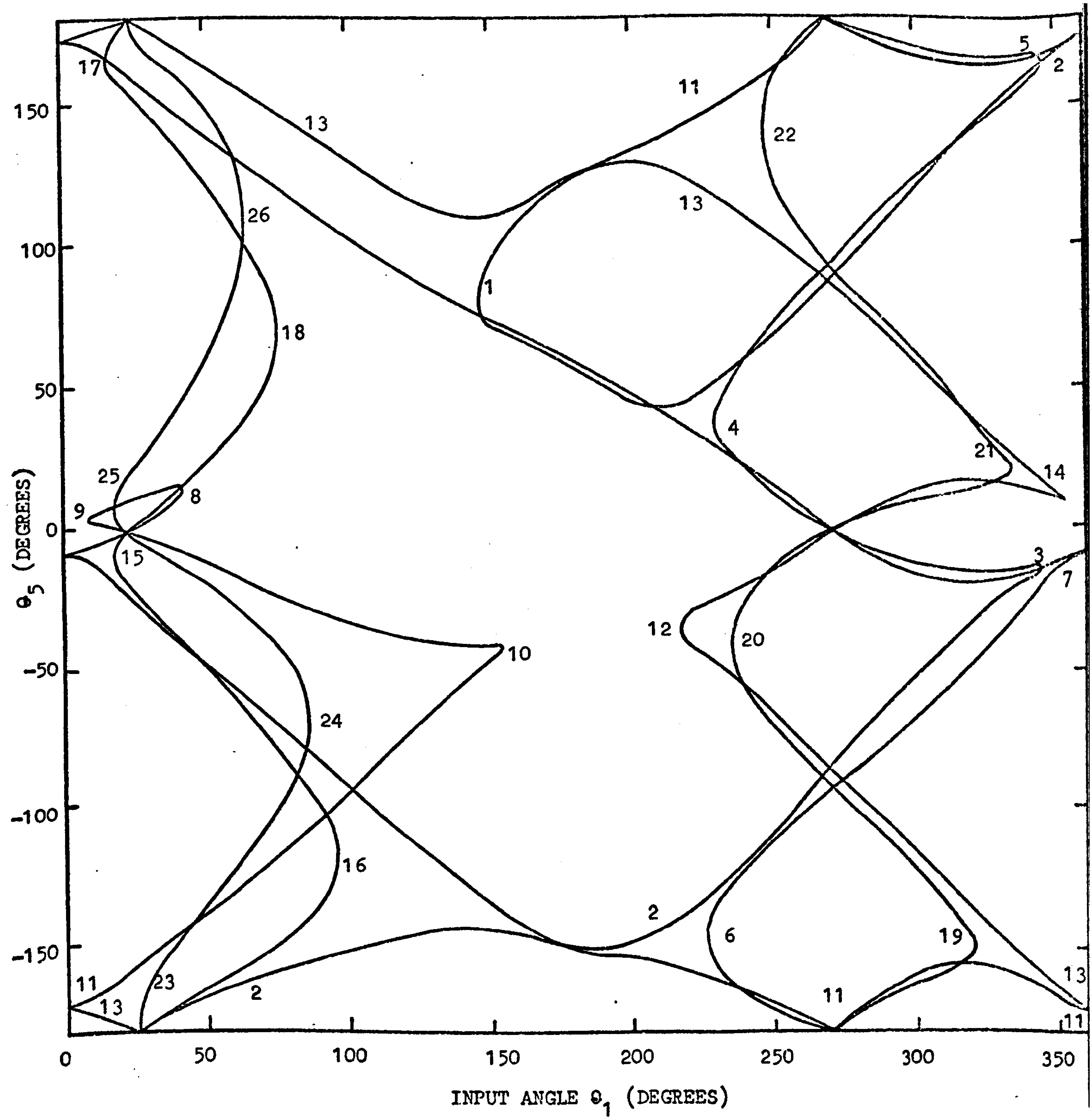


Figure 10.5 Graph of θ_5 vs θ_1 for the Six-Link 5R-C Mechanisms.

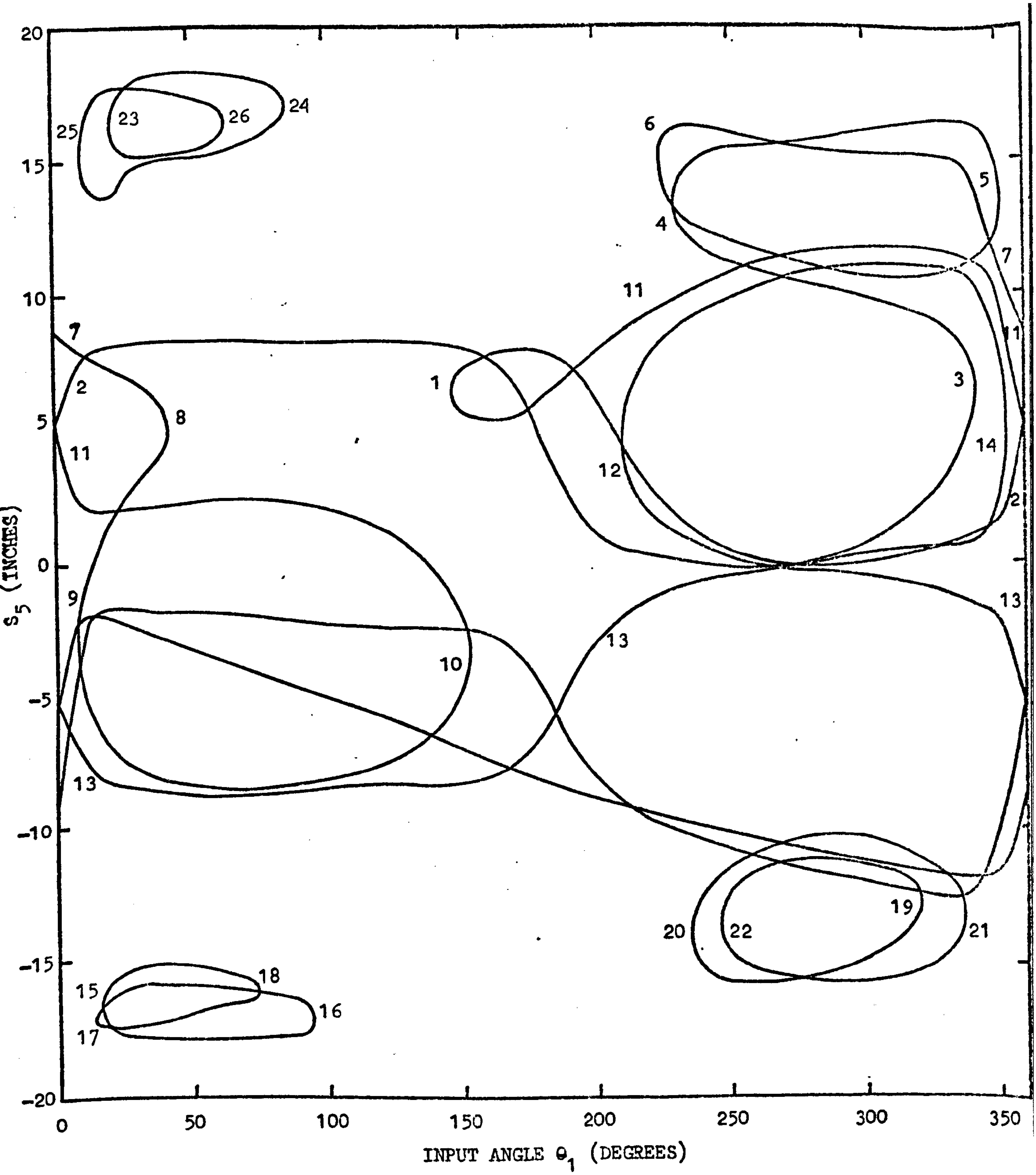


Figure 10.6 Graph of S_5 vs θ_1 for the Six-Link 5R-C Mechanisms.

CHAPTER 11

CONCLUSIONS

AND

SUGGESTIONS FOR FURTHER WORK

11.1 Introduction.

The objective of this dissertation has been to present a unified theory for the analysis of spatial mechanisms. Thus in Part I the basic theory is developed, whilst Part II is devoted to the application of the theory to the analysis of various specific mechanisms. The approach was based primarily on spherical trigonometry and led to the development of the important X-Y-Z notation detailed in Chapter 4 where loop equations are derived for spherical polygons simply by adding a succession of spherical triangles to one another and using the triangle laws. In particular, the three basic laws derived for each spherical polygon (i.e. the sine, sine-cosine and cosine laws) have proved to be of fundamental importance, whilst the notation (whose significance in representing Direction Cosines is outlined in Chapter 4) has greatly facilitated the writing of these laws, since it reduces their length and complexity enormously. In addition, certain, previously unsuspected, fundamental half-tangent laws (Chapter 5) have arisen naturally as expressions for the common root between pairs of sine and sine-cosine laws, and these play a crucial role in the analysis of the five and six-link (particular the 5R-C) spatial mechanisms presented in Part II.

Having derived the various trigonometrical laws and expressions describing spherical polygons in Chapters 4 and 5 the author has then shown how one may "dualize" (see Chapter 3) these to yield a corresponding scheme of laws applicable to spatial polygons, via the all-important Principle of Transference (proved in Chapter 3). These dual laws have then been used successfully to analyse four, five and six-link spatial mechanisms in Part II, and the results presented there are in exact agreement with the preliminary geometrical predictions of Chapter 2. In the latter chapter an attempt was made to classify certain groups of mechanisms collectively as being derived from particular basic structures (see Table II) and this categorisation led to various predictions about the degrees of the input-output equations of these spatial mechanisms.

Finally, the author decided that it would prove more fruitful to analyse the 5R-C mechanisms (Chapter 10) rather than to continue with an analysis of the 5R-2P seven-link mechanisms in Part II since the former posed a significantly different problem. However, the method of analysis for the 5R-2P mechanisms appears to be reasonably straightforward and involves the elimination of a single unknown between two equations.

There are three distinct 5R-2P mechanisms and these have been recently analysed by Keen [22] who obtained a degree eight equation for the RPPRRRR, a degree twelve for the RPRPRRR and a degree sixteen equation for the RPRRPRR. An outline of the method of analysis for these seven-link mechanisms is given below.

The chief advantages of the method presented here over the numerous other approaches appearing in the literature may therefore be listed as follows:-

(i) The method is of general applicability.

(ii) The degrees of the input-output equations derived in Part II are the same as or lower than any derived elsewhere for the particular mechanisms analysed there.

(iii) A physical significance has been attached (see Chapter 2) to the degree of an input-output equation (or number of closures of a mechanism), and to the phenomenon of different closures having equations of different degrees.

(iv) Using the special values of the parameters adopted in Chapter 2 for ease of visualisation, one may obtain input-output equations having all-real solutions without recourse to tedious random-search techniques. This process also verifies that no extraneous roots have been introduced into the analysis, and illustrates the relationships that exist between structures and mechanisms.

(v) The equations that one must manipulate, though still of an extremely complex nature, have been reduced to a concise and manageable form by means of the X-Y-Z notation derived in Chapter 4, and have been clearly categorised into a natural scheme of sine, sine-cosine and cosine laws, in analogy with the case of the spherical triangle.

(vi) The fundamental half-tangent laws (see Chapter 5) are linear in the half-tangent of a single angular displacement and hence one may begin an analysis with equations of a lower degree than was previously possible.

(vii) The problem formulation for obtaining the maximum number of closures for each mechanism, has been shown to fall into one of four possible types. Thus, the four-link RCCC requires no elimination procedure to obtain its input-output equation; all five-link 3R-2C, six-link 4R-P-C and seven-link 5R-2P mechanisms require the elimination of a single unknown between two equations; input-output equations for six-link 5R-C mechanisms are obtained by eliminating two unknowns from four equations; the 7R mechanism appears to require the elimination of three unknowns from eight equations (see below).

(viii) For the first time a six-link mechanism (i.e. the 5R-C) with a significantly different problem formulation (the elimination of two unknown angular displacements) has been tackled and successfully analysed (see Chapter 10).

There now appear to be three main areas in which further work may be carried out. The first, and most important at present, is an analysis of all seven-link (including the 7R) spatial mechanisms together with the RRRRRC inversion of the 5R-C mechanisms, since this will then complete the analysis.

Secondly, an investigation of the gross motion of five, six and seven-link spatial mechanisms, similar to that carried out by Gilmartin [50] for the four-link mechanism, is warranted. In particular, the prediction of limit positions and the derivation of conditions for complete rotability of the input link of a mechanism are of importance to the designer.

Finally, it is clear that an essential factor in the practical use of a spatial mechanism is a thorough knowledge and understanding of its dynamic behaviour. Thus, an obvious field for further work is an investigation of the velocities, accelerations, forces, torques etc., present in the components of a spatial mechanism in motion.

In the following sections brief outlines are given of suggested approaches to these areas of further research, based on the unified theory presented here.

11.2. Analysis of Seven-Link Mechanisms.

The first area of research mentioned above is an analysis of the seven-link spatial mechanisms, and there are two distinct groups of these:- the 5R-2P mechanisms (which require the elimination of a single unknown between two equations) and the 6R-P and 7R mechanisms which pose more difficult problems

11.2.1 The 5R-2P Seven-Link Spatial Mechanisms.

For the RPPRRRR mechanism (Figure 2.29) one may write three different primary equations relating the input (θ_1) output (θ_7) and a single extraneous angular displacement, since the angles, θ_{22} and θ_{33} are constant mechanism proportions. These three equations are (see Appendix III):-

$$Z_{71234} = \cos\alpha_{56}. \tag{11.1}$$

$$Z_{67123} = \cos\alpha_{45} \tag{11.2}$$

$$Z_{7123} = \bar{Z}_5 \tag{11.3}$$

One must now use the secondary equation:-

$$Z_{017654} = - a_{23}\sin\alpha_{23} \tag{11.4}$$

where:-

$$\begin{aligned} Z_{017654} = {} & a_{12}Y_{45671} \\ & + S_{11}\sin\alpha_{12}X_{45671} \\ & + a_{71}[(X_{456}\sin\theta_7 + Y_{456}\cos\theta_7)\bar{Z}_1 + Z_{456}\bar{Y}_1] \\ & - S_{77}[(X_{456}\sin\theta_7 + Y_{456}\cos\theta_7)\bar{X}_1 + X_{4567}\bar{Y}_1] \\ & + a_{67}[(X_{45}\sin\theta_6 + Y_{45}\cos\theta_6)Z_{17} + Z_{45}Y_{17}] \\ & + S_{66}[(Y_{17}Y_{45} - X_{17}X_{45})\sin\theta_6 - (Y_{17}X_{45} + X_{17}Y_{45})\cos\theta_6] \\ & + a_{56}[(X_{17}\sin\theta_6 + Y_{17}\cos\theta_6)Z_{45} + Z_{17}Y_{45}] \\ & - S_{55}[(X_{176}\sin\theta_5 + Y_{176}\cos\theta_5)X_4 + X_{1765}Y_4] \\ & + a_{45}[(X_{176}\sin\theta_5 + Y_{176}\cos\theta_5)Z_4 + Z_{176}Y_4] \\ & + S_{44}\sin\alpha_{34}X_{17654} \\ & + a_{34}Y_{17654} \end{aligned} \qquad (11.5)$$

since this alone contains all the fixed parameters. The problem is then to transform (11.4) into an equation in the three angular displacements θ_1, θ_7 and either θ_4, θ_5 or θ_6 only. It would then be possible to eliminate the latter extraneous variable from the transformed version of (11.4) and the appropriate primary equation (i.e. (11.1), (11.2) or (11.3)), and obtain the input-output equation for the RPPRRRR mechanism.

In a similar manner the input-output equation for the RPRPRRR mechanism (Figure 2.30) may be obtained by eliminating a single unknown between one of the following three primary equations (see Appendix III):-

$$Z_{71234} = \cos\alpha_{56} \qquad (11.6)$$

$$Z_{6712} = \bar{Z}_4 \qquad (11.7)$$

$$Z_{712} = Z_{54} \qquad (11.8)$$

and the appropriate transformed version of the following secondary equation:-

$$Z_{01765} = Z_{03} \qquad (11.9)$$

where:-

$$
\begin{aligned}
Z_{01765} = \; & a_{12}Y_{5671} \\
& + S_{11}\sin\alpha_{12}X_{5671} \\
& + a_{71}[(X_{56}\sin\theta_7 + Y_{56}\cos\theta_7)\bar{Z}_1 + Z_{56}\bar{Y}_1] \\
& - S_{77}[(X_{56}\sin\theta_7 + Y_{56}\cos\theta_7)\bar{X}_1 + X_{567}\bar{Y}_1] \\
& + a_{67}\operatorname{cosec}\alpha_{67}(Z_{17}Z_{56} - \bar{Z}_1 Z_5) \\
& - S_{66}[(X_{17}\sin\theta_6 + Y_{17}\cos\theta_6)X_5 + X_{176}Y_5] \\
& + a_{56}[(X_{17}\sin\theta_6 + Y_{17}\cos\theta_6)Z_5 + Z_{17}Y_5] \\
& + S_{55}\sin\alpha_{45}X_{1765} \\
& + a_{45}Y_{1765}
\end{aligned}
\tag{11.10}
$$

and:-

$$
\begin{aligned}
Z_{03} = \; & a_{23}\bar{Y}_3 \\
& + S_{33}\sin\alpha_{23}\sin\alpha_{34}\sin\theta_3 \\
& + a_{34}Y_3
\end{aligned}
\tag{11.11}
$$

(Note that the angular displacements θ_{22} and θ_{44} are constants for the RPRPRRR mechanism).

Finally, for the RPRRPRR mechanism (Figure 2.31), the input-output equation may be obtained by eliminating a single unknown (θ_{22} and θ_{55} are constants) between one of the following three primary equations (see Appendix III):-

$$Z_{7123} = \bar{Z}_5 \tag{11.12}$$

$$Z_{56712} = \cos\alpha_{34} \tag{11.13}$$

$$Z_{712} = Z_{54} \tag{11.14}$$

and the appropriate transformed version of the following secondary equation:-

$$Z_{0176} = Z_{034} \tag{11.15}$$

where:-

$$
\begin{aligned}
Z_{0176} = \; & a_{12}Y_{671} \\
& + S_{11}\sin\alpha_{12}X_{671} \\
& + a_{71}[(X_6\sin\theta_7 + Y_6\cos\theta_7)\bar{Z}_1 + Z_6\bar{Y}_1] \\
& + S_{77}[(\bar{Y}_1Y_6 - \bar{X}_1X_6)\sin\theta_7 - (\bar{Y}_1X_6 + \bar{X}_1Y_6)\cos\theta_7] \\
& + a_{67}[(\bar{X}_1\sin\theta_7 + \bar{Y}_1\cos\theta_7)Z_6 + \bar{Z}_1Y_6] \\
& + S_{66}\sin\alpha_{56}X_{176} \\
& + a_{56}Y_{176}
\end{aligned}
\qquad (11.16)
$$

and:-

$$
\begin{aligned}
Z_{034} = \; & a_{23}Y_{43} \\
& + S_{33}\sin\alpha_{23}X_{43} \\
& + a_{34}\operatorname{cosec}\alpha_{34}(Z_3\bar{Z}_4 - \cos\alpha_{23}\cos\alpha_{45}) \\
& + S_{44}\sin\alpha_{45}X_{34} \\
& + a_{45}Y_{34}
\end{aligned}
\qquad (11.17)
$$

11.2.2. The 6R-P and 7R Seven-Link Spatial Mechanisms.

It seems likely that the input-output equation for the RRRRRPR seven-link mechanism (Figure 2.32) may be obtained using a similar procedure to that used for the 5R-C mechanisms (Chapter 10) since the problem formulation appears to be the same (i.e. the elimination of two unknowns from four equations). This is because one may write primary equations in the input (θ_1), output (θ_7) and two extraneous angular displacements (θ_{66} is constant), and hence one would expect to obtain a scheme of equations, similar to equations (10.30), where the coefficients $a_i, \ldots\ldots, f_i$, are now functions of θ_1 and θ_7.

Now the input-output equation for the six-link RRRRRC mechanism (Figure 2.28) is clearly of degree sixteen in the output angle (θ_5) if the predictions of Chapter 2 are correct (see Table II), since fixing the <u>input angle</u> (θ_6) reduces the mechanism to the RRCRR structure, which has sixteen assemblies. However, if one holds the <u>output angle</u> constant one obtains the RRRRRP structure which has n_4 assemblies say (see Table II). Furthermore, the RRRRRPR mechanism also reduces to the latter structure for a fixed input angle (θ_1). Consequently the RRRRRPR seven-link mechanism has an input-output equation of the same degree, n_4, in the output angle (θ_7), as that of the equation of the six-link RRRRRC mechanism in the input angle (θ_6).

Finally, the most difficult and perhaps the most important and significant (from the point of view of the number of possible reductions to other mechanisms) spatial mechanism to analyse is the RRRRRRR or 7R seven-link mechanism (Figure 2.33). Now since one may write various primary equations involving the input (θ_1), output (θ_7) and three extraneous angular displacements (see Appendix III) it seems clear that one is faced with the elimination of at least three unknowns. It is suggested by the author that the correct problem formulation for the 7R mechanism is the elimination of three unknowns from eight equations. The exact form of these equations is not yet clear although two possible schemes suggest themselves, in analogy with equations (10.30) (see Chapter 10). Thus, if it is possible to derive a set of eight equations of the following form:-

$$\begin{aligned}&[(a_i x^2 + b_i x + c_i)y + (d_i x^2 + e_i x + f_i)]z \\ &+ [(g_i x^2 + h_i x + k_i)y + (l_i x^2 + m_i x + n_i)] = 0 \qquad (11.18)\end{aligned}$$

where a_i, b_i,..., n_i (i = 1, 2,..., 8) are functions of the input and output angles (θ_1 and θ_7) only, and x, y, z are each the half-tangent of an extraneous angular variable, then by multiplying the system (11.18) throughout by x, one obtains a set of sixteen non-homogeneous equations in the fifteen variables, x^3yz, x^3y, x^3z, x^2yz, x^3, x^2y, x^2z, xyz, x^2, xy, yz, xz, x, y and z. By taking the determinant of this system, treated as a set of linear equations, one then obtains a (16 x 16)-determinant of the coefficients a_i,..., n_i, as the eliminant. Hence, if the coefficients are each quadratic in input and output angles the input-output equation for the 7R mechanism would be of degree <u>thirty-two</u> in these.

Alternatively, the following system may arise:-

$$\left[(a_i x^2 + b_i x + c_i)y^2 + (d_i x^2 + e_i x + f_i)y + (g_i x^2 + h_i x + k_i)\right] z$$
$$+ \left[(l_i x^2 + m_i x + n_i)y^2 + (p_i x^2 + q_i x + r_i)y + (s_i x^2 + t_i x + u_i)\right] = 0 \tag{11.19}$$

where i = 1, 2,...., 8., and the coefficients are again functions of Θ_1 and Θ_7 only. If this is the case, then multiplying the system (11.19) throughout by x, y and xy in turn, produces a set of thirty-two non-homogeneous equations in the thirty-one variables, x^3y^3z, x^3y^3, x^3y^2z, x^2y^3z, x^3y^2, x^2y^3, x^3yz, x^2y^2z, xy^3z, x^3y, x^3z, xy^3, y^3z, x^2y^2, x^2yz, xy^2z, x^3, y^3, x^2y, x^2z, xy^2, y^2z, xyz, x^2, y^2, xy, xz, yz, x, y and z. By treating the latter as linear unknowns and eliminating, one obtains a (32 x 32)-determinant of the coefficients a_i,...., u_i, as the eliminant of the system. This clearly leads to a degree <u>sixty-four</u> input-output equation for the 7R mechanism if the coefficients are quadratic in Θ_1 and Θ_7.

As justification for the two suggested systems (equations (11.18) and (11.19)) one may list the following two points:-

(i) It may be possible to transform the intermediate half-tangent laws mentioned in Chapter 5 (see equation (5.56)) into the form of equations (11.18) since they are initially in a similar form, and, in addition, there exist <u>eight</u> such suitable equations of the type (5.56).

(ii) It is possible to write immediately <u>four</u> equations of the form of (11.19) for the 7R mechanism since these are the heptagon equations corresponding to equations (10.30) for the hexagon. Furthermore it may be possible to reduce a suitable cyclic permutation of these, to the same form and hence obtain <u>eight</u> equations in all.

It is the opinion of the author that if the 6R-P mechanism proves to have an input-output equation of degree sixteen (i.e. $n_4 = 16$) then it seems likely that the 7R mechanism should have an equation of degree thirty-two. Current research work appears to indicate that this may be correct, and that the set of equations (11.19) is the more probable alternative. If this is the case then there must occur certain proportionalities amongst the coefficients in order that the final eliminant may reduce from degree sixty-four to degree thirty-two (see also equations (8.24) and (8.25) of Chapter 8).

Thus, from the above discussion and Chapters 6-10 it seems probable that for the elimination of n unknowns one requires 2^n equations.

11.3. The Gross Motion of Spatial Mechanisms.

One of the most important attributes of an algebraic (as opposed to a numerical) approach to the analysis of spatial mechanisms, is that it can provide a firm foundation for the further study of such important areas of linkage behaviour as the following:-

(i) The determination of those proportions which lead to overclosure.

(ii) The derivation of criteria for rotability.

(iii) The determination of type, limit positions and number of branches of the input-output curves.

(iv) The determination of transmission characteristics etc.

In particular, a knowledge of the degree of an input-output equation should theoretically enable one to predict the maximum number of turning points (or limit positions) that the graph of the equation can have. Thus, treating the input-output relationship as a polynomial in the output variable alone, it is clear that a limit position exists at those values of the input for which this polynomial has at least two equal roots. This will only occur if the polynomial and its first derivative share a common root. Hence one requires the discriminant of the polynomial to be zero (see Bôcher [2]). This is simply the eliminant of the polynomial and its first derivative. Thus, in the case of a quadratic one has:-

$$f(x) = a_2x^2 + a_1x + a_0 = 0 \qquad (11.20)$$

$$f'(x) = 2a_2x + a_1 = 0 \qquad (11.21)$$

and D(f), the discriminant of f, is given by:-

$$D(f) \equiv E(f, f') = \begin{vmatrix} a_2 & a_1 & a_0 \\ 0 & 2a_2 & a_1 \\ 2a_2 & a_1 & 0 \end{vmatrix} = 0 \qquad (11.22)$$

However, one may always reduce this determinant by one order since the leading coefficients of (11.20) and (11.21) are proportional, and this is true for any polynomial (see also Chapter 5 and equations (8.24) and (8.25)). Thus a polynomial of degree n in the output variable has a discriminant of order 2(n - 1) in the coefficients. In the case of a spatial mechanism, the latter coefficients are also polynomial expressions, of degree m, in the input variable and hence the discriminant is a polynomial of degree 2.(n - 1).m in the input. Consequently there are at most 2.(n - 1).m limit positions.

For the RCRCR five-link mechanism for example (Chapter 6 and Figure 2.24) one has n = m = 4, and hence the above discussion suggests that there exist a maximum of twenty-four possible limit positions for this mechanism. It has been suggested by Gilmartin [50] that this number may be too high, and that extraneous roots may have been introduced into the analysis, although the present author is of the opinion that this is not the case and that it is possible to reconcile four closures with twenty-four limit positions without producing any contradictions.

Now, unlike the case of the planar four-link mechanism, for spatial five, six and seven-link mechanisms, the broad classifications of "crank-rocker", "rocker-crank" and "double-rocker" are somewhat inadequate terms as can be appreciated from a consideration of the graphs in Chapters 7-10 of this dissertation and those obtained by Habib-Olahi [19] and Keen [22] etc. It is therefore suggested that a more meaningful categorisation be developed for spatial linkages.

It may be useful to consider Figure 11.1 in this respect. Thus, normally, one plots input-output curves on a finite rectangular plane of side 2π and obtains various different types of curve, such as A (double-rocker), B (crank-rocker) and C (rocker-crank). Since the co-ordinate axes represent input or output angles in radians, the curves would repeat, at intervals of 2π on a plane of infinite extent and hence it suffices to consider a finite plane, (i.e. $(0, 2\pi) \times (0, 2\pi)$) as shown in Figure 11.1. The latter may, therefore, be mapped onto the surface of a right circular torus in the manner shown, by identifying the four corners, P_1, P_2, P_3 and P_4.

Following this procedure, it is clear that all three curves (A, B and C) may now be considered to be closed. However, they differ in their topological properties. Thus only curve A can be "shrunk to a point" (i.e. it is homotopic to zero), whereas this is not possible with B or C. Furthermore none of these curves can be continuously deformed into another. Thus there are at least these three distinct fundamental types of curve, as well as others of a type that may "spiral" the torus by various numbers of turns.

It is suggested that perhaps a classification of spatial mechanisms, based on the above topological considerations, may be developed and related to conditions for rotability, etc.

Finally, having shown in Chapter 2 that two inversions of the same spatial mechanism may have differing numbers of closures, the question arises as to whether or not there exist any true invariants for a given closed kinematic loop, irrespective of the particular inversion. Perhaps the maximum number of loops (or branches) of the graph of an input-output relationship is such an invariant?

11.4. The Dynamics of Spatial Mechanisms.

In the long-term the dynamics of spatial mechanisms will be of fundamental importance to the designer. It is thus essential to be able to manipulate expressions for the angular velocities, accelerations, forces, torques, etc., present in the links of a spatial mechanism. The author suggests that initially one may be able to obtain reasonably simple expressions for the derivatives, with respect to time, of the various X-Y-Z terms presented in this dissertation, and perhaps this may form a foundation on which to proceed.

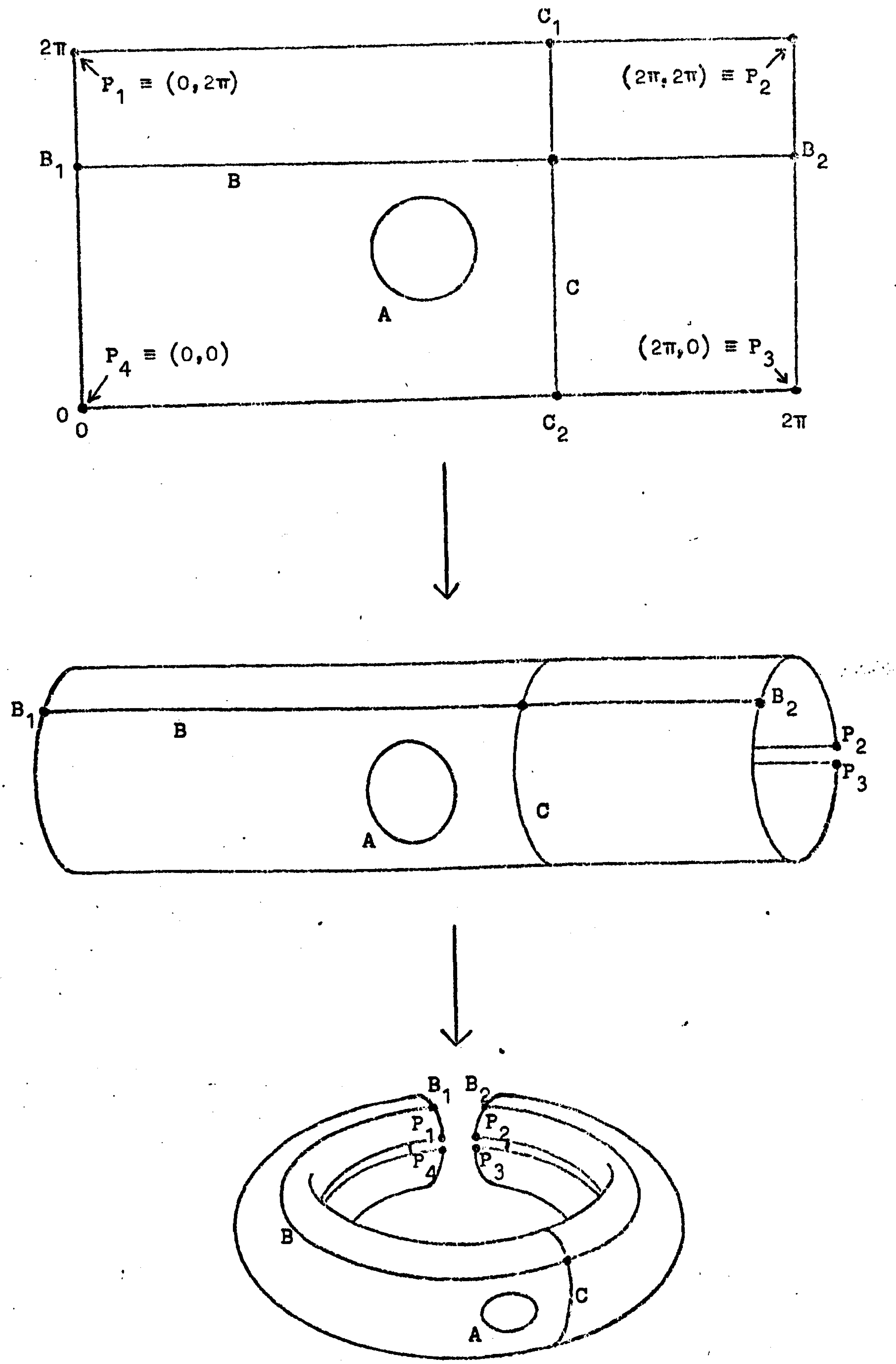


Figure 11.1 Mapping the Finite Plane, $(0,2\pi) \times (0,2\pi)$, onto the Surface of a Right Circular Torus with Circular Cross-Section.

APPENDICES

APPENDIX I

A PRELIMINARY OUTLINE

OF

A POSSIBLE ALGEBRAIC APPROACH

TO THE PROBLEM OF PREDICTING

THE CLOSURES OF SPATIAL MECHANISMS

A. Basic Concepts.

It is suggested in Chapter 2 that the number of closures of a spatial mechanism may be predicted in advance using algebraic and projective geometry and it is the intention of the author to outline in this appendix a possible procedure to achieve this end. It is necessary therefore to base the geometry of three-dimensional Euclidean space on the fundamental concepts of 'point', 'line' and 'plane', where these objects are treated as elements of the projective space thus produced. (see Semple and Roth [32]). In particular, various line ensembles (i.e. systems of lines) are of importance in this context. (For present purposes, a 'line' is considered to be an infinite straight line in the usually accepted sense).

Now it is clear from an algebraic point of view that there exist ∞^4 distinct lines in Euclidean space of three dimensions, since four independent parameters are required to specify any given line. The validity of this may also be seen geometrically since:-

(i) There are an ∞^2 of lines through any one point.

(ii) There are an ∞^3 of points in three-dimensional space.

(iii) Each line passes through an ∞^1 of points.

Hence from (i), (ii) and (iii) there are clearly ∞^4 ($= \infty^2 \times \infty^3/\infty^1$) distinct lines in all (see also Hunt [21] and Woo and Freudenstein [42]).

Now, certain preferred subsets of this ∞^4 system of lines have been studied in detail by several authors (see Semple and Roth [32]), and three particular types are of special interest here: the line series, the congruence and the complex.

A.1. The Line Series.

The line series or regulus is an ∞^1 algebraic ensemble of lines. In other words, each line of the ensemble requires a single parameter to specify its position. Such systems of lines include ruled surfaces (planes, cylinders, hyperboloids of one sheet, etc).

A.2. The Congruence.

The congruence is an algebraic system of ∞^2 lines. A finite number, m, of these pass through a generic point, whilst a finite number, n, lie in a generic plane. The integers m and n are termed the order and class of the congruence respectively, and are collectively referred to as the indices of the system.

A congruence is usually symbolised by $K^{(m,n)}$ and the point star, $K^{(1,0)}$, the ruled plane, $K^{(0,1)}$, and the linear congruence, $K^{(1,1)}$, are particular examples.

A.3. The Complex.

The complex is an algebraic ensemble of ∞^3 lines. One of the more important types of complex is the linear complex dealt with by Hunt [21].

The most interesting of the above three types of line ensemble, from the point of view of this dissertation, is the congruence, and there exists an important theorem, applicable to systems of the latter, which is stated below.

A.4. Halphen's Theorem.

Halphen's theorem determines the number of lines common to two congruences. The theorem states that:-

The number of rays (lines) common to two congruences, $K_1^{(M,N)}$ and $K_2^{(m,n)}$, in general position, is Mm + Nn.

B. Algebraic Prediction of the Number of Closures of a Spatial Mechanism.

From the definition of a congruence as an ∞^2 ensemble of lines it is clear that each line of the system requires two parameters to specify it. Now, of the six open spatial chains dealt with in Chapter 2. there are four (i.e. the Cc, RRc, PRc and RPc) which generate a two-parameter family of lines. Thus the Cc open chain (Figure 2.7) consists of a single link, a_{12}, constrained so that it can both rotate about and slide along $\hat{\underline{S}}_1$ and since there is a one-one correspondence between the points of the cylinder thus generated and the tangent line vectors, $\hat{\underline{S}}_2$, the latter clearly require two parameters for their specification. Consequently, the tangent line vector field on the surface of the cylinder forms an ∞^2 ensemble of lines and is, therefore, a congruence. It is suggested by the author that this congruence has indices, (2, 2), and this is justified somewhat by reference to Figure I.1 which illustrates the two lines passing through a generic point and the two lines lying in a generic plane for the Cc open chain.

In a similar manner the RRc open chain (Figure 2.8) also produces a system of lines (in one-one correspondence with the points on the surface of a skew torus) requiring two co-ordinates for their specification and in this case it is suggested that the resulting congruence has indices $m = n = 4$. Again Figure I.2 illustrates the four lines passing through a generic point and the four lines lying in a generic plane.

Using similar reasoning to the above it is suggested that both the PRc (Figure 2.11) and RPc (Figure 2.12) open chains generate congruences with indices (2, 2).

Finally, the CRc chain (Figure 2.10) requires three parameters (one sliding and two angular displacements) to specify the position of its free end and hence its associated line vectors generate an ∞^3 system of lines (i.e a complex), whilst the Rc chain (Figure 2.6) requires only a single parameter and its line vectors thus generate a regulus (a hyperboloid of one sheet in this case).

Now if the m and n values conjectured above are correct one may apply Halphen's Theorem and list the following three results:-

(i) Two Cc chains should have eight (= 2.2 + 2.2) lines in common.

(ii) A Cc, PRc or RPc chain and an RRc chain should have sixteen (= 2.4 + 2.4) lines in common.

(iii) Two RRc chains should have thirty-two (= 4.4 + 4.4) lines in common.

At first sight, however, these results do not appear to be consistent with the number of assembly configurations for the various structures thus produced (see Table II., Chapter 2). However, the following two observations are valid:-

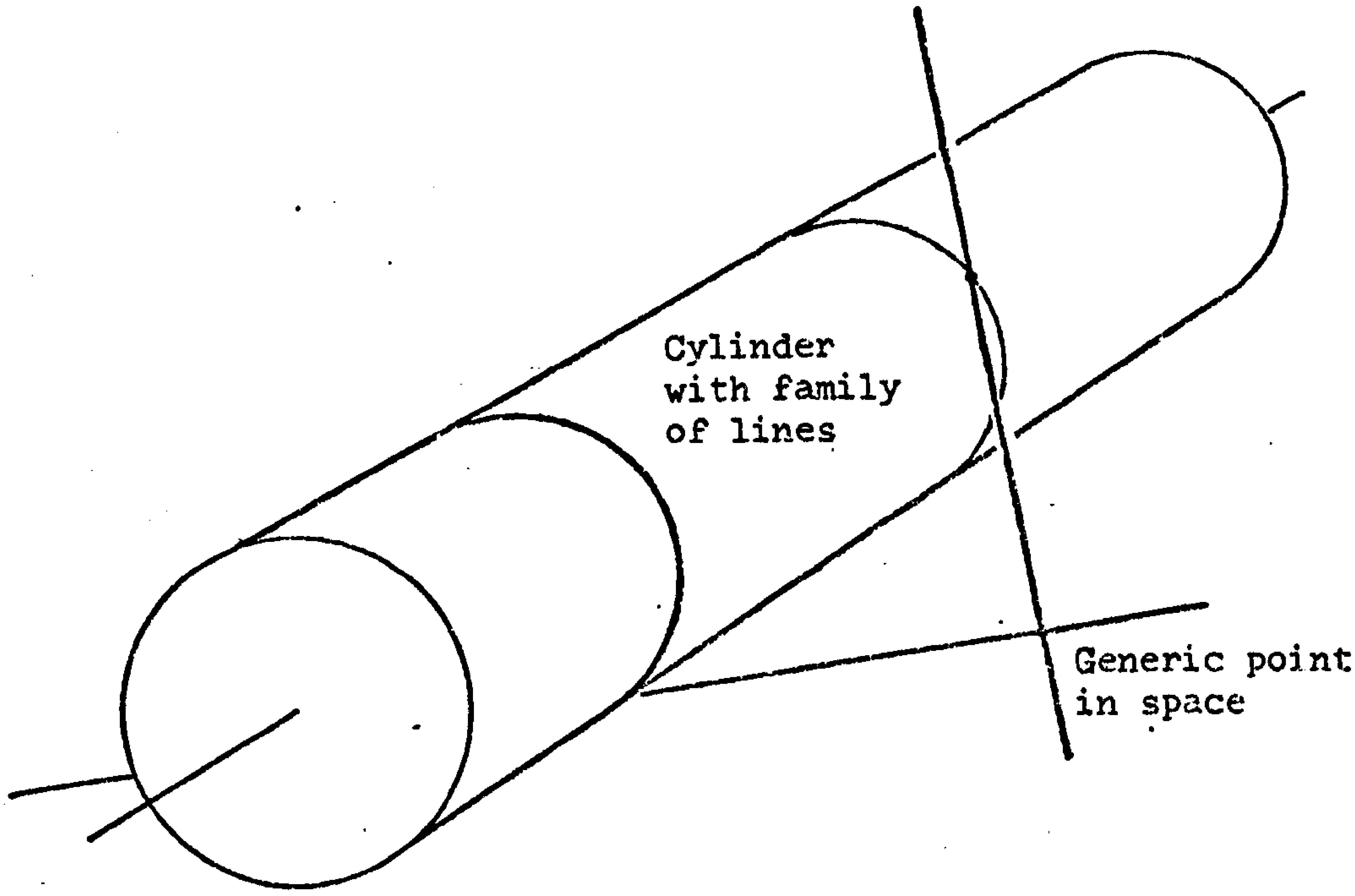
(i) A Cc chain produces a cylinder of infinite extent and hence possesses tangent lines at infinity. (A similar situation also occurs for the PRc and RPc chains).

(ii) The systems of lines generated by open spatial chains must be considered to be systems of <u>directed</u> lines in order that mechanism proportions (particularly the twist angles) should remain constant.

Consequently, it appears that of the eight common lines between two Cc chains, four lie at infinity, and of the remaining four, only two are acceptable as common <u>directed</u> lines. In the case of a Cc, PRc or RPc chain with an RRc chain, however, the latter does not produce any lines at infinity, and hence there can be no common lines at infinity. Nevertheless, of the sixteen common lines between the two chains, only half are acceptable common directed lines and one must therefore discount eight (see Chapter 2). Finally, in the case of two RRc chains, neither congruence possesses lines at infinity (the two torii are of finite extent) and hence cannot have any common lines at infinity. However, one must still discount sixteen (i.e. half) of the thirty-two common lines since these do not preserve direction. (see Figure 2.20). The unacceptable lines are symmetrically positioned with respect to the acceptable ones in Figure 2.20.

Finally it must be noted that the number of assemblies of the RCRC structure depends on the number of common lines between a regulus (Rc chain) and a complex (CRc chain) and thus Halphen's Theorem cannot be applied in this case. In a similar manner the assemblies of the six-link 4R-2P, 5R-P and 6R structures depend on the number of common lines between a congruence (the RRr chain, say) and a complex (the RRRr chain, say) and the theorem is again not applicable.

The approach outlined in this appendix to the problem of proving rigourously the results summarised in Table II is clearly a preliminary one, although the author is of the opinion that these algebraic suggestions are correct particularly concerning the order and class of the congruences considered.



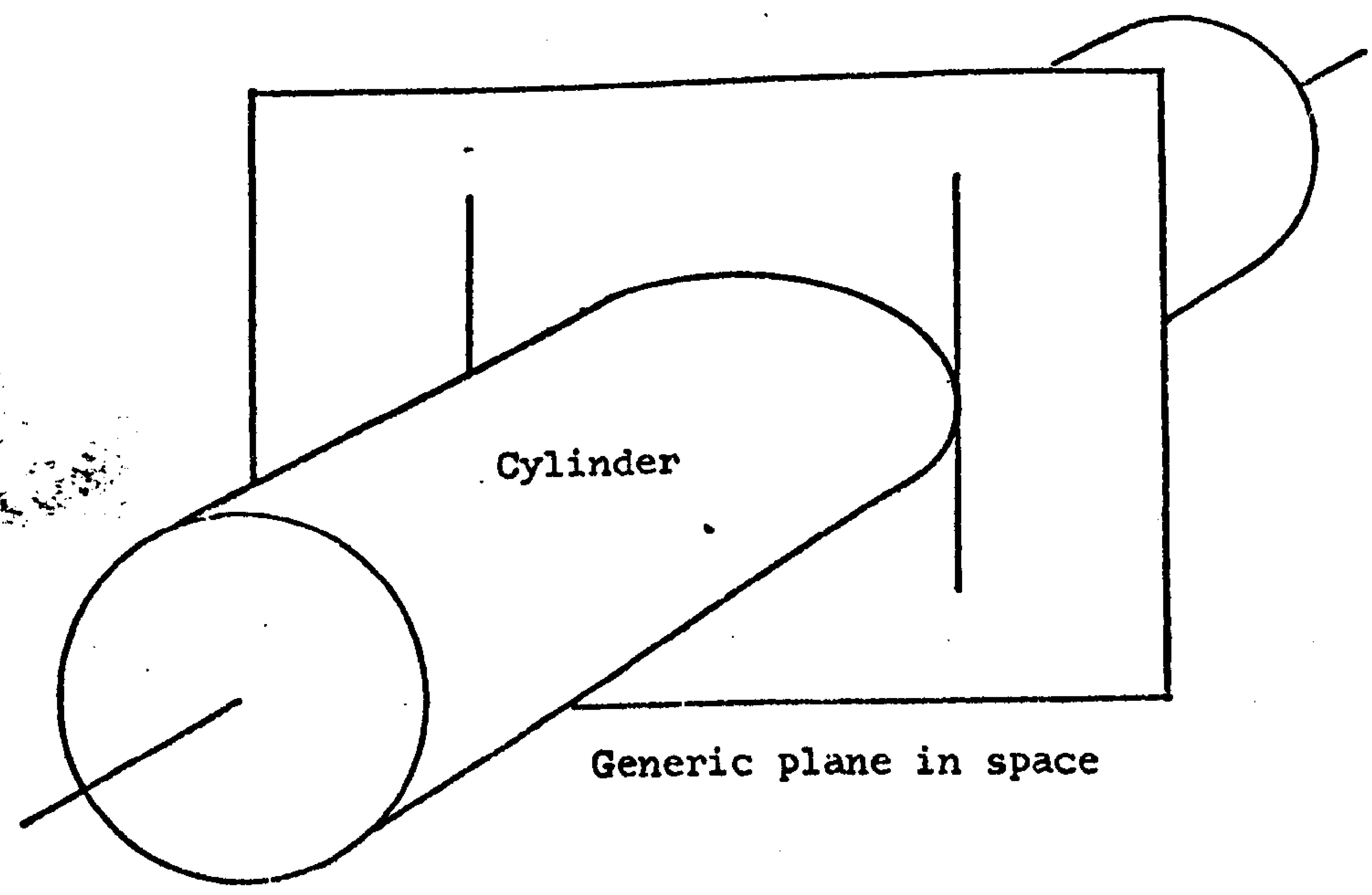
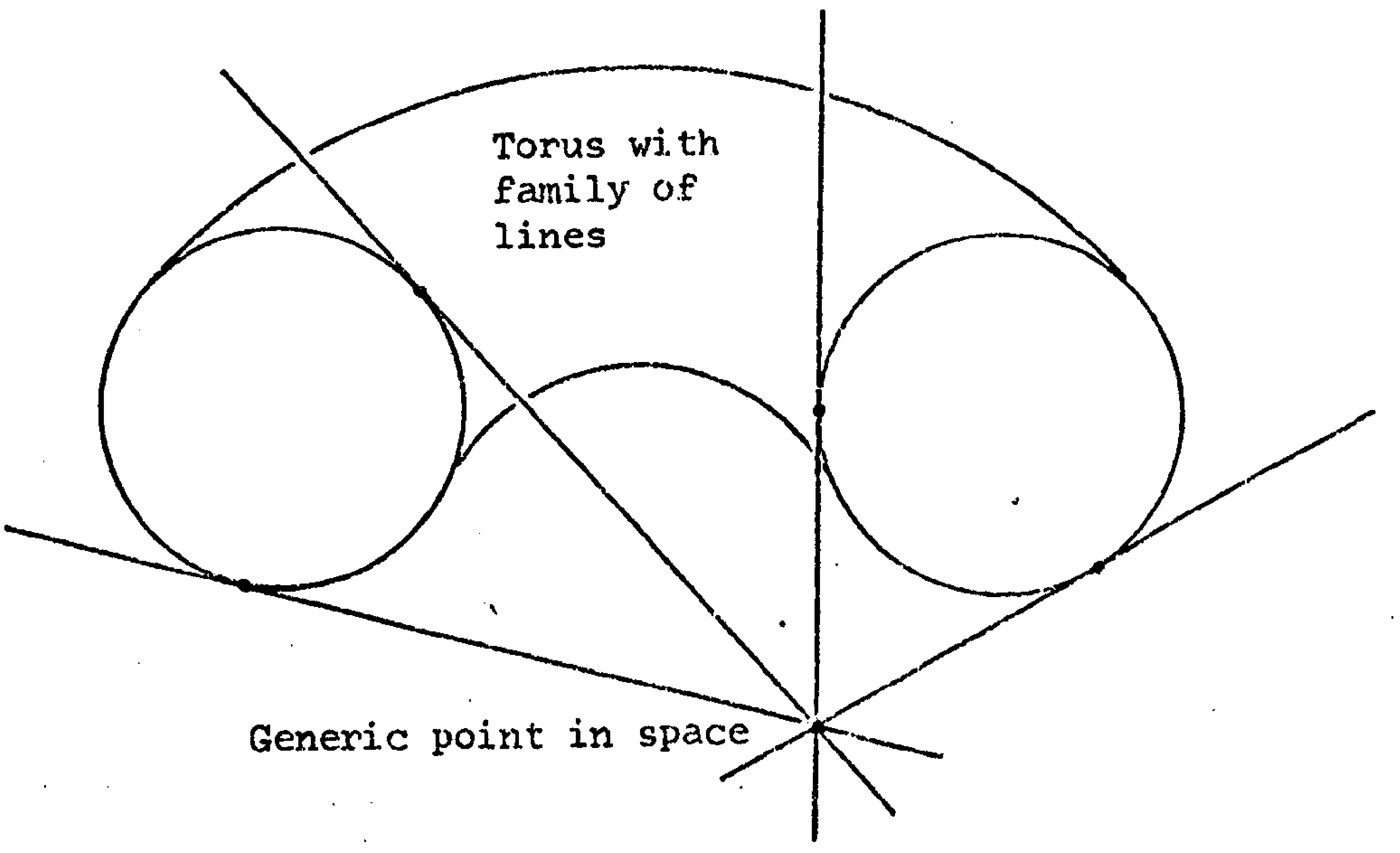


Figure I.1 Illustration of the Two Lines Passing Through a Generic Point and the Two Lines Lying in a Generic Plane for the Congruence Produced by the Cc Open Spatial Chain.



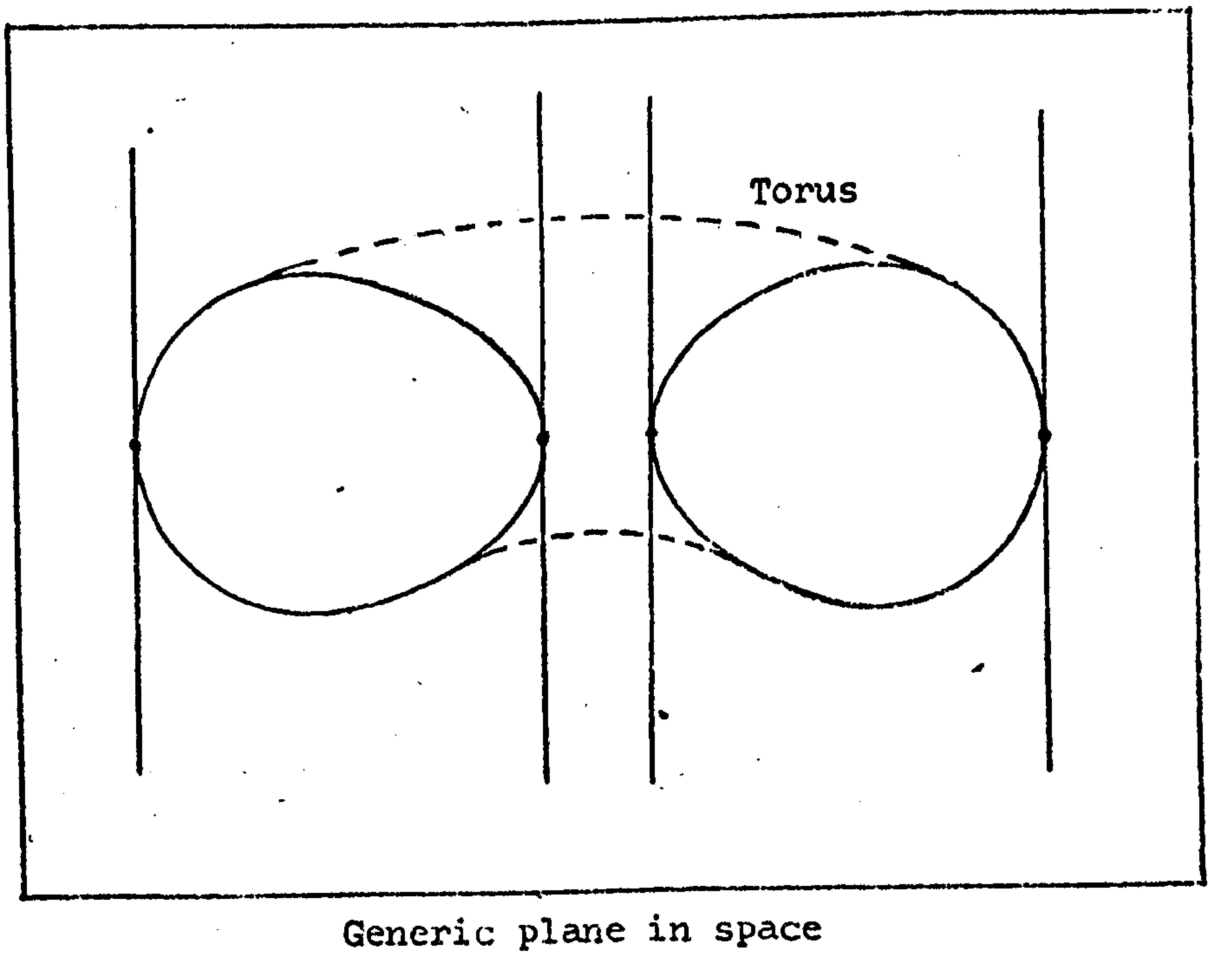


Figure I.2 Illustration of the Four Lines Passing Through a Generic Point and the Four Lines Lying in a Generic Plane for the Congruence Produced by the RRc Open Spatial Chain.

APPENDIX II

FUNDAMENTAL ALGEBRAIC CONCEPTS

A. Relations.

A binary relation, R, on a set, A, is a subset of A x A (the Cartesian product of A with itself), and hence consists of a set of ordered pairs, (a, b), where 'a' and 'b' belong to A. An equivalence relation, R, on A is a relation whose elements satisfy the following three axioms:-

E1. (a, a) is an element of R for each element, 'a', of A.

E2. If (a, b) is an element of R then so is (b, a).

E3. If (a, b) and (b, c) are elements of R then so is (a, c).

Alternatively one may think of a binary relation as a means of combining two elements of A, by defining 'b' to be related to 'a' (written a R b) if and only if (a, b) is an element of R. The above three axioms for an equivalence relation may now be more conveniently restated as follows:-

E1'. a R a for each element, 'a', of A.

E2'. If a R b then b R a.

E3'. If a R b and b R c then a R c.

The first of these properties is called reflexivity, the second, symmetry, and the third, transitivity.

Now if A is a set and if R is an equivalence relation on A, then the equivalence class of an element, 'a', of A is defined to be the subset of all those elements of A which are related to 'a'. In other words the equivalence class of 'a' is the set of all x such that a R x, and is written as [a] .

It is proved in Herstein [20] that: "The distinct equivalence classes of an equivalence relation on A provide a decomposition of A into a union of mutually disjoint subsets. Conversely, given a decomposition of A as a union of mutually disjoint, non-empty subsets, one can define an equivalence relation on A for which these subsets are the distinct equivalence classes."

B. Algebraic Structures.

For present purposes an algebraic structure is a set whose elements may be combined by one or more binary operations which satisfy certain axioms.

There are five such structures which are of interest here: groups, rings, fields, vector spaces and algebras.

B.1. Groups.

Thus a non-empty set of elements, G, forms a group, if in G there is defined a binary operation, termed the product and denoted by ".", such that the following four axioms are satisfied.

G1. If a and b are elements of G then so is a.b (i.e. G is closed under the product).

G2. If a, b and c belong to G then a.(b.c) = (a.b).c (associative law).

G3. There exists an element, e, of G such that a.e = e.a = a for all elements, a, of G (i.e. G has an identity element).

G4. For each element, a, of G there exists an element, a^{-1}, such that $a.a^{-1} = a^{-1}.a = e$ (i.e. every element has an inverse in G).

A group G is an abelian group (i.e. commutative) if for every a and b in G one has, a.b = b.a.

B.2. Rings.

A non-empty set, R, is an associative ring if there are defined, in R, two binary operations, denoted by "+" and "." respectively, such that for all a, b and c in R, the following four axioms are satisfied:-

R1. The set, R, forms an abelian group under the operation of "+" (termed addition), with identity element denoted by, 0, (zero) and inverses denoted by minus signs (-a, -b, etc.)

R2. a.b is in R (i.e. closure under multiplication, ".").

R3. a.(b.c) = (a.b).c (multiplication is associative).

R4. a.(b + c) = a.b + a.c and (b + c).a = b.a + c.a (the two distributive laws).

If the multiplication of R is such that a.b = b.a for every a and b in R then R is called a commutative ring.

Furthermore, if there is an element, 1, in R such that a.1 = 1.a = a for every a in R, then R is a ring with unit element.

Finally, if R is a commutative ring then any non-zero element, 'a', of R is said to be a zero-divisor if there exists another non-zero element, 'b', such that a.b = 0.

B.3. Fields.

A non-empty set, F, forms a field if there are defined in F two binary operations, "+" and "." such that the following two axioms are satisfied:-

F1. The set, F, is a commutative ring with unit element under the operations "+" and ".".

F2. The non-zero elements of F form an abelian group under multiplication.

B.4. Vector Spaces.

A non-empty set, V, is said to be a vector space over a field F if V is an abelian group under an operation, "+", and if for each α in F and v in V there is defined an element (written αv) in V subject to the following four axioms:-

V1. $\alpha(v + w) = \alpha v + \alpha w$

V2. $(\alpha + \beta)v = \alpha v + \beta v$

V3. $\alpha(\beta v) = (\alpha\beta)v$

V4. $1v = v$

for all elements α, β of F and v, w of V.

B.5. Algebras.

A non-empty set, A, is called an algebra over a field F, if the following three axioms are satisfied by its elements:-

A1. The elements of A form an associative ring under two operations, "+" and ".".

A2. The elements of A form a vector space over F.

A3. For all elements, a and b, of A and α of F one has the condition:-

$$\alpha(a\, b) = (\alpha a)b = a(\alpha b)$$

C. The Structure of Rings.

A ring, R, whose elements satisfy the axioms R1-R4 (see above) may possess certain important subsets termed ideals whose properties are relevant to the study of dual numbers (see Chapter 3).

Thus a non-empty subset, U, of R is said to be a two-sided ideal of R if the following two conditions are valid:-

I1. U is a subgroup (i.e. a subset which is a group) of R under addition.

I2. For every element, u, of U and r of R, both u.r and r.u are in U.

The condition I2 simply asserts that U "absorbs" multiplication from the right or left by arbitrary ring elements.

Now, given an ideal, U, of a ring, R, one may construct the set, R/U, of all distinct cosets of U in R, obtained by considering U as a subgroup of R under addition (Note: if U is a subgroup of R and 'a' is an element of R, then a + U, the set of all elements of the form a + u, where u belongs to U, is termed a coset of U in R). By defining an appropriate addition and multiplication on R/U (see equation (3.27) of Chapter 3, and Herstein [20]) the latter has the structure of a ring and is termed the quotient ring of U in R.

Finally, a maximal ideal of a ring, R, is an ideal, U($\neq$ R), such that U is not properly contained by any other ideal of R, apart from R itself. It is proved in Herstein [20] that: "If U is an ideal of R then it is a maximal ideal if and only if R/U is a field". This result is of importance in Chapter 3.

APPENDIX III

LIST OF TRIGONOMETRICAL LAWS

FOR

SPHERICAL AND SPATIAL POLYGONS

A. Spherical or Primary Laws.

All cyclic permutations of the sine, sine-cosine and cosine laws for the spherical triangle, quadrilateral, pentagon, hexagon and heptagon may now be listed (see Chapter 4.), together with the relevant definitions and subsidiary formulae.

A.1. The Spherical Triangle.

(i) Definitions.

With reference to Figure 4.2(a) and equations (4.6) and (4.7), one may define the following basic forms for a spherical triangle (where q, i and j are in ascending consecutive cyclic order):-

$$\begin{aligned} X_i &= \sin\alpha_{qi}\sin\theta_i \\ Y_i &= -(\cos\alpha_{qi}\sin\alpha_{ij} + \sin\alpha_{qi}\cos\alpha_{ij}\cos\theta_i) \\ Z_i &= (\cos\alpha_{qi}\cos\alpha_{ij} - \sin\alpha_{qi}\sin\alpha_{ij}\cos\theta_i) \end{aligned} \tag{III.1}$$

$$\begin{aligned} \bar{X}_i &= \sin\alpha_{ij}\sin\theta_i \\ \bar{Y}_i &= -(\cos\alpha_{ij}\sin\alpha_{qi} + \sin\alpha_{ij}\cos\alpha_{qi}\cos\theta_i) \\ \bar{Z}_i &= (\cos\alpha_{ij}\cos\alpha_{qi} - \sin\alpha_{ij}\sin\alpha_{qi}\cos\theta_i) \end{aligned} \tag{III.2}$$

These definitions are applicable to any dyad in a spherical polygon.

(ii) Fundamental Laws.

Using the above definitions one may now list all cyclic permutations of the basic sine, sine-cosine and cosine laws for a spherical triangle (equations (4.12) and (4.13)) with the aid of Figure 4.2(b) (see Chapter 3.), as follows:-

Sine Laws	Sine-Cosine Laws	Cosine Laws.
$X_1 = \sin\alpha_{23}\sin\theta_2$	$Y_1 = \sin\alpha_{23}\cos\theta_2$	$Z_1 = \cos\alpha_{23}$
$X_2 = \sin\alpha_{31}\sin\theta_3$	$Y_2 = \sin\alpha_{31}\cos\theta_3$	$Z_2 = \cos\alpha_{31}$
$X_3 = \sin\alpha_{12}\sin\theta_1$	$Y_3 = \sin\alpha_{12}\cos\theta_1$	$Z_3 = \cos\alpha_{12}$

$$\bar{X}_1 = \sin\alpha_{23}\sin\theta_3 \qquad \bar{Y}_1 = \sin\alpha_{23}\cos\theta_3 \qquad \bar{Z}_1 = \cos\alpha_{23}$$
$$\bar{X}_2 = \sin\alpha_{31}\sin\theta_1 \qquad \bar{Y}_2 = \sin\alpha_{31}\cos\theta_1 \qquad \bar{Z}_2 = \cos\alpha_{31}$$
$$\bar{X}_3 = \sin\alpha_{12}\sin\theta_2 \qquad \bar{Y}_3 = \sin\alpha_{12}\cos\theta_2 \qquad \bar{Z}_3 = \cos\alpha_{12}$$

There are no subsidiary formulae for a spherical triangle.

A.2. The Spherical Quadrilateral.

(i) Definitions.

If i, j and k are in ascending consecutive cyclic order then one may make the following definitions (see equations (4.38) and (4.39)) for a spherical quadrilateral:-

$$\begin{aligned} X_{ij} &= X_i\cos\theta_j - Y_i\sin\theta_j \\ Y_{ij} &= \cos\alpha_{jk}(X_i\sin\theta_j + Y_i\cos\theta_j) - \sin\alpha_{jk} Z_i \\ Z_{ij} &= \sin\alpha_{jk}(X_i\sin\theta_j + Y_i\cos\theta_j) + \cos\alpha_{jk} Z_i \end{aligned} \qquad \text{(III.3)}$$

$$\begin{aligned} X_{kj} &= \bar{X}_k\cos\theta_j - \bar{Y}_k\sin\theta_j \\ Y_{kj} &= \cos\alpha_{ij}(\bar{X}_k\sin\theta_j + \bar{Y}_k\cos\theta_j) - \sin\alpha_{ij} \bar{Z}_k \\ Z_{kj} &= \sin\alpha_{ij}(\bar{X}_k\sin\theta_j + \bar{Y}_k\cos\theta_j) + \cos\alpha_{ij} \bar{Z}_k \end{aligned} \qquad \text{(III.4)}$$

where X_i, Y_i, Z_i are given by definitions (III.1) for the triangle, and $\bar{X}_k$, $\bar{Y}_k$, $\bar{Z}_k$ are obtained from (III.2) (see previous section).

The above definitions are applicable to any three adjacent links in a spherical polygon.

(ii) Fundamental Laws.

All cyclic permutations of the laws for a spherical quadrilateral may now be listed as follows, (with the aid of Figure 4.4(b)):-

Sine Laws	Sine-Cosine Laws	Cosine Laws.
$X_{12} = \sin\alpha_{34}\sin\theta_3$	$Y_{12} = \sin\alpha_{34}\cos\theta_3$	$Z_{12} = \cos\alpha_{34}$
$X_{23} = \sin\alpha_{41}\sin\theta_4$	$Y_{23} = \sin\alpha_{41}\cos\theta_4$	$Z_{23} = \cos\alpha_{41}$
$X_{34} = \sin\alpha_{12}\sin\theta_1$	$Y_{34} = \sin\alpha_{12}\cos\theta_1$	$Z_{34} = \cos\alpha_{12}$
$X_{41} = \sin\alpha_{23}\sin\theta_2$	$Y_{41} = \sin\alpha_{23}\cos\theta_2$	$Z_{41} = \cos\alpha_{23}$

$$X_{21} = \sin\alpha_{34}\sin\theta_4 \qquad Y_{21} = \sin\alpha_{34}\cos\theta_4 \qquad Z_{21} = \cos\alpha_{34}$$
$$X_{32} = \sin\alpha_{41}\sin\theta_1 \qquad Y_{32} = \sin\alpha_{41}\cos\theta_1 \qquad Z_{32} = \cos\alpha_{41}$$
$$X_{43} = \sin\alpha_{12}\sin\theta_2 \qquad Y_{43} = \sin\alpha_{12}\cos\theta_2 \qquad Z_{43} = \cos\alpha_{12}$$
$$X_{14} = \sin\alpha_{23}\sin\theta_3 \qquad Y_{14} = \sin\alpha_{23}\cos\theta_3 \qquad Z_{14} = \cos\alpha_{23}$$

(iii) Subsidiary Formulae.

The following subsidiary formulae may be derived from the above fundamental laws for a spherical quadrilateral:-

Sine Laws	Sine-Cosine Laws	Cosine Laws.
$X_1\cos\theta_2 - Y_1\sin\theta_2 = \bar{X}_3$	$X_1\sin\theta_2 + Y_1\cos\theta_2 = -\bar{Y}_3$	$Z_1 = \bar{Z}_3$
$X_2\cos\theta_3 - Y_2\sin\theta_3 = \bar{X}_4$	$X_2\sin\theta_3 + Y_2\cos\theta_3 = -\bar{Y}_4$	$Z_2 = \bar{Z}_4$
$X_3\cos\theta_4 - Y_3\sin\theta_4 = \bar{X}_1$	$X_3\sin\theta_4 + Y_3\cos\theta_4 = -\bar{Y}_1$	$Z_3 = \bar{Z}_1$
$X_4\cos\theta_1 - Y_4\sin\theta_1 = \bar{X}_2$	$X_4\sin\theta_1 + Y_4\cos\theta_1 = -\bar{Y}_2$	$Z_4 = \bar{Z}_2$
$\bar{X}_2\cos\theta_1 - \bar{Y}_2\cos\theta_1 = X_4$	$\bar{X}_2\sin\theta_1 + \bar{Y}_2\cos\theta_1 = -Y_4$	$\bar{Z}_2 = Z_4$
$\bar{X}_3\cos\theta_2 - \bar{Y}_3\sin\theta_2 = X_1$	$\bar{X}_3\sin\theta_2 + \bar{Y}_3\cos\theta_2 = -Y_1$	$\bar{Z}_3 = Z_1$
$\bar{X}_4\cos\theta_3 - \bar{Y}_4\sin\theta_3 = X_2$	$\bar{X}_4\sin\theta_3 + \bar{Y}_4\cos\theta_3 = -Y_2$	$\bar{Z}_4 = Z_2$
$\bar{X}_1\cos\theta_4 - \bar{Y}_1\sin\theta_4 = X_3$	$\bar{X}_1\sin\theta_4 + \bar{Y}_1\cos\theta_4 = -Y_3$	$\bar{Z}_1 = Z_3$

A.3. The Spherical Pentagon.

(i) Definitions.

If i, j, k and l are in ascending consecutive cyclic order, then one may define, for a spherical pentagon (see equations (4.54)):-

$$\begin{aligned} X_{ijk} &= X_{ij}\cos\theta_k - Y_{ij}\sin\theta_k \\ Y_{ijk} &= \cos\alpha_{kl}(X_{ij}\sin\theta_k + Y_{ij}\cos\theta_k) - \sin\alpha_{kl}Z_{ij} \\ Z_{ijk} &= \sin\alpha_{kl}(X_{ij}\sin\theta_k + Y_{ij}\cos\theta_k) + \cos\alpha_{kl}Z_{ij} \end{aligned} \qquad \text{(III.5)}$$

$$\begin{aligned} X_{lkj} &= X_{lk}\cos\theta_j - Y_{lk}\sin\theta_k \\ Y_{lkj} &= \cos\alpha_{ij}(X_{lk}\sin\theta_j + Y_{lk}\cos\theta_j) - \sin\alpha_{ij}Z_{lk} \\ Z_{lkj} &= \sin\alpha_{ij}(X_{lk}\sin\theta_j + Y_{lk}\cos\theta_j) + \cos\alpha_{ij}Z_{lk} \end{aligned} \qquad \text{(III.6)}$$

where X_{ij}, Y_{ij}, Z_{ij} are given by definitions (III.3) and X_{lk}, Y_{lk}, Z_{lk} are obtained from (III.4) (see previous section).

The above definitions are applicable to any four adjacent links in a spherical polygon.

(ii) Fundamental Laws.

All cyclic permutations of the laws for a spherical pentagon may now be listed as follows (with the aid of Figure 4.5(b)):-

Sine Laws	Sine-Cosine Laws	Cosine Laws.
$X_{123} = \sin\alpha_{45}\sin\theta_4$	$Y_{123} = \sin\alpha_{45}\cos\theta_4$	$Z_{123} = \cos\alpha_{45}$
$X_{234} = \sin\alpha_{51}\sin\theta_5$	$Y_{234} = \sin\alpha_{51}\cos\theta_5$	$Z_{234} = \cos\alpha_{51}$
$X_{345} = \sin\alpha_{12}\sin\theta_1$	$Y_{345} = \sin\alpha_{12}\cos\theta_1$	$Z_{345} = \cos\alpha_{12}$
$X_{451} = \sin\alpha_{23}\sin\theta_2$	$Y_{451} = \sin\alpha_{23}\cos\theta_2$	$Z_{451} = \cos\alpha_{23}$
$X_{512} = \sin\alpha_{34}\sin\theta_3$	$Y_{512} = \sin\alpha_{34}\cos\theta_3$	$Z_{512} = \cos\alpha_{34}$
$X_{321} = \sin\alpha_{45}\sin\theta_5$	$Y_{321} = \sin\alpha_{45}\cos\theta_5$	$Z_{321} = \cos\alpha_{45}$
$X_{432} = \sin\alpha_{51}\sin\theta_1$	$Y_{432} = \sin\alpha_{51}\cos\theta_1$	$Z_{432} = \cos\alpha_{51}$
$X_{543} = \sin\alpha_{12}\sin\theta_2$	$Y_{543} = \sin\alpha_{12}\cos\theta_2$	$Z_{543} = \cos\alpha_{12}$
$X_{154} = \sin\alpha_{23}\sin\theta_3$	$Y_{154} = \sin\alpha_{23}\cos\theta_3$	$Z_{154} = \cos\alpha_{23}$
$X_{215} = \sin\alpha_{34}\sin\theta_4$	$Y_{215} = \sin\alpha_{34}\cos\theta_4$	$Z_{215} = \cos\alpha_{34}$

(iii) Subsidiary Formulae.

The following two groups of subsidiary formulae may be derived from the above fundamental laws for a spherical pentagon:-

Sine Laws	Sine-Cosine Laws	Cosine Laws.
Group 1.		
$X_{12}\cos\theta_3 - Y_{12}\sin\theta_3 = \bar{X}_4$	$X_{12}\sin\theta_3 + Y_{12}\cos\theta_3 = -\bar{Y}_4$	$Z_{12} = \bar{Z}_4$
$X_{23}\cos\theta_4 - Y_{23}\sin\theta_4 = \bar{X}_5$	$X_{23}\sin\theta_4 + Y_{23}\cos\theta_4 = -\bar{Y}_5$	$Z_{23} = \bar{Z}_5$
$X_{34}\cos\theta_5 - Y_{34}\sin\theta_5 = \bar{X}_1$	$X_{34}\sin\theta_5 + Y_{34}\cos\theta_5 = -\bar{Y}_1$	$Z_{34} = \bar{Z}_1$
$X_{45}\cos\theta_1 - Y_{45}\sin\theta_1 = \bar{X}_2$	$X_{45}\sin\theta_1 + Y_{45}\cos\theta_1 = -\bar{Y}_2$	$Z_{45} = \bar{Z}_2$
$X_{51}\cos\theta_2 - Y_{51}\sin\theta_2 = \bar{X}_3$	$X_{51}\sin\theta_2 + Y_{51}\cos\theta_2 = -\bar{Y}_3$	$Z_{51} = \bar{Z}_3$
$X_{32}\cos\theta_1 - Y_{32}\sin\theta_1 = X_5$	$X_{32}\sin\theta_1 + Y_{32}\cos\theta_1 = -Y_5$	$Z_{32} = Z_5$
$X_{43}\cos\theta_2 - Y_{43}\sin\theta_2 = X_1$	$X_{43}\sin\theta_2 + Y_{43}\cos\theta_2 = -Y_1$	$Z_{43} = Z_1$
$X_{54}\cos\theta_3 - Y_{54}\sin\theta_3 = X_2$	$X_{54}\sin\theta_3 + Y_{54}\cos\theta_3 = -Y_2$	$Z_{54} = Z_2$
$X_{15}\cos\theta_4 - Y_{15}\sin\theta_4 = X_3$	$X_{15}\sin\theta_4 + Y_{15}\cos\theta_4 = -Y_3$	$Z_{15} = Z_3$
$X_{21}\cos\theta_5 - Y_{21}\sin\theta_5 = X_4$	$X_{21}\sin\theta_5 + Y_{21}\cos\theta_5 = -Y_4$	$Z_{21} = Z_4$
Group 2.		
$X_{12} = \bar{X}_4\cos\theta_3 - \bar{Y}_4\sin\theta_3$	$-Y_{12} = \bar{X}_4\sin\theta_3 + \bar{Y}_4\cos\theta_3$	$Z_{12} = \bar{Z}_4$
$X_{23} = \bar{X}_5\cos\theta_4 - \bar{Y}_5\sin\theta_4$	$-Y_{23} = \bar{X}_5\sin\theta_4 + \bar{Y}_5\cos\theta_4$	$Z_{23} = \bar{Z}_5$
$X_{34} = \bar{X}_1\cos\theta_5 - \bar{Y}_1\sin\theta_5$	$-Y_{34} = \bar{X}_1\sin\theta_5 + \bar{Y}_1\cos\theta_5$	$Z_{34} = \bar{Z}_1$
$X_{45} = \bar{X}_2\cos\theta_1 - \bar{Y}_2\sin\theta_1$	$-Y_{45} = \bar{X}_2\sin\theta_1 + \bar{Y}_2\cos\theta_1$	$Z_{45} = \bar{Z}_2$
$X_{51} = \bar{X}_3\cos\theta_2 - \bar{Y}_3\sin\theta_2$	$-Y_{51} = \bar{X}_3\sin\theta_2 + \bar{Y}_3\cos\theta_2$	$Z_{51} = \bar{Z}_3$
$X_{32} = X_5\cos\theta_1 - Y_5\sin\theta_1$	$-Y_{32} = X_5\sin\theta_1 + Y_5\cos\theta_1$	$Z_{32} = Z_5$
$X_{43} = X_1\cos\theta_2 - Y_1\sin\theta_2$	$-Y_{43} = X_1\sin\theta_2 + Y_1\cos\theta_2$	$Z_{43} = Z_1$
$X_{54} = X_2\cos\theta_3 - Y_2\sin\theta_3$	$-Y_{54} = X_2\sin\theta_3 + Y_2\cos\theta_3$	$Z_{54} = Z_2$
$X_{15} = X_3\cos\theta_4 - Y_3\sin\theta_4$	$-Y_{15} = X_3\sin\theta_4 + Y_3\cos\theta_4$	$Z_{15} = Z_3$
$X_{21} = X_4\cos\theta_5 - Y_4\sin\theta_5$	$-Y_{21} = X_4\sin\theta_5 + Y_4\cos\theta_5$	$Z_{21} = Z_4$

A.4. The Spherical Hexagon.

(i) Definitions.

If i, j, k, l and m are in ascending consecutive cyclic order, then one may define, for a spherical hexagon (see equations (4.65)):-

$$X_{ijkl} = X_{ijk}\cos\theta_l - Y_{ijk}\sin\theta_l$$
$$Y_{ijkl} = \cos\alpha_{lm}(X_{ijk}\sin\theta_l + Y_{ijk}\cos\theta_l) - \sin\alpha_{lm}Z_{ijk}$$
$$Z_{ijkl} = \sin\alpha_{lm}(X_{ijk}\sin\theta_l + Y_{ijk}\cos\theta_l) + \cos\alpha_{lm}Z_{ijk} \qquad \text{(III.7)}$$

$$X_{mlkj} = X_{mlk}\cos\theta_j - Y_{mlk}\sin\theta_j$$
$$Y_{mlkj} = \cos\alpha_{ij}(X_{mlk}\sin\theta_j + Y_{mlk}\cos\theta_j) - \sin\alpha_{ij}Z_{mlk}$$
$$Z_{mlkj} = \sin\alpha_{ij}(X_{mlk}\sin\theta_j + Y_{mlk}\cos\theta_j) + \cos\alpha_{ij}Z_{mlk} \qquad \text{(III.8)}$$

where X_{ijk}, Y_{ijk}, Z_{ijk} are given by definitions (III.5) and X_{mlk}, Y_{mlk}, Z_{mlk} are obtained from (III.6) (see previous section).

The above definitions are applicable to any five adjacent links in a spherical polygon.

(ii) Fundamental Laws.

All cyclic permutations of the laws for a spherical hexagon may now be listed as follows (with the aid of Figure 4.6(b)):-

Sine Laws	Sine-Cosine Laws	Cosine Laws.
$X_{1234} = \sin\alpha_{56}\sin\theta_5$	$Y_{1234} = \sin\alpha_{56}\cos\theta_5$	$Z_{1234} = \cos\alpha_{56}$
$X_{2345} = \sin\alpha_{61}\sin\theta_6$	$Y_{2345} = \sin\alpha_{61}\cos\theta_6$	$Z_{2345} = \cos\alpha_{61}$
$X_{3456} = \sin\alpha_{12}\sin\theta_1$	$Y_{3456} = \sin\alpha_{12}\cos\theta_1$	$Z_{3456} = \cos\alpha_{12}$
$X_{4561} = \sin\alpha_{23}\sin\theta_2$	$Y_{4561} = \sin\alpha_{23}\cos\theta_2$	$Z_{4561} = \cos\alpha_{23}$
$X_{5612} = \sin\alpha_{34}\sin\theta_3$	$Y_{5612} = \sin\alpha_{34}\cos\theta_3$	$Z_{5612} = \cos\alpha_{34}$
$X_{6123} = \sin\alpha_{45}\sin\theta_4$	$Y_{6123} = \sin\alpha_{45}\cos\theta_4$	$Z_{6123} = \cos\alpha_{45}$
$X_{4321} = \sin\alpha_{56}\sin\theta_6$	$Y_{4321} = \sin\alpha_{56}\cos\theta_6$	$Z_{4321} = \cos\alpha_{56}$
$X_{5432} = \sin\alpha_{61}\sin\theta_1$	$Y_{5432} = \sin\alpha_{61}\cos\theta_1$	$Z_{5432} = \cos\alpha_{61}$
$X_{6543} = \sin\alpha_{12}\sin\theta_2$	$Y_{6543} = \sin\alpha_{12}\cos\theta_2$	$Z_{6543} = \cos\alpha_{12}$
$X_{1654} = \sin\alpha_{23}\sin\theta_3$	$Y_{1654} = \sin\alpha_{23}\cos\theta_3$	$Z_{1654} = \cos\alpha_{23}$
$X_{2165} = \sin\alpha_{34}\sin\theta_4$	$Y_{2165} = \sin\alpha_{34}\cos\theta_4$	$Z_{2165} = \cos\alpha_{34}$
$X_{3216} = \sin\alpha_{45}\sin\theta_5$	$Y_{3216} = \sin\alpha_{45}\cos\theta_5$	$Z_{3216} = \cos\alpha_{45}$

(iii) Subsidiary Formulae.

The following three groups of subsidiary formulae may be derived from the above fundamental laws for a spherical hexagon:-

Sine Laws	Sine-Cosine Laws	Cosine Laws.
Group 1.		
$X_{123}\cos\theta_4 - Y_{123}\sin\theta_4 = \bar{X}_5$	$X_{123}\sin\theta_4 + Y_{123}\cos\theta_4 = -\bar{Y}_5$	$Z_{123} = \bar{Z}_5$
$X_{234}\cos\theta_5 - Y_{234}\sin\theta_5 = \bar{X}_6$	$X_{234}\sin\theta_5 + Y_{234}\cos\theta_5 = -\bar{Y}_6$	$Z_{234} = \bar{Z}_6$
$X_{345}\cos\theta_6 - Y_{345}\sin\theta_6 = \bar{X}_1$	$X_{345}\sin\theta_6 + Y_{345}\cos\theta_6 = -\bar{Y}_1$	$Z_{345} = \bar{Z}_1$
$X_{456}\cos\theta_1 - Y_{456}\sin\theta_1 = \bar{X}_2$	$X_{456}\sin\theta_1 + Y_{456}\cos\theta_1 = -\bar{Y}_2$	$Z_{456} = \bar{Z}_2$
$X_{561}\cos\theta_2 - Y_{561}\sin\theta_2 = \bar{X}_3$	$X_{561}\sin\theta_2 + Y_{561}\cos\theta_2 = -\bar{Y}_3$	$Z_{561} = \bar{Z}_3$
$X_{612}\cos\theta_3 - Y_{612}\sin\theta_3 = \bar{X}_4$	$X_{612}\sin\theta_3 + Y_{612}\cos\theta_3 = -\bar{Y}_4$	$Z_{612} = \bar{Z}_4$
$X_{432}\cos\theta_1 - Y_{432}\sin\theta_1 = X_6$	$X_{432}\sin\theta_1 + Y_{432}\cos\theta_1 = -Y_6$	$Z_{432} = Z_6$
$X_{543}\cos\theta_2 - Y_{543}\sin\theta_2 = X_1$	$X_{543}\sin\theta_2 + Y_{543}\cos\theta_2 = -Y_1$	$Z_{543} = Z_1$
$X_{654}\cos\theta_3 - Y_{654}\sin\theta_3 = X_2$	$X_{654}\sin\theta_3 + Y_{654}\cos\theta_3 = -Y_2$	$Z_{654} = Z_2$
$X_{165}\cos\theta_4 - Y_{165}\sin\theta_4 = X_3$	$X_{165}\sin\theta_4 + Y_{165}\cos\theta_4 = -Y_3$	$Z_{165} = Z_3$
$X_{216}\cos\theta_5 - Y_{216}\sin\theta_5 = X_4$	$X_{216}\sin\theta_5 + Y_{216}\cos\theta_5 = -Y_4$	$Z_{216} = Z_4$
$X_{321}\cos\theta_6 - Y_{321}\sin\theta_6 = X_5$	$X_{321}\sin\theta_6 + Y_{321}\cos\theta_6 = -Y_5$	$Z_{321} = Z_5$
Group 2.		
$X_{123} = \bar{X}_5\cos\theta_4 - \bar{Y}_5\sin\theta_4$	$-Y_{123} = \bar{X}_5\sin\theta_4 + \bar{Y}_5\cos\theta_4$	$Z_{123} = \bar{Z}_5$
$X_{234} = \bar{X}_6\cos\theta_5 - \bar{Y}_6\sin\theta_5$	$-Y_{234} = \bar{X}_6\sin\theta_5 + \bar{Y}_6\cos\theta_5$	$Z_{234} = \bar{Z}_6$
$X_{345} = \bar{X}_1\cos\theta_6 - \bar{Y}_1\sin\theta_6$	$-Y_{345} = \bar{X}_1\sin\theta_6 + \bar{Y}_1\cos\theta_6$	$Z_{345} = \bar{Z}_1$
$X_{456} = \bar{X}_2\cos\theta_1 - \bar{Y}_2\sin\theta_1$	$-Y_{456} = \bar{X}_2\sin\theta_1 + \bar{Y}_2\cos\theta_1$	$Z_{456} = \bar{Z}_2$
$X_{561} = \bar{X}_3\cos\theta_2 - \bar{Y}_3\sin\theta_2$	$-Y_{561} = \bar{X}_3\sin\theta_2 + \bar{Y}_3\cos\theta_2$	$Z_{561} = \bar{Z}_3$
$X_{612} = \bar{X}_4\cos\theta_3 - \bar{Y}_4\sin\theta_3$	$-Y_{612} = \bar{X}_4\sin\theta_3 + \bar{Y}_4\cos\theta_3$	$Z_{612} = \bar{Z}_4$

$$X_{432} = X_6\cos\theta_1 - Y_6\sin\theta_1 \qquad -Y_{432} = X_6\sin\theta_1 + Y_6\cos\theta_1 \qquad Z_{432} = Z_6$$
$$X_{543} = X_1\cos\theta_2 - Y_1\sin\theta_2 \qquad -Y_{543} = X_1\sin\theta_2 + Y_1\cos\theta_2 \qquad Z_{543} = Z_1$$
$$X_{654} = X_2\cos\theta_3 - Y_2\sin\theta_3 \qquad -Y_{654} = X_2\sin\theta_3 + Y_2\cos\theta_3 \qquad Z_{654} = Z_2$$
$$X_{165} = X_3\cos\theta_4 - Y_3\sin\theta_4 \qquad -Y_{165} = X_3\sin\theta_4 + Y_3\cos\theta_4 \qquad Z_{165} = Z_3$$
$$X_{216} = X_4\cos\theta_5 - Y_4\sin\theta_5 \qquad -Y_{216} = X_4\sin\theta_5 + Y_4\cos\theta_5 \qquad Z_{216} = Z_4$$
$$X_{321} = X_5\cos\theta_6 - Y_5\sin\theta_6 \qquad -Y_{321} = X_5\sin\theta_6 + Y_5\cos\theta_6 \qquad Z_{321} = Z_5$$

Group 3.

$$X_{12}\cos\theta_3 - Y_{12}\sin\theta_3 = X_{54} \qquad X_{12}\sin\theta_3 + Y_{12}\cos\theta_3 = -Y_{54} \qquad Z_{12} = Z_{54}$$
$$X_{23}\cos\theta_4 - Y_{23}\sin\theta_4 = X_{65} \qquad X_{23}\sin\theta_4 + Y_{23}\cos\theta_4 = -Y_{65} \qquad Z_{23} = Z_{65}$$
$$X_{34}\cos\theta_5 - Y_{34}\sin\theta_5 = X_{16} \qquad X_{34}\sin\theta_5 + Y_{34}\cos\theta_5 = -Y_{16} \qquad Z_{34} = Z_{16}$$
$$X_{45}\cos\theta_6 - Y_{45}\sin\theta_6 = X_{21} \qquad X_{45}\sin\theta_6 + Y_{45}\cos\theta_6 = -Y_{21} \qquad Z_{45} = Z_{21}$$
$$X_{56}\cos\theta_1 - Y_{56}\sin\theta_1 = X_{32} \qquad X_{56}\sin\theta_1 + Y_{56}\cos\theta_1 = -Y_{32} \qquad Z_{56} = Z_{32}$$
$$X_{61}\cos\theta_2 - Y_{61}\sin\theta_2 = X_{43} \qquad X_{61}\sin\theta_2 + Y_{61}\cos\theta_2 = -Y_{43} \qquad Z_{61} = Z_{43}$$

$$X_{43}\cos\theta_2 - Y_{43}\sin\theta_2 = X_{61} \qquad X_{43}\sin\theta_2 + Y_{43}\cos\theta_2 = -Y_{61} \qquad Z_{43} = Z_{61}$$
$$X_{54}\cos\theta_3 - Y_{54}\sin\theta_3 = X_{12} \qquad X_{54}\sin\theta_3 + Y_{54}\cos\theta_3 = -Y_{12} \qquad Z_{54} = Z_{12}$$
$$X_{65}\cos\theta_4 - Y_{65}\sin\theta_4 = X_{23} \qquad X_{65}\sin\theta_4 + Y_{65}\cos\theta_4 = -Y_{23} \qquad Z_{65} = Z_{23}$$
$$X_{16}\cos\theta_5 - Y_{16}\sin\theta_5 = X_{34} \qquad X_{16}\sin\theta_5 + Y_{16}\cos\theta_5 = -Y_{34} \qquad Z_{16} = Z_{34}$$
$$X_{21}\cos\theta_6 - Y_{21}\sin\theta_6 = X_{45} \qquad X_{21}\sin\theta_6 + Y_{21}\cos\theta_6 = -Y_{45} \qquad Z_{21} = Z_{45}$$
$$X_{32}\cos\theta_1 - Y_{32}\sin\theta_1 = X_{56} \qquad X_{32}\sin\theta_1 + Y_{32}\cos\theta_1 = -Y_{56} \qquad Z_{32} = Z_{56}$$

A.5. The Spherical Heptagon.

(i) Definitions.

If i, j, k, l, m and n are in ascending consecutive cyclic order, then one may define, for a spherical heptagon (see equations (4.77)):-

$$X_{ijklm} = X_{ijkl}\cos\theta_m - Y_{ijkl}\sin\theta_m$$
$$Y_{ijklm} = \cos\alpha_{mn}(X_{ijkl}\sin\theta_m + Y_{ijkl}\cos\theta_m) - \sin\alpha_{mn}Z_{ijkl}$$
$$Z_{ijklm} = \sin\alpha_{mn}(X_{ijkl}\sin\theta_m + Y_{ijkl}\cos\theta_m) + \cos\alpha_{mn}Z_{ijkl} \qquad \text{(III.9)}$$

$$X_{nmlkj} = X_{nmlk}\cos\theta_j - Y_{nmlk}\sin\theta_j$$

$$Y_{nmlkj} = \cos\alpha_{ij}(X_{nmlk}\sin\theta_j + Y_{nmlk}\cos\theta_j) - \sin\alpha_{ij} Z_{nmlk}$$

$$Z_{nmlkj} = \sin\alpha_{ij}(X_{nmlk}\sin\theta_j + Y_{nmlk}\cos\theta_j) + \cos\alpha_{ij} Z_{nmlk} \qquad \text{(III.10)}$$

where X_{ijkl}, Y_{ijkl}, Z_{ijkl} are given by definitions (III.7) and X_{nmlk}, Y_{nmlk}, Z_{nmlk} are obtained from (III.8) (see previous section).

The above definitions are applicable to any six adjacent links in a spherical polygon.

(ii) Fundamental Laws.

All cyclic permutations of the laws for a spherical heptagon may now be listed as follows (with the aid of Figure 4.7(b)):-

Sine Laws	Sine-Cosine Laws	Cosine Laws.
$X_{12345} = \sin\alpha_{67}\sin\theta_6$	$Y_{12345} = \sin\alpha_{67}\cos\theta_6$	$Z_{12345} = \cos\alpha_{67}$
$X_{23456} = \sin\alpha_{71}\sin\theta_7$	$Y_{23456} = \sin\alpha_{71}\cos\theta_7$	$Z_{23456} = \cos\alpha_{71}$
$X_{34567} = \sin\alpha_{12}\sin\theta_1$	$Y_{34567} = \sin\alpha_{12}\cos\theta_1$	$Z_{34567} = \cos\alpha_{12}$
$X_{45671} = \sin\alpha_{23}\sin\theta_2$	$Y_{45671} = \sin\alpha_{23}\cos\theta_2$	$Z_{45671} = \cos\alpha_{23}$
$X_{56712} = \sin\alpha_{34}\sin\theta_3$	$Y_{56712} = \sin\alpha_{34}\cos\theta_3$	$Z_{56712} = \cos\alpha_{34}$
$X_{67123} = \sin\alpha_{45}\sin\theta_4$	$Y_{67123} = \sin\alpha_{45}\cos\theta_4$	$Z_{67123} = \cos\alpha_{45}$
$X_{71234} = \sin\alpha_{56}\sin\theta_5$	$Y_{71234} = \sin\alpha_{56}\cos\theta_5$	$Z_{71234} = \cos\alpha_{56}$
$X_{54321} = \sin\alpha_{67}\sin\theta_7$	$Y_{54321} = \sin\alpha_{67}\cos\theta_7$	$Z_{54321} = \cos\alpha_{67}$
$X_{65432} = \sin\alpha_{71}\sin\theta_1$	$Y_{65432} = \sin\alpha_{71}\cos\theta_1$	$Z_{65432} = \cos\alpha_{71}$
$X_{76543} = \sin\alpha_{12}\sin\theta_2$	$Y_{76543} = \sin\alpha_{12}\cos\theta_2$	$Z_{76543} = \cos\alpha_{12}$
$X_{17654} = \sin\alpha_{23}\sin\theta_3$	$Y_{17654} = \sin\alpha_{23}\cos\theta_3$	$Z_{17654} = \cos\alpha_{23}$
$X_{21765} = \sin\alpha_{34}\sin\theta_4$	$Y_{21765} = \sin\alpha_{34}\cos\theta_4$	$Z_{21765} = \cos\alpha_{34}$
$X_{32176} = \sin\alpha_{45}\sin\theta_5$	$Y_{32176} = \sin\alpha_{45}\cos\theta_5$	$Z_{32176} = \cos\alpha_{45}$
$X_{43217} = \sin\alpha_{56}\sin\theta_6$	$Y_{43217} = \sin\alpha_{56}\cos\theta_6$	$Z_{43217} = \cos\alpha_{56}$

(iii) Subsidiary Formulae.

The following four groups of subsidiary formulae may be derived from the above fundamental laws for a spherical heptagon:-

Group 1.

Sine Laws	Sine-Cosine Laws	Cosine Laws.
$X_{1234}\cos\theta_5 - Y_{1234}\sin\theta_5 = \bar{X}_6$	$X_{1234}\sin\theta_5 + Y_{1234}\cos\theta_5 = -\bar{Y}_6$	$Z_{1234} = \bar{Z}_6$
$X_{2345}\cos\theta_6 - Y_{2345}\sin\theta_6 = \bar{X}_7$	$X_{2345}\sin\theta_6 + Y_{2345}\cos\theta_6 = -\bar{Y}_7$	$Z_{2345} = \bar{Z}_7$
$X_{3456}\cos\theta_7 - Y_{3456}\sin\theta_7 = \bar{X}_1$	$X_{3456}\sin\theta_7 + Y_{3456}\cos\theta_7 = -\bar{Y}_1$	$Z_{3456} = \bar{Z}_1$
$X_{4567}\cos\theta_1 - Y_{4567}\sin\theta_1 = \bar{X}_2$	$X_{4567}\sin\theta_1 + Y_{4567}\cos\theta_1 = -\bar{Y}_2$	$Z_{4567} = \bar{Z}_2$
$X_{5671}\cos\theta_2 - Y_{5671}\sin\theta_2 = \bar{X}_3$	$X_{5671}\sin\theta_2 + Y_{5671}\cos\theta_2 = -\bar{Y}_3$	$Z_{5671} = \bar{Z}_3$
$X_{6712}\cos\theta_3 - Y_{6712}\sin\theta_3 = \bar{X}_4$	$X_{6712}\sin\theta_3 + Y_{6712}\cos\theta_3 = -\bar{Y}_4$	$Z_{6712} = \bar{Z}_4$
$X_{7123}\cos\theta_4 - Y_{7123}\sin\theta_4 = \bar{X}_5$	$X_{7123}\sin\theta_4 + Y_{7123}\cos\theta_4 = -\bar{Y}_5$	$Z_{7123} = \bar{Z}_5$

$X_{5432}\cos\theta_1 - Y_{5432}\sin\theta_1 = X_7$	$X_{5432}\sin\theta_1 + Y_{5432}\cos\theta_1 = -Y_7$	$Z_{5432} = Z_7$
$X_{6543}\cos\theta_2 - Y_{6543}\sin\theta_2 = X_1$	$X_{6543}\sin\theta_2 + Y_{6543}\cos\theta_2 = -Y_1$	$Z_{6543} = Z_1$
$X_{7654}\cos\theta_3 - Y_{7654}\sin\theta_3 = X_2$	$X_{7654}\sin\theta_3 + Y_{7654}\cos\theta_3 = -Y_2$	$Z_{7654} = Z_2$
$X_{1765}\cos\theta_4 - Y_{1765}\sin\theta_4 = X_3$	$X_{1765}\sin\theta_4 + Y_{1765}\cos\theta_4 = -Y_3$	$Z_{1765} = Z_3$
$X_{2176}\cos\theta_5 - Y_{2176}\sin\theta_5 = X_4$	$X_{2176}\sin\theta_5 + Y_{2176}\cos\theta_5 = -Y_4$	$Z_{2176} = Z_4$
$X_{3217}\cos\theta_6 - Y_{3217}\sin\theta_6 = X_5$	$X_{3217}\sin\theta_6 + Y_{3217}\cos\theta_6 = -Y_5$	$Z_{3217} = Z_5$
$X_{4321}\cos\theta_7 - Y_{4321}\sin\theta_7 = X_6$	$X_{4321}\sin\theta_7 + Y_{4321}\cos\theta_7 = -Y_6$	$Z_{4321} = Z_6$

Group 2.

$X_{1234} = \bar{X}_6\cos\theta_5 - \bar{Y}_6\sin\theta_5$	$-Y_{1234} = \bar{X}_6\sin\theta_5 + \bar{Y}_6\cos\theta_5$	$Z_{1234} = \bar{Z}_6$
$X_{2345} = \bar{X}_7\cos\theta_6 - \bar{Y}_7\sin\theta_6$	$-Y_{2345} = \bar{X}_7\sin\theta_6 + \bar{Y}_7\cos\theta_6$	$Z_{2345} = \bar{Z}_7$
$X_{3456} = \bar{X}_1\cos\theta_7 - \bar{Y}_1\sin\theta_7$	$-Y_{3456} = \bar{X}_1\sin\theta_7 + \bar{Y}_1\cos\theta_7$	$Z_{3456} = \bar{Z}_1$
$X_{4567} = \bar{X}_2\cos\theta_1 - \bar{Y}_2\sin\theta_1$	$-Y_{4567} = \bar{X}_2\sin\theta_1 + \bar{Y}_2\cos\theta_1$	$Z_{4567} = \bar{Z}_2$
$X_{5671} = \bar{X}_3\cos\theta_2 - \bar{Y}_3\sin\theta_2$	$-Y_{5671} = \bar{X}_3\sin\theta_2 + \bar{Y}_3\cos\theta_2$	$Z_{5671} = \bar{Z}_3$
$X_{6712} = \bar{X}_4\cos\theta_3 - \bar{Y}_4\sin\theta_3$	$-Y_{6712} = \bar{X}_4\sin\theta_3 + \bar{Y}_4\cos\theta_3$	$Z_{6712} = \bar{Z}_4$
$X_{7123} = \bar{X}_5\cos\theta_4 - \bar{Y}_5\sin\theta_4$	$-Y_{7123} = \bar{X}_5\sin\theta_4 + \bar{Y}_5\cos\theta_4$	$Z_{7123} = \bar{Z}_5$

$$X_{5432} = X_7\cos\theta_1 - Y_7\sin\theta_1 \qquad -Y_{5432} = X_7\sin\theta_1 + Y_7\cos\theta_1 \qquad Z_{5432} = Z_7$$
$$X_{6543} = X_1\cos\theta_2 - Y_1\sin\theta_2 \qquad -Y_{6543} = X_1\sin\theta_2 + Y_1\cos\theta_2 \qquad Z_{6543} = Z_1$$
$$X_{7654} = X_2\cos\theta_3 - Y_2\sin\theta_3 \qquad -Y_{7654} = X_2\sin\theta_3 + Y_2\cos\theta_3 \qquad Z_{7654} = Z_2$$
$$X_{1765} = X_3\cos\theta_4 - Y_3\sin\theta_4 \qquad -Y_{1765} = X_3\sin\theta_4 + Y_3\cos\theta_4 \qquad Z_{1765} = Z_3$$
$$X_{2176} = X_4\cos\theta_5 - Y_4\sin\theta_5 \qquad -Y_{2176} = X_4\sin\theta_5 + Y_4\cos\theta_5 \qquad Z_{2176} = Z_4$$
$$X_{3217} = X_5\cos\theta_6 - Y_5\sin\theta_6 \qquad -Y_{3217} = X_5\sin\theta_6 + Y_5\cos\theta_6 \qquad Z_{3217} = Z_5$$
$$X_{4321} = X_6\cos\theta_7 - Y_6\sin\theta_7 \qquad -Y_{4321} = X_6\sin\theta_7 + Y_6\cos\theta_7 \qquad Z_{4321} = Z_6$$

Group 3.

$$X_{123}\cos\theta_4 - Y_{123}\sin\theta_4 = X_{65} \qquad X_{123}\sin\theta_4 + Y_{123}\cos\theta_4 = -Y_{65} \qquad Z_{123} = Z_{65}$$
$$X_{234}\cos\theta_5 - Y_{234}\sin\theta_5 = X_{76} \qquad X_{234}\sin\theta_5 + Y_{234}\cos\theta_5 = -Y_{76} \qquad Z_{234} = Z_{76}$$
$$X_{345}\cos\theta_6 - Y_{345}\sin\theta_6 = X_{17} \qquad X_{345}\sin\theta_6 + Y_{345}\cos\theta_6 = -Y_{17} \qquad Z_{345} = Z_{17}$$
$$X_{456}\cos\theta_7 - Y_{456}\sin\theta_7 = X_{21} \qquad X_{456}\sin\theta_7 + Y_{456}\cos\theta_7 = -Y_{21} \qquad Z_{456} = Z_{21}$$
$$X_{567}\cos\theta_1 - Y_{567}\sin\theta_1 = X_{32} \qquad X_{567}\sin\theta_1 + Y_{567}\cos\theta_1 = -Y_{32} \qquad Z_{567} = Z_{32}$$
$$X_{671}\cos\theta_2 - Y_{671}\sin\theta_2 = X_{43} \qquad X_{671}\sin\theta_2 + Y_{671}\cos\theta_2 = -Y_{43} \qquad Z_{671} = Z_{43}$$
$$X_{712}\cos\theta_3 - Y_{712}\sin\theta_3 = X_{54} \qquad X_{712}\sin\theta_3 + Y_{712}\cos\theta_3 = -Y_{54} \qquad Z_{712} = Z_{54}$$

$$X_{543}\cos\theta_2 - Y_{543}\sin\theta_2 = X_{71} \qquad X_{543}\sin\theta_2 + Y_{543}\cos\theta_2 = -Y_{71} \qquad Z_{543} = Z_{71}$$
$$X_{654}\cos\theta_3 - Y_{654}\sin\theta_3 = X_{12} \qquad X_{654}\sin\theta_3 + Y_{654}\cos\theta_3 = -Y_{12} \qquad Z_{654} = Z_{12}$$
$$X_{765}\cos\theta_4 - Y_{765}\sin\theta_4 = X_{23} \qquad X_{765}\sin\theta_4 + Y_{765}\cos\theta_4 = -Y_{23} \qquad Z_{765} = Z_{23}$$
$$X_{176}\cos\theta_5 - Y_{176}\sin\theta_5 = X_{34} \qquad X_{176}\sin\theta_5 + Y_{176}\cos\theta_5 = -Y_{34} \qquad Z_{176} = Z_{34}$$
$$X_{217}\cos\theta_6 - Y_{217}\sin\theta_6 = X_{45} \qquad X_{217}\sin\theta_6 + Y_{217}\cos\theta_6 = -Y_{45} \qquad Z_{217} = Z_{45}$$
$$X_{321}\cos\theta_7 - Y_{321}\sin\theta_7 = X_{56} \qquad X_{321}\sin\theta_7 + Y_{321}\cos\theta_7 = -Y_{56} \qquad Z_{321} = Z_{56}$$
$$X_{432}\cos\theta_1 - Y_{432}\sin\theta_1 = X_{67} \qquad X_{432}\sin\theta_1 + Y_{432}\cos\theta_1 = -Y_{67} \qquad Z_{432} = Z_{67}$$

Group 4.

$$X_{123} = X_{65}\cos\theta_4 - Y_{65}\sin\theta_4 \qquad -Y_{123} = X_{65}\sin\theta_4 + Y_{65}\cos\theta_4 \qquad Z_{123} = Z_{65}$$
$$X_{234} = X_{76}\cos\theta_5 - Y_{76}\sin\theta_5 \qquad -Y_{234} = X_{76}\sin\theta_5 + Y_{76}\cos\theta_5 \qquad Z_{234} = Z_{76}$$
$$X_{345} = X_{17}\cos\theta_6 - Y_{17}\sin\theta_6 \qquad -Y_{345} = X_{17}\sin\theta_6 + Y_{17}\cos\theta_6 \qquad Z_{345} = Z_{17}$$
$$X_{456} = X_{21}\cos\theta_7 - Y_{21}\sin\theta_7 \qquad -Y_{456} = X_{21}\sin\theta_7 + Y_{21}\cos\theta_7 \qquad Z_{456} = Z_{21}$$
$$X_{567} = X_{32}\cos\theta_1 - Y_{32}\sin\theta_1 \qquad -Y_{567} = X_{32}\sin\theta_1 + Y_{32}\cos\theta_1 \qquad Z_{567} = Z_{32}$$
$$X_{671} = X_{43}\cos\theta_2 - Y_{43}\sin\theta_2 \qquad -Y_{671} = X_{43}\sin\theta_2 + Y_{43}\cos\theta_2 \qquad Z_{671} = Z_{43}$$
$$X_{712} = X_{54}\cos\theta_3 - Y_{54}\sin\theta_3 \qquad -Y_{712} = X_{54}\sin\theta_3 + Y_{54}\cos\theta_3 \qquad Z_{712} = Z_{54}$$

$$X_{543} = X_{71}\cos\theta_2 - Y_{71}\sin\theta_2 \qquad -Y_{543} = X_{71}\sin\theta_2 + Y_{71}\cos\theta_2 \qquad Z_{543} = Z_{71}$$
$$X_{654} = X_{12}\cos\theta_3 - Y_{12}\sin\theta_3 \qquad -Y_{654} = X_{12}\sin\theta_3 + Y_{12}\cos\theta_3 \qquad Z_{654} = Z_{12}$$
$$X_{765} = X_{23}\cos\theta_4 - Y_{23}\sin\theta_4 \qquad -Y_{765} = X_{23}\sin\theta_4 + Y_{23}\cos\theta_4 \qquad Z_{765} = Z_{23}$$
$$X_{176} = X_{34}\cos\theta_5 - Y_{34}\sin\theta_5 \qquad -Y_{176} = X_{34}\sin\theta_5 + Y_{34}\cos\theta_5 \qquad Z_{176} = Z_{34}$$
$$X_{217} = X_{45}\cos\theta_6 - Y_{45}\sin\theta_6 \qquad -Y_{217} = X_{45}\sin\theta_6 + Y_{45}\cos\theta_6 \qquad Z_{217} = Z_{45}$$
$$X_{321} = X_{56}\cos\theta_7 - Y_{56}\sin\theta_7 \qquad -Y_{321} = X_{56}\sin\theta_7 + Y_{56}\cos\theta_7 \qquad Z_{321} = Z_{56}$$
$$X_{432} = X_{67}\cos\theta_1 - Y_{67}\sin\theta_1 \qquad -Y_{432} = X_{67}\sin\theta_1 + Y_{67}\cos\theta_1 \qquad Z_{432} = Z_{67}$$

B. Secondary Laws.

Introducing the dual symbol (see Chapter 3) into the above laws, and expanding into primary and secondary parts, produces a series of secondary laws in addition to the primary laws given in the previous sections. The writing of these laws is greatly facilitated by designating the secondary part of $\hat{X}_{6123}$, for example, as X_{06123} (see equation (4.93)) and one then obtains an X_0, Y_0 and Z_0 expression corresponding to each X, Y and Z expression defined above.

Thus, if q, i, j, k, l, m and n are positive integers in ascending consecutive cyclic order, one may list the following series of definitions, with the aid of the rules for manipulating functions of a dual variable presented in Chapter 3:-

$$\hat{X}_i = X_i + \epsilon X_{0i} \qquad \hat{\bar{X}}_i = \bar{X}_i + \epsilon \bar{X}_{0i}$$
$$\hat{Y}_i = Y_i + \epsilon Y_{0i} \qquad \hat{\bar{Y}}_i = \bar{Y}_i + \epsilon \bar{Y}_{0i}$$
$$\hat{Z}_i = Z_i + \epsilon Z_{0i} \qquad \hat{\bar{Z}}_i = \bar{Z}_i + \epsilon \bar{Z}_{0i}$$

$$\hat{X}_{ij} = X_{ij} + \epsilon X_{0ij}$$
$$\hat{Y}_{ij} = Y_{ij} + \epsilon Y_{0ij}$$
$$\hat{Z}_{ij} = Z_{ij} + \epsilon Z_{0ij}$$

$$\hat{X}_{ijk} = X_{ijk} + \epsilon X_{0ijk}$$
$$\hat{Y}_{ijk} = Y_{ijk} + \epsilon Y_{0ijk}$$
$$\hat{Z}_{ijk} = Z_{ijk} + \epsilon Z_{0ijk}$$

$$\hat{X}_{ijkl} = X_{ijkl} + \epsilon X_{0ijkl}$$
$$\hat{Y}_{ijkl} = Y_{ijkl} + \epsilon Y_{0ijkl}$$
$$\hat{Z}_{ijkl} = Z_{ijkl} + \epsilon Z_{0ijkl}$$

$$\hat{X}_{ijklm} = X_{ijklm} + \epsilon X_{0ijklm}$$
$$\hat{Y}_{ijklm} = Y_{ijklm} + \epsilon Y_{0ijklm}$$
$$\hat{Z}_{ijklm} = Z_{ijklm} + \epsilon Z_{0ijklm}$$

where the primary parts are defined in Part A. of this appendix and the secondary parts are defined below:-

$$X_{0i} = a_{qi} \cos\alpha_{qi} \sin\theta_i + S_i \sin\alpha_{qi} \cos\theta_i$$

$$Y_{0i} = a_{qi}(\sin\alpha_{qi} \sin\alpha_{ij} - \cos\alpha_{qi} \cos\alpha_{ij} \cos\theta_i) - a_{ij}(\cos\alpha_{qi} \cos\alpha_{ij} - \sin\alpha_{qi} \sin\alpha_{ij} \cos\theta_i) + S_i \sin\alpha_{qi} \cos\alpha_{ij} \sin\theta_i$$

$$Z_{0i} = - a_{qi}(\sin\alpha_{qi} \cos\alpha_{ij} + \cos\alpha_{qi} \sin\alpha_{ij} \cos\theta_i) - a_{ij}(\cos\alpha_{qi} \sin\alpha_{ij} + \sin\alpha_{qi} \cos\alpha_{ij} \cos\theta_i) + S_i \sin\alpha_{qi} \sin\alpha_{ij} \sin\theta_i$$

$$\bar{X}_{0i} = a_{ij}\cos\alpha_{ij}\sin\theta_i + S_i\sin\alpha_{ij}\cos\theta_i$$

$$\begin{aligned}\bar{Y}_{0i} = {} & a_{ij}(\sin\alpha_{ij}\sin\alpha_{qi} - \cos\alpha_{ij}\cos\alpha_{qi}\cos\theta_i) \\ & - a_{qi}(\cos\alpha_{ij}\cos\alpha_{qi} - \sin\alpha_{ij}\sin\alpha_{qi}\cos\theta_i) \\ & + S_i\sin\alpha_{ij}\cos\alpha_{qi}\sin\theta_i\end{aligned}$$

$$\begin{aligned}\bar{Z}_{0i} = {} & - a_{ij}(\sin\alpha_{ij}\cos\alpha_{qi} + \cos\alpha_{ij}\sin\alpha_{qi}\cos\theta_i) \\ & - a_{qi}(\cos\alpha_{ij}\sin\alpha_{qi} + \sin\alpha_{ij}\cos\alpha_{qi}\cos\theta_i) \\ & + S_i\sin\alpha_{ij}\sin\alpha_{qi}\sin\theta_i\end{aligned}$$

$$\begin{aligned}X_{0ij} = {} & (X_{0i}\cos\theta_j - Y_{0i}\sin\theta_j) \\ & - S_j(X_i\sin\theta_j + Y_i\cos\theta_j)\end{aligned}$$

$$\begin{aligned}Y_{0ij} = {} & \cos\alpha_{jk}(X_{0i}\sin\theta_j + Y_{0i}\cos\theta_j) - \sin\alpha_{jk} Z_{0i} \\ & + S_j\cos\alpha_{jk}(X_i\cos\theta_j - Y_i\sin\theta_j) \\ & - a_{jk}[\sin\alpha_{jk}(X_i\sin\theta_j + Y_i\cos\theta_j) + \cos\alpha_{jk} Z_i]\end{aligned}$$

$$\begin{aligned}Z_{0ij} = {} & \sin\alpha_{jk}(X_{0i}\sin\theta_j + Y_{0i}\cos\theta_j) + \cos\alpha_{jk} Z_{0i} \\ & + S_j\sin\alpha_{jk}(X_i\cos\theta_j - Y_i\sin\theta_j) \\ & + a_{jk}[\cos\alpha_{jk}(X_i\sin\theta_j + Y_i\cos\theta_j) - \sin\alpha_{jk} Z_i]\end{aligned}$$

$$\begin{aligned}X_{0ji} = {} & (\bar{X}_{0j}\cos\theta_i - \bar{Y}_{0j}\sin\theta_i) \\ & - S_i(\bar{X}_j\sin\theta_i + \bar{Y}_j\cos\theta_i)\end{aligned}$$

$$\begin{aligned}Y_{0ji} = {} & \cos\alpha_{qi}(\bar{X}_{0j}\sin\theta_i + \bar{Y}_{0j}\cos\theta_i) - \sin\alpha_{qi}\bar{Z}_{0j} \\ & + S_i\cos\alpha_{qi}(\bar{X}_j\cos\theta_i - \bar{Y}_j\sin\theta_i) \\ & - a_{qi}[\sin\alpha_{qi}(\bar{X}_j\sin\theta_i + \bar{Y}_j\cos\theta_i) + \cos\alpha_{qi}\bar{Z}_j]\end{aligned}$$

$$\begin{aligned}Z_{0ji} = {} & \sin\alpha_{qi}(\bar{X}_{0j}\sin\theta_i + \bar{Y}_{0j}\cos\theta_i) + \cos\alpha_{qi}\bar{Z}_{0j} \\ & + S_i\sin\alpha_{qi}(\bar{X}_j\cos\theta_i - \bar{Y}_j\sin\theta_i) \\ & + a_{qi}[\cos\alpha_{qi}(\bar{X}_j\sin\theta_i + \bar{Y}_j\cos\theta_i) - \sin\alpha_{qi}\bar{Z}_j]\end{aligned}$$

$$X_{0ijk} = (X_{0ij}\cos\theta_k - Y_{0ij}\sin\theta_k) - S_k(X_{ij}\sin\theta_k + Y_{ij}\cos\theta_k)$$

$$Y_{0ijk} = \cos\alpha_{kl}(X_{0ij}\sin\theta_k + Y_{0ij}\cos\theta_k) - \sin\alpha_{kl} Z_{0ij} + S_k\cos\alpha_{kl}(X_{ij}\cos\theta_k - Y_{ij}\sin\theta_k) - a_{kl}[\sin\alpha_{kl}(X_{ij}\sin\theta_k + Y_{ij}\cos\theta_k) + \cos\alpha_{kl} Z_{ij}]$$

$$Z_{0ijk} = \sin\alpha_{kl}(X_{0ij}\sin\theta_k + Y_{0ij}\cos\theta_k) + \cos\alpha_{kl} Z_{0ij} + S_k\sin\alpha_{kl}(X_{ij}\cos\theta_k - Y_{ij}\sin\theta_k) + a_{kl}[\cos\alpha_{kl}(X_{ij}\sin\theta_k + Y_{ij}\cos\theta_k) - \sin\alpha_{kl} Z_{ij}]$$

$$X_{0ijkl} = (X_{0ijk}\cos\theta_l - Y_{0ijk}\sin\theta_l) - S_l(X_{ijk}\sin\theta_l + Y_{ijk}\cos\theta_l)$$

$$Y_{0ijkl} = \cos\alpha_{lm}(X_{0ijk}\sin\theta_l + Y_{0ijk}\cos\theta_l) - \sin\alpha_{lm} Z_{0ijk} + S_l\cos\alpha_{lm}(X_{ijk}\cos\theta_l - Y_{ijk}\sin\theta_l) - a_{lm}[\sin\alpha_{lm}(X_{ijk}\sin\theta_l + Y_{ijk}\cos\theta_l) + \cos\alpha_{lm} Z_{ijk}]$$

$$Z_{0ijkl} = \sin\alpha_{lm}(X_{0ijk}\sin\theta_l + Y_{0ijk}\cos\theta_l) + \cos\alpha_{lm} Z_{0ijk} + S_l\sin\alpha_{lm}(X_{ijk}\cos\theta_l - Y_{ijk}\sin\theta_l) + a_{lm}[\cos\alpha_{lm}(X_{ijk}\sin\theta_l + Y_{ijk}\cos\theta_l) - \sin\alpha_{lm} Z_{ijk}]$$

$$X_{0ijklm} = (X_{0ijkl}\cos\theta_m - Y_{0ijkl}\sin\theta_m) - S_m(X_{ijkl}\sin\theta_m + Y_{ijkl}\cos\theta_m)$$

$$Y_{0ijklm} = \cos\alpha_{mn}(X_{0ijkl}\sin\theta_m + Y_{0ijkl}\cos\theta_m) - \sin\alpha_{mn} Z_{0ijkl} + S_m\cos\alpha_{mn}(X_{ijkl}\cos\theta_m - Y_{ijkl}\sin\theta_m) - a_{mn}[\sin\alpha_{mn}(X_{ijkl}\sin\theta_m + Y_{ijkl}\cos\theta_m) + \cos\alpha_{mn} Z_{ijkl}]$$

$$Z_{0ijklm} = \sin\alpha_{mn}(X_{0ijkl}\sin\theta_m + Y_{0ijkl}\cos\theta_m) + \cos\alpha_{mn} Z_{0ijkl} + S_m\sin\alpha_{mn}(X_{ijkl}\cos\theta_m - Y_{ijkl}\sin\theta_m) + a_{mn}[\cos\alpha_{mn}(X_{ijkl}\sin\theta_m + Y_{ijkl}\cos\theta_m) - \sin\alpha_{mn} Z_{ijkl}]$$

Now from identities (4.8), (4.42), (4.55), (4.66) and (4.78) (see Chapter 4.) it is clear that the primary Z expressions are symmetric with respect to their suffices and hence one would expect the corresponding Z_0 expressions to be symmetric also. This is indeed the case, and after expanding and using the various sine, sine-cosine and cosine laws it is possible to rewrite these Z_0 expressions alternatively in the symmetric forms shown below. Thus:-

$$\begin{aligned} Z_{0i} = \; & a_{qi}\bar{Y}_i \\ & + S_i \sin\alpha_{qi} \sin\alpha_{ij} \sin\theta_i \\ & + a_{ij} Y_i \end{aligned}$$

$$\begin{aligned} Z_{0ij} = \; & a_{qi} Y_{ji} \\ & + S_i \sin\alpha_{qi} X_{ji} \\ & + a_{ij} \operatorname{cosec}\alpha_{ij} (Z_i \bar{Z}_j - \cos\alpha_{qi} \cos\alpha_{jk}) \\ & + S_j \sin\alpha_{jk} X_{ij} \\ & + a_{jk} Y_{ij} \end{aligned}$$

$$\begin{aligned} Z_{0ijk} = \; & a_{qi} Y_{kji} \\ & + S_i \sin\alpha_{qi} X_{kji} \\ & + a_{ij}[(\bar{X}_k \sin\theta_j + \bar{Y}_k \cos\theta_j) Z_i + \bar{Z}_k Y_i] \\ & + S_j[(Y_i \bar{Y}_k - X_i \bar{X}_k) \sin\theta_j - (Y_i \bar{X}_k + X_i \bar{Y}_k) \cos\theta_j] \\ & + a_{jk}[(X_i \sin\theta_j + Y_i \cos\theta_j) \bar{Z}_k + Z_i \bar{Y}_k] \\ & + S_k \sin\alpha_{kl} X_{ijk} \\ & + a_{kl} Y_{ijk} \end{aligned}$$

$$
\begin{aligned}
Z_{0ijkl} = \; & a_{qi}Y_{lkji} \\
& + S_i \sin\alpha_{qi} X_{lkji} \\
& + a_{ij}[(X_{lk}\sin\theta_j + Y_{lk}\cos\theta_j)Z_i + Z_{lk}Y_i] \\
& - S_j[(X_{lk}\sin\theta_j + Y_{lk}\cos\theta_j)X_i + X_{lkj}Y_i] \\
& + a_{jk}\operatorname{cosec}\alpha_{jk}(Z_{ij}Z_{lk} - Z_i\bar{\bar{Z}}_l) \\
& - S_k[(X_{ij}\sin\theta_k + Y_{ij}\cos\theta_k)\bar{X}_l + X_{ijk}\bar{Y}_l] \\
& + a_{kl}[(X_{ij}\sin\theta_k + Y_{ij}\cos\theta_k)\bar{Z}_l + Z_{ij}\bar{Y}_l] \\
& + S_l \sin\alpha_{lm} X_{ijkl} \\
& + a_{lm}Y_{ijkl}
\end{aligned}
$$

$$
\begin{aligned}
Z_{0ijklm} = \; & a_{qi}Y_{mlkji} \\
& + S_i \sin\alpha_{qi} X_{mlkji} \\
& + a_{ij}[(X_{mlk}\sin\theta_j + Y_{mlk}\cos\theta_j)Z_i + Z_{mlk}Y_i] \\
& - S_j[(X_{mlk}\sin\theta_j + Y_{mlk}\cos\theta_j)X_i + X_{mlkj}Y_i] \\
& + a_{jk}[(X_{ml}\sin\theta_k + Y_{ml}\cos\theta_k)Z_{ij} + Z_{ml}Y_{ij}] \\
& + S_k[(Y_{ij}Y_{ml} - X_{ij}X_{ml})\sin\theta_k - (Y_{ij}X_{ml} + X_{ij}Y_{ml})\cos\theta_k] \\
& + a_{kl}[(X_{ij}\sin\theta_k + Y_{ij}\cos\theta_k)Z_{ml} + Z_{ij}Y_{ml}] \\
& - S_l[(X_{ijk}\sin\theta_l + Y_{ijk}\cos\theta_l)\bar{X}_m + X_{ijkl}\bar{Y}_m] \\
& + a_{lm}[(X_{ijk}\sin\theta_l + Y_{ijk}\cos\theta_l)\bar{Z}_m + Z_{ijk}\bar{Y}_m] \\
& + S_m \sin\alpha_{mn} X_{ijklm} \\
& + a_{mn}Y_{ijklm}
\end{aligned}
$$

APPENDIX IV

LIST OF HALF-TANGENT LAWS

FOR

SPHERICAL POLYGONS

A. Fundamental Half-Tangent Laws.

All cyclic permutations of the fundamental half-tangent laws for the spherical triangle, quadrilateral, pentagon, hexagon and heptagon may now be listed (see Chapter 5). In each case x_i is given by:-

$$x_i \equiv \tan(\theta_i/2)$$

A.1. The Spherical Triangle.

The fundamental half-tangent laws for the spherical triangle (Figure 4.1) may be listed as follows:-

$$X_3x_1 + (Y_3 - \sin\alpha_{12}) = 0 \qquad (Y_3 + \sin\alpha_{12})x_1 - X_3 = 0$$
$$X_1x_2 + (Y_1 - \sin\alpha_{23}) = 0 \qquad (Y_1 + \sin\alpha_{23})x_2 - X_1 = 0$$
$$X_2x_3 + (Y_2 - \sin\alpha_{31}) = 0 \qquad (Y_2 + \sin\alpha_{31})x_3 - X_2 = 0$$

$$\bar{X}_2x_1 + (\bar{Y}_2 - \sin\alpha_{31}) = 0 \qquad (\bar{Y}_2 + \sin\alpha_{31})x_1 - \bar{X}_2 = 0$$
$$\bar{X}_3x_2 + (\bar{Y}_3 - \sin\alpha_{12}) = 0 \qquad (\bar{Y}_3 + \sin\alpha_{12})x_2 - \bar{X}_3 = 0$$
$$\bar{X}_1x_3 + (\bar{Y}_1 - \sin\alpha_{23}) = 0 \qquad (\bar{Y}_1 + \sin\alpha_{23})x_3 - \bar{X}_1 = 0$$

A.2. The Spherical Quadrilateral.

The two groups of fundamental half-tangent laws for the spherical quadrilateral (Figure 4.4) may be listed as follows:-

Group 1.

$$X_{34}x_1 + (Y_{34} - \sin\alpha_{12}) = 0 \qquad (Y_{34} + \sin\alpha_{12})x_1 - X_{34} = 0$$
$$X_{41}x_2 + (Y_{41} - \sin\alpha_{23}) = 0 \qquad (Y_{41} + \sin\alpha_{23})x_2 - X_{41} = 0$$
$$X_{12}x_3 + (Y_{12} - \sin\alpha_{34}) = 0 \qquad (Y_{12} + \sin\alpha_{34})x_3 - X_{12} = 0$$
$$X_{23}x_4 + (Y_{23} - \sin\alpha_{41}) = 0 \qquad (Y_{23} + \sin\alpha_{41})x_4 - X_{23} = 0$$

$$X_{32}x_1 + (Y_{32} - \sin\alpha_{41}) = 0 \qquad (Y_{32} + \sin\alpha_{41})\,x_1 - X_{32} = 0$$
$$X_{43}x_2 + (Y_{43} - \sin\alpha_{12}) = 0 \qquad (Y_{43} + \sin\alpha_{12})\,x_2 - X_{43} = 0$$
$$X_{14}x_3 + (Y_{14} - \sin\alpha_{23}) = 0 \qquad (Y_{14} + \sin\alpha_{23})\,x_3 - X_{14} = 0$$
$$X_{21}x_4 + (Y_{21} - \sin\alpha_{34}) = 0 \qquad (Y_{21} + \sin\alpha_{34})\,x_4 - X_{21} = 0$$

Group 2.

$$(X_4 + \bar{X}_2)x_1 + (Y_4 + \bar{Y}_2) = 0 \qquad (Y_4 - \bar{Y}_2)x_1 - (X_4 - \bar{X}_2) = 0$$
$$(X_1 + \bar{X}_3)x_2 + (Y_1 + \bar{Y}_3) = 0 \qquad (Y_1 - \bar{Y}_3)x_2 - (X_1 - \bar{X}_3) = 0$$
$$(X_2 + \bar{X}_4)x_3 + (Y_2 + \bar{Y}_4) = 0 \qquad (Y_2 - \bar{Y}_4)x_3 - (X_2 - \bar{X}_4) = 0$$
$$(X_3 + \bar{X}_1)x_4 + (Y_3 + \bar{Y}_1) = 0 \qquad (Y_3 - \bar{Y}_1)x_4 - (X_3 - \bar{X}_1) = 0$$

A.3. The Spherical Pentagon.

The two groups of fundamental half-tangent laws for the spherical pentagon (Figure 4.5) may be listed as follows:-

Group 1.

$$X_{345}x_1 + (Y_{345} - \sin\alpha_{12}) = 0 \qquad (Y_{345} + \sin\alpha_{12})\,x_1 - X_{345} = 0$$
$$X_{451}x_2 + (Y_{451} - \sin\alpha_{23}) = 0 \qquad (Y_{451} + \sin\alpha_{23})\,x_2 - X_{451} = 0$$
$$X_{512}x_3 + (Y_{512} - \sin\alpha_{34}) = 0 \qquad (Y_{512} + \sin\alpha_{34})\,x_3 - X_{512} = 0$$
$$X_{123}x_4 + (Y_{123} - \sin\alpha_{45}) = 0 \qquad (Y_{123} + \sin\alpha_{45})\,x_4 - X_{123} = 0$$
$$X_{234}x_5 + (Y_{234} - \sin\alpha_{51}) = 0 \qquad (Y_{234} + \sin\alpha_{51})\,x_5 - X_{234} = 0$$

$$X_{432}x_1 + (Y_{432} - \sin\alpha_{51}) = 0 \qquad (Y_{432} + \sin\alpha_{51})\,x_1 - X_{432} = 0$$
$$X_{543}x_2 + (Y_{543} - \sin\alpha_{12}) = 0 \qquad (Y_{543} + \sin\alpha_{12})\,x_2 - X_{543} = 0$$
$$X_{154}x_3 + (Y_{154} - \sin\alpha_{23}) = 0 \qquad (Y_{154} + \sin\alpha_{23})\,x_3 - X_{154} = 0$$
$$X_{215}x_4 + (Y_{215} - \sin\alpha_{34}) = 0 \qquad (Y_{215} + \sin\alpha_{34})\,x_4 - X_{215} = 0$$
$$X_{321}x_5 + (Y_{321} - \sin\alpha_{45}) = 0 \qquad (Y_{321} + \sin\alpha_{45})\,x_5 - X_{321} = 0$$

Group 2.

$$(X_{45} + \bar{X}_2)x_1 + (Y_{45} + \bar{Y}_2) = 0 \qquad (Y_{45} - \bar{Y}_2)x_1 - (X_{45} - \bar{X}_2) = 0$$
$$(X_{51} + \bar{X}_3)x_2 + (Y_{51} + \bar{Y}_3) = 0 \qquad (Y_{51} - \bar{Y}_3)x_2 - (X_{51} - \bar{X}_3) = 0$$
$$(X_{12} + \bar{X}_4)x_3 + (Y_{12} + \bar{Y}_4) = 0 \qquad (Y_{12} - \bar{Y}_4)x_3 - (X_{12} - \bar{X}_4) = 0$$
$$(X_{23} + \bar{X}_5)x_4 + (Y_{23} + \bar{Y}_5) = 0 \qquad (Y_{23} - \bar{Y}_5)x_4 - (X_{23} - \bar{X}_5) = 0$$
$$(X_{34} + \bar{X}_1)x_5 + (Y_{34} + \bar{Y}_1) = 0 \qquad (Y_{34} - \bar{Y}_1)x_5 - (X_{34} - \bar{X}_1) = 0$$

$$(X_{32} + X_5)x_1 + (Y_{32} + Y_5) = 0 \qquad (Y_{32} - Y_5)x_1 - (X_{32} - X_5) = 0$$
$$(X_{43} + X_1)x_2 + (Y_{43} + Y_1) = 0 \qquad (Y_{43} - Y_1)x_2 - (X_{43} - X_1) = 0$$
$$(X_{54} + X_2)x_3 + (Y_{54} + Y_2) = 0 \qquad (Y_{54} - Y_2)x_3 - (X_{54} - X_2) = 0$$
$$(X_{15} + X_3)x_4 + (Y_{15} + Y_3) = 0 \qquad (Y_{15} - Y_3)x_4 - (X_{15} - X_3) = 0$$
$$(X_{21} + X_4)x_5 + (Y_{21} + Y_4) = 0 \qquad (Y_{21} - Y_4)x_5 - (X_{21} - X_4) = 0$$

A.4. The Spherical Hexagon.

The three groups of fundamental half-tangent laws for the spherical hexagon (Figure 4.6) may be listed as follows:-

Group 1.

$$X_{3456}x_1 + (Y_{3456} - \sin\alpha_{12}) = 0 \qquad (Y_{3456} + \sin\alpha_{12})\,x_1 - X_{3456} = 0$$
$$X_{4561}x_2 + (Y_{4561} - \sin\alpha_{23}) = 0 \qquad (Y_{4561} + \sin\alpha_{23})\,x_2 - X_{4561} = 0$$
$$X_{5612}x_3 + (Y_{5612} - \sin\alpha_{34}) = 0 \qquad (Y_{5612} + \sin\alpha_{34})\,x_3 - X_{5612} = 0$$
$$X_{6123}x_4 + (Y_{6123} - \sin\alpha_{45}) = 0 \qquad (Y_{6123} + \sin\alpha_{45})\,x_4 - X_{6123} = 0$$
$$X_{1234}x_5 + (Y_{1234} - \sin\alpha_{56}) = 0 \qquad (Y_{1234} + \sin\alpha_{56})\,x_5 - X_{1234} = 0$$
$$X_{2345}x_6 + (Y_{2345} - \sin\alpha_{61}) = 0 \qquad (Y_{2345} + \sin\alpha_{61})\,x_6 - X_{2345} = 0$$

$$X_{5432}x_1 + (Y_{5432} - \sin\alpha_{61}) = 0 \qquad (Y_{5432} + \sin\alpha_{61})\,x_1 - X_{5432} = 0$$
$$X_{6543}x_2 + (Y_{6543} - \sin\alpha_{12}) = 0 \qquad (Y_{6543} + \sin\alpha_{12})\,x_2 - X_{6543} = 0$$
$$X_{1654}x_3 + (Y_{1654} - \sin\alpha_{23}) = 0 \qquad (Y_{1654} + \sin\alpha_{23})\,x_3 - X_{1654} = 0$$
$$X_{2165}x_4 + (Y_{2165} - \sin\alpha_{34}) = 0 \qquad (Y_{2165} + \sin\alpha_{34})\,x_4 - X_{2165} = 0$$
$$X_{3216}x_5 + (Y_{3216} - \sin\alpha_{45}) = 0 \qquad (Y_{3216} + \sin\alpha_{45})\,x_5 - X_{3216} = 0$$
$$X_{4321}x_6 + (Y_{4321} - \sin\alpha_{56}) = 0 \qquad (Y_{4321} + \sin\alpha_{56})\,x_6 - X_{4321} = 0$$

<u>Group 2.</u>

$$(X_{456} + \bar{X}_2)x_1 + (Y_{456} + \bar{Y}_2) = 0 \qquad (Y_{456} - \bar{Y}_2)x_1 - (X_{456} - \bar{X}_2) = 0$$
$$(X_{561} + \bar{X}_3)x_2 + (Y_{561} + \bar{Y}_3) = 0 \qquad (Y_{561} - \bar{Y}_3)x_2 - (X_{561} - \bar{X}_3) = 0$$
$$(X_{612} + \bar{X}_4)x_3 + (Y_{612} + \bar{Y}_4) = 0 \qquad (Y_{612} - \bar{Y}_4)x_3 - (X_{612} - \bar{X}_4) = 0$$
$$(X_{123} + \bar{X}_5)x_4 + (Y_{123} + \bar{Y}_5) = 0 \qquad (Y_{123} - \bar{Y}_5)x_4 - (X_{123} - \bar{X}_5) = 0$$
$$(X_{234} + \bar{X}_6)x_5 + (Y_{234} + \bar{Y}_6) = 0 \qquad (Y_{234} - \bar{Y}_6)x_5 - (X_{234} - \bar{X}_6) = 0$$
$$(X_{345} + \bar{X}_1)x_6 + (Y_{345} + \bar{Y}_1) = 0 \qquad (Y_{345} - \bar{Y}_1)x_6 - (X_{345} - \bar{X}_1) = 0$$

$$(X_{432} + X_6)x_1 + (Y_{432} + Y_6) = 0 \qquad (Y_{432} - Y_6)x_1 - (X_{432} - X_6) = 0$$
$$(X_{543} + X_1)x_2 + (Y_{543} + Y_1) = 0 \qquad (Y_{543} - Y_1)x_2 - (X_{543} - X_1) = 0$$
$$(X_{654} + X_2)x_3 + (Y_{654} + Y_2) = 0 \qquad (Y_{654} - Y_2)x_3 - (X_{654} - X_2) = 0$$
$$(X_{165} + X_3)x_4 + (Y_{165} + Y_3) = 0 \qquad (Y_{165} - Y_3)x_4 - (X_{165} - X_3) = 0$$
$$(X_{216} + X_4)x_5 + (Y_{216} + Y_4) = 0 \qquad (Y_{216} - Y_4)x_5 - (X_{216} - X_4) = 0$$
$$(X_{321} + X_5)x_6 + (Y_{321} + Y_5) = 0 \qquad (Y_{321} - Y_5)x_6 - (X_{321} - X_5) = 0$$

<u>Group 3.</u>

$$(X_{56} + X_{32})x_1 + (Y_{56} + Y_{32}) = 0 \qquad (Y_{56} - Y_{32})x_1 - (X_{56} - X_{32}) = 0$$
$$(X_{61} + X_{43})x_2 + (Y_{61} + Y_{43}) = 0 \qquad (Y_{61} - Y_{43})x_2 - (X_{61} - X_{43}) = 0$$
$$(X_{12} + X_{54})x_3 + (Y_{12} + Y_{54}) = 0 \qquad (Y_{12} - Y_{54})x_3 - (X_{12} - X_{54}) = 0$$
$$(X_{23} + X_{65})x_4 + (Y_{23} + Y_{65}) = 0 \qquad (Y_{23} - Y_{65})x_4 - (X_{23} - X_{65}) = 0$$
$$(X_{34} + X_{16})x_5 + (Y_{34} + Y_{16}) = 0 \qquad (Y_{34} - Y_{16})x_5 - (X_{34} - X_{16}) = 0$$
$$(X_{45} + X_{21})x_6 + (Y_{45} + Y_{21}) = 0 \qquad (Y_{45} - Y_{21})x_6 - (X_{45} - X_{21}) = 0$$

A.5. The Spherical Heptagon.

The three groups of fundamental half-tangent laws for the spherical heptagon (Figure 4.7) may be listed as follows:-

Group 1.

$$X_{34567}x_1 + (Y_{34567} - \sin\alpha_{12}) = 0 \qquad (Y_{34567} + \sin\alpha_{12})x_1 - X_{34567} = 0$$
$$X_{45671}x_2 + (Y_{45671} - \sin\alpha_{23}) = 0 \qquad (Y_{45671} + \sin\alpha_{23})x_2 - X_{45671} = 0$$
$$X_{56712}x_3 + (Y_{56712} - \sin\alpha_{34}) = 0 \qquad (Y_{56712} + \sin\alpha_{34})x_3 - X_{56712} = 0$$
$$X_{67123}x_4 + (Y_{67123} - \sin\alpha_{45}) = 0 \qquad (Y_{67123} + \sin\alpha_{45})x_4 - X_{67123} = 0$$
$$X_{71234}x_5 + (Y_{71234} - \sin\alpha_{56}) = 0 \qquad (Y_{71234} + \sin\alpha_{56})x_5 - X_{71234} = 0$$
$$X_{12345}x_6 + (Y_{12345} - \sin\alpha_{67}) = 0 \qquad (Y_{12345} + \sin\alpha_{67})x_6 - X_{12345} = 0$$
$$X_{23456}x_7 + (Y_{23456} - \sin\alpha_{71}) = 0 \qquad (Y_{23456} + \sin\alpha_{71})x_7 - X_{23456} = 0$$

$$X_{65432}x_1 + (Y_{65432} - \sin\alpha_{71}) = 0 \qquad (Y_{65432} + \sin\alpha_{71})x_1 - X_{65432} = 0$$
$$X_{76543}x_2 + (Y_{76543} - \sin\alpha_{12}) = 0 \qquad (Y_{76543} + \sin\alpha_{12})x_2 - X_{76543} = 0$$
$$X_{17654}x_3 + (Y_{17654} - \sin\alpha_{23}) = 0 \qquad (Y_{17654} + \sin\alpha_{23})x_3 - X_{17654} = 0$$
$$X_{21765}x_4 + (Y_{21765} - \sin\alpha_{34}) = 0 \qquad (Y_{21765} + \sin\alpha_{34})x_4 - X_{21765} = 0$$
$$X_{32176}x_5 + (Y_{32176} - \sin\alpha_{45}) = 0 \qquad (Y_{32176} + \sin\alpha_{45})x_5 - X_{32176} = 0$$
$$X_{43217}x_6 + (Y_{43217} - \sin\alpha_{56}) = 0 \qquad (Y_{43217} + \sin\alpha_{56})x_6 - X_{43217} = 0$$
$$X_{54321}x_7 + (Y_{54321} - \sin\alpha_{67}) = 0 \qquad (Y_{54321} + \sin\alpha_{67})x_7 - X_{54321} = 0$$

Group 2.

$$(X_{4567} + \bar{X}_2)x_1 + (Y_{4567} + \bar{Y}_2) = 0 \qquad (Y_{4567} - \bar{Y}_2)x_1 - (X_{4567} - \bar{X}_2) = 0$$
$$(X_{5671} + \bar{X}_3)x_2 + (Y_{5671} + \bar{Y}_3) = 0 \qquad (Y_{5671} - \bar{Y}_3)x_2 - (X_{5671} - \bar{X}_3) = 0$$
$$(X_{6712} + \bar{X}_4)x_3 + (Y_{6712} + \bar{Y}_4) = 0 \qquad (Y_{6712} - \bar{Y}_4)x_3 - (X_{6712} - \bar{X}_4) = 0$$
$$(X_{7123} + \bar{X}_5)x_4 + (Y_{7123} + \bar{Y}_5) = 0 \qquad (Y_{7123} - \bar{Y}_5)x_4 - (X_{7123} - \bar{X}_5) = 0$$
$$(X_{1234} + \bar{X}_6)x_5 + (Y_{1234} + \bar{Y}_6) = 0 \qquad (Y_{1234} - \bar{Y}_6)x_5 - (X_{1234} - \bar{X}_6) = 0$$
$$(X_{2345} + \bar{X}_7)x_6 + (Y_{2345} + \bar{Y}_7) = 0 \qquad (Y_{2345} - \bar{Y}_7)x_6 - (X_{2345} - \bar{X}_7) = 0$$
$$(X_{3456} + \bar{X}_1)x_7 + (Y_{3456} + \bar{Y}_1) = 0 \qquad (Y_{3456} - \bar{Y}_1)x_7 - (X_{3456} - \bar{X}_1) = 0$$

$$(X_{5432} + X_7)x_1 + (Y_{5432} + Y_7) = 0 \qquad (Y_{5432} - Y_7)x_1 - (X_{5432} - X_7) = 0$$
$$(X_{6543} + X_1)x_2 + (Y_{6543} + Y_1) = 0 \qquad (Y_{6543} - Y_1)x_2 - (X_{6543} - X_1) = 0$$
$$(X_{7654} + X_2)x_3 + (Y_{7654} + Y_2) = 0 \qquad (Y_{7654} - Y_2)x_3 - (X_{7654} - X_2) = 0$$
$$(X_{1765} + X_3)x_4 + (Y_{1765} + Y_3) = 0 \qquad (Y_{1765} - Y_3)x_4 - (X_{1765} - X_3) = 0$$
$$(X_{2176} + X_4)x_5 + (Y_{2176} + Y_4) = 0 \qquad (Y_{2176} - Y_4)x_5 - (X_{2176} - X_4) = 0$$
$$(X_{3217} + X_5)x_6 + (Y_{3217} + Y_5) = 0 \qquad (Y_{3217} - Y_5)x_6 - (X_{3217} - X_5) = 0$$
$$(X_{4321} + X_6)x_7 + (Y_{4321} + Y_6) = 0 \qquad (Y_{4321} - Y_6)x_7 - (X_{4321} - X_6) = 0$$

Group 3.

$$(X_{567} + X_{32})x_1 + (Y_{567} + Y_{32}) = 0 \qquad (Y_{567} - Y_{32})x_1 - (X_{567} - X_{32}) = 0$$
$$(X_{671} + X_{43})x_2 + (Y_{671} + Y_{43}) = 0 \qquad (Y_{671} - Y_{43})x_2 - (X_{671} - X_{43}) = 0$$
$$(X_{712} + X_{54})x_3 + (Y_{712} + Y_{54}) = 0 \qquad (Y_{712} - Y_{54})x_3 - (X_{712} - X_{54}) = 0$$
$$(X_{123} + X_{65})x_4 + (Y_{123} + Y_{65}) = 0 \qquad (Y_{123} - Y_{65})x_4 - (X_{123} - X_{65}) = 0$$
$$(X_{234} + X_{76})x_5 + (Y_{234} + Y_{76}) = 0 \qquad (Y_{234} - Y_{76})x_5 - (X_{234} - X_{76}) = 0$$
$$(X_{345} + X_{17})x_6 + (Y_{345} + Y_{17}) = 0 \qquad (Y_{345} - Y_{17})x_6 - (X_{345} - X_{17}) = 0$$
$$(X_{456} + X_{21})x_7 + (Y_{456} + Y_{21}) = 0 \qquad (Y_{456} - Y_{21})x_7 - (X_{456} - X_{21}) = 0$$

$$(X_{432} + X_{67})x_1 + (Y_{432} + Y_{67}) = 0 \qquad (Y_{432} - Y_{67})x_1 - (X_{432} - X_{67}) = 0$$
$$(X_{543} + X_{71})x_2 + (Y_{543} + Y_{71}) = 0 \qquad (Y_{543} - Y_{71})x_2 - (X_{543} - X_{71}) = 0$$
$$(X_{654} + X_{12})x_3 + (Y_{654} + Y_{12}) = 0 \qquad (Y_{654} - Y_{12})x_3 - (X_{654} - X_{12}) = 0$$
$$(X_{765} + X_{23})x_4 + (Y_{765} + Y_{23}) = 0 \qquad (Y_{765} - Y_{23})x_4 - (X_{765} - X_{23}) = 0$$
$$(X_{176} + X_{34})x_5 + (Y_{176} + Y_{34}) = 0 \qquad (Y_{176} - Y_{34})x_5 - (X_{176} - X_{34}) = 0$$
$$(X_{217} + X_{45})x_6 + (Y_{217} + Y_{45}) = 0 \qquad (Y_{217} - Y_{45})x_6 - (X_{217} - X_{45}) = 0$$
$$(X_{321} + X_{56})x_7 + (Y_{321} + Y_{56}) = 0 \qquad (Y_{321} - Y_{56})x_7 - (X_{321} - X_{56}) = 0$$

B. Further Half-Tangent Laws.

It has been explained in Chapter 5. that it is possible to derive an extensive series of further half-tangent laws from the fundamental half-tangent laws for each spherical polygon (see above). In particular there are $2^{(n-1)}$ distinct further laws, each in (n - 1) half-tangents, for an n-sided spherical polygon. Thus the four further laws in the two half-tangents x_1 and x_2 for a spherical triangle (Figure 4.1) are given by equations (5.45), (5.46), (5.47) and (5.48) (see Chapter 5).

For the spherical quadrilateral (Figure 4.4), there are eight further laws in the three half-tangents x_1, x_2 and x_3, and equation (5.55) represents one of these laws. A second example may be written:-

$$\begin{aligned}
&\sin\alpha_{12}[\cos(\alpha_{23}+\alpha_{34}) - Z_1]x_1x_2 \\
&+ [\sin\alpha_{12}\cos\alpha_{34} - \sin(\alpha_{23}+\alpha_{12})Z_1 + \sin\alpha_{23}\cos\alpha_{41}]x_1x_3 \\
&+ \sin\alpha_{23}[\cos(\alpha_{12}+\alpha_{41}) - Z_1]x_2x_3 = 0
\end{aligned}$$

For the pentagon (Figure 4.5) there are sixteen further half-tangent laws in any four half-tangents, whilst for the spherical hexagon (Figure 4.6) there exist thirty-two such laws in, say, the five half-tangents, x_1, x_2, x_3, x_4 and x_5. Three examples of the latter may be listed as follows:-

$$\begin{aligned}
&x_5[\cos(\alpha_{45}-\alpha_{56}) - \bar{Z}_5]\sin\alpha_{12}\sin\alpha_{23}\sin\alpha_{34} \\
&+ x_4[\sin(\alpha_{45}-\alpha_{34})\bar{Z}_5 - \sin\alpha_{45}Z_{12} + \sin\alpha_{34}\cos\alpha_{56}]\sin\alpha_{12}\sin\alpha_{23} \\
&+ x_3[\sin(\alpha_{34}-\alpha_{23})Z_{12} - \sin\alpha_{34}Z_1 + \sin\alpha_{23}\bar{Z}_5]\sin\alpha_{12}\sin\alpha_{45} \\
&+ x_2[\sin(\alpha_{23}-\alpha_{12})Z_1 + \sin\alpha_{12}Z_{54} - \sin\alpha_{23}\cos\alpha_{61}]\sin\alpha_{34}\sin\alpha_{45} \\
&+ x_1[-\cos(\alpha_{12}-\alpha_{61}) + Z_1]\sin\alpha_{23}\sin\alpha_{34}\sin\alpha_{45} = 0
\end{aligned}$$

$$\begin{aligned}
&x_1x_2x_3x_4[\bar{Z}_5 - \cos(\alpha_{45}+\alpha_{56})]\sin\alpha_{12}\sin\alpha_{23}\sin\alpha_{34} \\
&+ x_1x_2x_3x_5[\sin(\alpha_{34}+\alpha_{45})\bar{Z}_5 - \sin\alpha_{45}Z_{12} - \sin\alpha_{34}\cos\alpha_{56}]\sin\alpha_{12}\sin\alpha_{23} \\
&+ x_1x_2x_4x_5[\sin\alpha_{23}\bar{Z}_5 - \sin(\alpha_{23}+\alpha_{34})Z_{12} + \sin\alpha_{34}Z_1]\sin\alpha_{12}\sin\alpha_{45} \\
&+ x_1x_3x_4x_5[\sin(\alpha_{12}+\alpha_{23})Z_1 - \sin\alpha_{12}Z_{54} - \sin\alpha_{23}\cos\alpha_{61}]\sin\alpha_{34}\sin\alpha_{45} \\
&+ x_2x_3x_4x_5[Z_1 - \cos(\alpha_{61}+\alpha_{12})]\sin\alpha_{23}\sin\alpha_{34}\sin\alpha_{45} = 0
\end{aligned}$$

$$x_1x_2x_3[\sin\alpha_{23}\cos\alpha_{61} - \sin\alpha_{12}Z_{54} - \sin(\alpha_{23} - \alpha_{12})Z_1]\sin\alpha_{34}\sin\alpha_{45}$$
$$+ x_1x_3x_4[\sin\alpha_{34}\cos\alpha_{56} - \sin\alpha_{45}Z_{12} + \sin(\alpha_{45} - \alpha_{34})\bar{Z}_5]\sin\alpha_{12}\sin\alpha_{23}$$
$$+ x_1x_3x_5[\cos(\alpha_{45} + \alpha_{56}) - \bar{Z}_5]\sin\alpha_{12}\sin\alpha_{23}\sin\alpha_{34}$$
$$+ x_1[\sin(\alpha_{34} + \alpha_{23})Z_{12} - \sin\alpha_{34}Z_1 - \sin\alpha_{23}\bar{Z}_5]\sin\alpha_{12}\sin\alpha_{45}$$
$$+ x_3[Z_1 - \cos(\alpha_{12} + \alpha_{61})]\sin\alpha_{23}\sin\alpha_{34}\sin\alpha_{45} = 0$$

Finally, one may derive the following equation as a representative of the sixty-four possible further half-tangent laws in the six half-tangents x_1, x_2, x_3, x_4, x_5 and x_6, for the spherical heptagon (Figure 4.7):-

$$x_1x_2x_3x_4x_5[Z_{1234} - \cos(\alpha_{56} + \alpha_{67})]\sin\alpha_{12}\sin\alpha_{23}\sin\alpha_{34}\sin\alpha_{45}$$
$$+ x_1x_2x_3x_4x_6(\cos\alpha_{56}Z_{1234} + \sin\alpha_{56}Y_{1234} - \cos\alpha_{67})\sin\alpha_{12}\sin\alpha_{23}\sin\alpha_{34}\sin\alpha_{45}$$
$$- x_1x_2x_3x_5x_6(\cos\alpha_{45}Z_{123} + \sin\alpha_{45}Y_{123} - Z_{1234})\sin\alpha_{12}\sin\alpha_{23}\sin\alpha_{34}\sin\alpha_{56}$$
$$+ x_1x_2x_4x_5x_6(\cos\alpha_{34}Z_{12} + \sin\alpha_{34}Y_{12} - Z_{123})\sin\alpha_{12}\sin\alpha_{23}\sin\alpha_{45}\sin\alpha_{56}$$
$$- x_1x_3x_4x_5x_6(\cos\alpha_{23}Z_1 + \sin\alpha_{23}Y_1 - Z_{12})\sin\alpha_{12}\sin\alpha_{34}\sin\alpha_{45}\sin\alpha_{56}$$
$$+ x_2x_3x_4x_5x_6[\cos(\alpha_{12} + \alpha_{71}) - Z_1]\sin\alpha_{23}\sin\alpha_{34}\sin\alpha_{45}\sin\alpha_{56} = 0$$

APPENDIX V

LIST OF COEFFICIENTS

FOR

THE RCRPRR MECHANISM

The coefficients for the systems of equations, (7.24) and (7.25), for the RCRPRR six-link spatial mechanism (see Chapter 7) may be listed as follows:-

$$p_{22} = Z_4(\cos\alpha_{56}\bar{Z}_1 - \sin\alpha_{56}\bar{Y}_1) - Y_4(\sin\alpha_{56}\bar{Z}_1 + \cos\alpha_{56}\bar{Y}_1) + X_4\bar{X}_1 - \cos\alpha_{23}$$

$$p_{12} = 2.[(\sin\alpha_{56}Z_4 + \cos\alpha_{56}Y_4)\bar{X}_1 + X_4\bar{Y}_1]$$

$$p_{02} = Z_4(\cos\alpha_{56}\bar{Z}_1 + \sin\alpha_{56}\bar{Y}_1) - Y_4(\sin\alpha_{56}\bar{Z}_1 - \cos\alpha_{56}\bar{Y}_1) - X_4\bar{X}_1 - \cos\alpha_{23}$$

$$p_{21} = 2.[Y_4\bar{X}_1 + X_4(\sin\alpha_{56}\bar{Z}_1 + \cos\alpha_{56}\bar{Y}_1)]$$

$$p_{11} = 4.(Y_4\bar{Y}_1 - \cos\alpha_{56}X_4\bar{X}_1)$$

$$p_{01} = -2.[Y_4\bar{X}_1 - X_4(\sin\alpha_{56}\bar{Z}_1 - \cos\alpha_{56}\bar{Y}_1)]$$

$$p_{20} = Z_4(\cos\alpha_{56}\bar{Z}_1 - \sin\alpha_{56}\bar{Y}_1) + Y_4(\sin\alpha_{56}\bar{Z}_1 + \cos\alpha_{56}\bar{Y}_1) - X_4\bar{X}_1 - \cos\alpha_{23}$$

$$p_{10} = 2.[(\sin\alpha_{56}Z_4 - \cos\alpha_{56}Y_4)\bar{X}_1 - X_4\bar{Y}_1]$$

$$p_{00} = Z_4(\cos\alpha_{56}\bar{Z}_1 + \sin\alpha_{56}\bar{Y}_1) + Y_4(\sin\alpha_{56}\bar{Z}_1 - \cos\alpha_{56}\bar{Y}_1) + X_4\bar{X}_1 - \cos\alpha_{23}$$

$$q_{22} = h_1\bar{X}_1 - h_2\bar{Y}_1 + h_3\bar{Z}_1 - h_4 + \sin(\alpha_{56} - \alpha_{45})\bar{Y}_{01} - \cos(\alpha_{56} - \alpha_{45})\bar{Z}_{01}$$

$$q_{12} = 2.[h_2\bar{X}_1 + h_1\bar{Y}_1 - \sin(\alpha_{56} - \alpha_{45})\bar{X}_{01}]$$

$$q_{02} = -h_1\bar{X}_1 + h_2\bar{Y}_1 + h_3\bar{Z}_1 - h_4 - \sin(\alpha_{56} - \alpha_{45})\bar{Y}_{01} - \cos(\alpha_{56} - \alpha_{45})\bar{Z}_{01}$$

$$q_{21} = -2.(K_2\bar{X}_1 - h_5\bar{Y}_1 - h_6\bar{Z}_1 - \sin\alpha_{45}\bar{X}_{01})$$

$$q_{11} = -4.(h_5\bar{X}_1 + K_2\bar{Y}_1 - \sin\alpha_{45}\bar{Y}_{01})$$

$$q_{01} = 2.(K_2\bar{X}_1 - h_5\bar{Y}_1 + h_6\bar{Z}_1 - \sin\alpha_{45}\bar{X}_{01})$$

$$q_{20} = -H_1\bar{X}_1 - H_2\bar{Y}_1 + H_3\bar{Z}_1 - h_4$$
$$+ \sin(\alpha_{56} + \alpha_{45})\bar{Y}_{01} - \cos(\alpha_{56} + \alpha_{45})\bar{Z}_{01}$$
$$q_{10} = 2.[H_2\bar{X}_1 - H_1\bar{Y}_1 - \sin(\alpha_{56} + \alpha_{45})\bar{X}_{01}]$$
$$q_{00} = H_1\bar{X}_1 + H_2\bar{Y}_1 + H_3\bar{Z}_1 - h_4$$
$$- \sin(\alpha_{56} + \alpha_{45})\bar{Y}_{01} - \cos(\alpha_{56} + \alpha_{45})\bar{Z}_{01}$$

where:-

$$h_1 = S_{33}\sin\alpha_{34} - S_{55}\sin\alpha_{45} + S_{66}\sin(\alpha_{56} - \alpha_{45})$$
$$h_2 = K_1\sin\alpha_{56} - K_2\cos\alpha_{56} - a_{56}\cos(\alpha_{56} - \alpha_{45})$$
$$h_3 = K_1\cos\alpha_{56} + K_2\sin\alpha_{56} + a_{56}\sin(\alpha_{56} - \alpha_{45})$$
$$h_4 = S_{33}\cos\alpha_{23}\,x_{44} + a_{23}(\sin\alpha_{23}\cos\alpha_{34} + \cos^2\alpha_{23}\cos\alpha_{34}\operatorname{cosec}\alpha_{23})$$
$$+ a_{34}(\sin\alpha_{34}\cos\alpha_{23} + \cos^2\alpha_{34}\cos\alpha_{23}\operatorname{cosec}\alpha_{34})$$
$$h_5 = (S_{33}\sin\alpha_{34} - S_{55}\sin\alpha_{45})\cos\alpha_{56} - S_{66}\sin\alpha_{45}$$
$$h_6 = (S_{33}\sin\alpha_{34} - S_{55}\sin\alpha_{45})\sin\alpha_{56}$$

$$H_1 = S_{33}\sin\alpha_{34} - S_{55}\sin\alpha_{45} - S_{66}\sin(\alpha_{56} + \alpha_{45})$$
$$H_2 = K_1\sin\alpha_{56} + K_2\cos\alpha_{56} - a_{56}\cos(\alpha_{56} + \alpha_{45})$$
$$H_3 = K_1\cos\alpha_{56} - K_2\sin\alpha_{56} + a_{56}\sin(\alpha_{56} + \alpha_{45})$$

$$K_1 = S_{33}\cos(\alpha_{45} - \alpha_{34})\,x_{44} + a_{45}\sin\alpha_{45}$$
$$+ \cos\alpha_{45}(a_{34}\cot\alpha_{34} + a_{23}\cot\alpha_{23})$$
$$K_2 = S_{33}\sin(\alpha_{45} - \alpha_{34})\,x_{44} - a_{45}\cos\alpha_{45}$$
$$+ \sin\alpha_{45}(a_{34}\cot\alpha_{34} + a_{23}\cot\alpha_{23})$$

and where:-

$$X_4 = \sin\alpha_{34}\sin\theta_{44}$$
$$Y_4 = -(\cos\alpha_{34}\sin\alpha_{45} + \sin\alpha_{34}\cos\alpha_{45}\cos\theta_{44})$$
$$Z_4 = (\cos\alpha_{34}\cos\alpha_{45} - \sin\alpha_{34}\sin\alpha_{45}\cos\theta_{44})$$

In the above coefficients, the term x_{44} is defined by:-

$$x_{44} \equiv \tan(\theta_{44}/2)$$

and the terms $\bar{X}_1$, $\bar{Y}_1$, $\bar{Z}_1$ and $\bar{X}_{01}$, $\bar{Y}_{01}$, $\bar{Z}_{01}$ are given in Appendix III. (see also Chapter 4).

APPENDIX VI

LIST OF COEFFICIENTS

FOR

THE RCRRPR MECHANISM

The coefficients for the systems of equations, (8.28) and (8.29), for the RCRRPR six-link spatial mechanism (see Chapter 8) may be listed as follows:-

$$p_{22} = \sin(\alpha_{34} - \alpha_{45}) \sin\theta_{55}\bar{X}_1 + h_1\bar{Y}_1 + h_2\bar{Z}_1 - \cos\alpha_{23}$$
$$p_{12} = -2.[h_1\bar{X}_1 - \sin(\alpha_{34} - \alpha_{45}) \sin\theta_{55}\bar{Y}_1]$$
$$p_{02} = - \sin(\alpha_{34} - \alpha_{45}) \sin\theta_{55}\bar{X}_1 - h_1\bar{Y}_1 + h_2\bar{Z}_1 - \cos\alpha_{23}$$

$$p_{21} = -2\sin\alpha_{34}(\cos\theta_{55}\bar{X}_1 - \cos\alpha_{56} \sin\theta_{55}\bar{Y}_1 - \sin\alpha_{56} \sin\theta_{55}\bar{Z}_1)$$
$$p_{11} = -4\sin\alpha_{34} (\cos\alpha_{56} \sin\theta_{55}\bar{X}_1 + \cos\theta_{55}\bar{Y}_1)$$
$$p_{01} = 2\sin\alpha_{34}(\cos\theta_{55}\bar{X}_1 - \cos\alpha_{56} \sin\theta_{55}\bar{Y}_1 + \sin\alpha_{56} \sin\theta_{55}\bar{Z}_1)$$

$$p_{20} = -\sin(\alpha_{34} + \alpha_{45}) \sin\theta_{55}\bar{X}_1 - h_3\bar{Y}_1 - h_4\bar{Z}_1 - \cos\alpha_{23}$$
$$p_{10} = 2.[h_3\bar{X}_1 - \sin(\alpha_{34} + \alpha_{45}) \sin\theta_{55}\bar{Y}_1]$$
$$p_{00} = \sin(\alpha_{34} + \alpha_{45}) \sin\theta_{55}\bar{X}_1 + h_3\bar{Y}_1 - h_4\bar{Z}_1 - \cos\alpha_{23}$$

$$q_{22} = H_1\bar{X}_1 - H_2\bar{Y}_1 - H_3\bar{Z}_1 + H_4 + L_2p_{21}$$
$$+ \sin\alpha_{34} \sin\alpha_{23} (\sin\alpha_{56} \bar{Y}_{01} - \cos\alpha_{56} \bar{Z}_{01})$$
$$q_{12} = 2.(H_2\bar{X}_1 + H_1\bar{Y}_1 - \sin\alpha_{34} \sin\alpha_{23} \sin\alpha_{56} \bar{X}_{01}) + L_2p_{11}$$
$$q_{02} = -H_1\bar{X}_1 + H_2\bar{Y}_1 - H_3 \bar{Z}_1 + H_4 + L_2p_{01}$$
$$- \sin\alpha_{34} \sin\alpha_{23} (\sin\alpha_{56} \bar{Y}_{01} + \cos\alpha_{56} \bar{Z}_{01})$$

$$q_{21} = K_1\bar{X}_1 - K_2\bar{Y}_1 - K_3\bar{Z}_1 + K_4 + L_0p_{22}$$
$$q_{11} = 2.(K_2\bar{X}_1 + K_1\bar{Y}_1) + L_0p_{12}$$
$$q_{01} = -K_1\bar{X}_1 + K_2\bar{Y}_1 - K_3\bar{Z}_1 + K_4 + L_0p_{02}$$

$$q_{20} = -J_1\bar{X}_1 + J_2\bar{Y}_1 + J_3\bar{Z}_1 + J_4 - L_1 p_{20}$$
$$+ \sin\alpha_{34}\sin\alpha_{23}(\sin\alpha_{56}\bar{Y}_{01} - \cos\alpha_{56}\bar{Z}_{01})$$
$$q_{10} = -2.(J_2\bar{X}_1 + J_1\bar{Y}_1 + \sin\alpha_{34}\sin\alpha_{23}\sin\alpha_{56}\bar{X}_{01}) - L_1 p_{10}$$
$$q_{00} = J_1\bar{X}_1 - J_2\bar{Y}_1 + J_3\bar{Z}_1 + J_4 - L_1 p_{00}$$
$$- \sin\alpha_{34}\sin\alpha_{23}(\sin\alpha_{56}\bar{Y}_{01} + \cos\alpha_{56}\bar{Z}_{01})$$

where:-

$$h_1 = \sin(\alpha_{34} - \alpha_{45})\cos\alpha_{56}\cos\theta_{55} - \cos(\alpha_{34} - \alpha_{45})\sin\alpha_{56}$$
$$h_2 = \sin(\alpha_{34} - \alpha_{45})\sin\alpha_{56}\cos\theta_{55} + \cos(\alpha_{34} - \alpha_{45})\cos\alpha_{56}$$
$$h_3 = \sin(\alpha_{34} + \alpha_{45})\cos\alpha_{56}\cos\theta_{55} + \cos(\alpha_{34} + \alpha_{45})\sin\alpha_{56}$$
$$h_4 = \sin(\alpha_{34} + \alpha_{45})\sin\alpha_{56}\cos\theta_{55} - \cos(\alpha_{34} + \alpha_{45})\cos\alpha_{56}$$

$$H_1 = \sin\alpha_{34}(L_2\cos\theta_{55} + S_{66}\sin\alpha_{23}\sin\alpha_{56}) + M_2X_5$$
$$H_2 = \sin\alpha_{34}\cos\alpha_{56}(L_2\sin\theta_{55} - a_{56}\sin\alpha_{23}) + M_2Y_5$$
$$H_3 = \sin\alpha_{34}\sin\alpha_{56}(L_2\sin\theta_{55} - a_{56}\sin\alpha_{23}) + M_2Z_5$$
$$H_4 = M_2\cos\alpha_{34}\cos\alpha_{23} + N_2$$

$$K_1 = L_1\sin\alpha_{34}\cos\theta_{55} + M_1X_5$$
$$K_2 = L_1\sin\alpha_{34}\cos\alpha_{56}\sin\theta_{55} + M_1Y_5$$
$$K_3 = L_1\sin\alpha_{34}\sin\alpha_{56}\sin\theta_{55} + M_1Z_5$$
$$K_4 = M_1\cos\alpha_{34}\cos\alpha_{23} + N_1$$

$$J_1 = \sin\alpha_{34}(L_0\cos\theta_{55} - S_{66}\sin\alpha_{23}\sin\alpha_{56}) - M_0X_5$$
$$J_2 = \sin\alpha_{34}\cos\alpha_{56}(L_0\sin\theta_{55} + a_{56}\sin\alpha_{23}) - M_0Y_5$$
$$J_3 = \sin\alpha_{34}\sin\alpha_{56}(L_0\sin\theta_{55} + a_{56}\sin\alpha_{23}) - M_0Z_5$$
$$J_4 = M_0\cos\alpha_{34}\cos\alpha_{23} + N_0$$

$$L_2 = \sin\alpha_{23}[S_{33}\sin(\alpha_{34} - \alpha_{45}) - S_{44}\sin\alpha_{45}]$$
$$L_1 = 2.(a_{23}\cos\alpha_{23}\sin\alpha_{45} + a_{45}\cos\alpha_{45}\sin\alpha_{23})$$
$$L_0 = \sin\alpha_{23}[S_{33}\sin(\alpha_{34} + \alpha_{45}) + S_{44}\sin\alpha_{45}]$$

$$M_2 = (a_{45} - a_{34})\sin\alpha_{23}\cos(\alpha_{34} - \alpha_{45})$$
$$\quad - a_{23}\cos\alpha_{23}\sin(\alpha_{34} - \alpha_{45})$$

$$M_1 = 2\sin\alpha_{23}\sin\alpha_{45}(S_{33} + S_{44}\cos\alpha_{34})$$

$$M_0 = -(a_{45} + a_{34})\sin\alpha_{23}\cos(\alpha_{34} + \alpha_{45})$$
$$\quad - a_{23}\cos\alpha_{23}\sin(\alpha_{34} + \alpha_{45})$$

$$N_2 = \sin\alpha_{23}\sin\alpha_{34}\big[(a_{45} - a_{34})\cos\alpha_{23}\sin(\alpha_{34} - \alpha_{45})$$
$$\quad - a_{23}\sin\alpha_{23}\cos(\alpha_{34} - \alpha_{45})\big]$$

$$N_1 = 2S_{44}\sin^2\alpha_{34}\sin\alpha_{23}\cos\alpha_{23}\sin\alpha_{45}$$

$$N_0 = -\sin\alpha_{23}\sin\alpha_{34}\big[(a_{45} + a_{34})\cos\alpha_{23}\sin(\alpha_{34} + \alpha_{45})$$
$$\quad + a_{23}\sin\alpha_{23}\cos(\alpha_{34} + \alpha_{45})\big]$$

and where:-

$$X_5 = \sin\alpha_{45}.\sin\theta_{55}$$
$$Y_5 = -(\cos\alpha_{45}\sin\alpha_{56} + \sin\alpha_{45}\cos\alpha_{56}\cos\theta_{55})$$
$$Z_5 = (\cos\alpha_{45}\cos\alpha_{56} - \sin\alpha_{45}\sin\alpha_{56}\cos\theta_{55})$$

In the above coefficients, the terms $\bar{X}_1$, $\bar{Y}_1$, $\bar{Z}_1$ and $\bar{\bar{X}}_{01}$, $\bar{Y}_{01}$, $\bar{Z}_{01}$ are given in Appendix III. (see also Chapter 4).

APPENDIX VII

LIST OF COEFFICIENTS

FOR

THE RRRPCR MECHANISM

The coefficients for the systems of equations, (9.42) and (9.43), for the RRRPCR six-link spatial mechanism (see Chapter 9) may be listed as follows:-

$$p_{22} = X_4\bar{X}_1 - h_2\bar{Y}_1 + h_1\bar{Z}_1 - \cos\alpha_{23}$$
$$p_{12} = 2.(h_2\bar{X}_1 + X_4\bar{Y}_1)$$
$$p_{02} = -X_4\bar{X}_1 + h_2\bar{Y}_1 + h_1\bar{Z}_1 - \cos\alpha_{23}$$

$$p_{21} = 2.(Y_4\bar{X}_1 + h_3X_4)$$
$$p_{11} = -4.(\cos\alpha_{56}X_4\bar{X}_1 - Y_4\bar{Y}_1)$$
$$p_{01} = -2.(Y_4\bar{X}_1 - h_4X_4)$$

$$p_{20} = -X_4\bar{X}_1 - h_6\bar{Y}_1 + h_5\bar{Z}_1 - \cos\alpha_{23}$$
$$p_{10} = 2.(h_6\bar{X}_1 - X_4\bar{Y}_1)$$
$$p_{00} = X_4\bar{X}_1 + h_6\bar{Y}_1 + h_5\bar{Z}_1 - \cos\alpha_{23}$$

$$q_{22} = a_{23}(h_2\bar{Z}_1 + h_1\bar{Y}_1)\mathrm{cosec}\alpha_{23} + S_{33}\sin\alpha_{34}\sin\theta_{44} + S_{22}\bar{X}_1 + a_{34}\cos\theta_{44} + a_{12}\cos\theta_1 + (a_{45} - a_{56} + a_{61})$$
$$q_{12} = 2.(S_{22}\bar{Y}_1 - a_{23}h_1\bar{X}_1\mathrm{cosec}\alpha_{23} - S_{11}\sin\alpha_{61} + a_{12}\cos\alpha_{61}\sin\theta_1)$$
$$q_{02} = a_{23}(h_2\bar{Z}_1 - h_1\bar{Y}_1)\mathrm{cosec}\alpha_{23} + S_{33}\sin\alpha_{34}\sin\theta_{44} - S_{22}\bar{X}_1 + a_{34}\cos\theta_{44} - a_{12}\cos\theta_1 + (a_{45} - a_{56} - a_{61})$$

$$q_{21} = 2.[S_{22}h_3 - a_{23}Z_4\bar{X}_1\mathrm{cosec}\alpha_{23} + a_{12}\cos(\alpha_{56} - \alpha_{61})\sin\theta_1 + S_{11}\sin(\alpha_{56} - \alpha_{61}) + S_{66}\sin\alpha_{56}]$$
$$q_{11} = -4.(S_{22}\cos\alpha_{56}\bar{X}_1 + a_{23}Z_4\bar{Y}_1\mathrm{cosec}\alpha_{23} + a_{12}\cos\alpha_{56}\cos\theta_1 + a_{61}\cos\alpha_{56})$$
$$q_{01} = 2.[S_{22}h_4 + a_{23}Z_4\bar{X}_1\mathrm{cosec}\alpha_{23} - a_{12}\cos(\alpha_{56} + \alpha_{61})\sin\theta_1 + S_{11}\sin(\alpha_{56} + \alpha_{61}) + S_{66}\sin\alpha_{56}]$$

$$q_{20} = -a_{23}(h_6\bar{Z}_1 + h_5\bar{Y}_1)\operatorname{cosec}\alpha_{23} + S_{33}\sin\alpha_{34}\sin\theta_{44} - S_{22}\bar{X}_1$$
$$+a_{34}\cos\theta_{44} - a_{12}\cos\theta_1 + (a_{45} + a_{56} - a_{61})$$
$$q_{10} = -2.(S_{22}\bar{Y}_1 - a_{23}h_5\bar{X}_1\operatorname{cosec}\alpha_{23} - S_{11}\sin\alpha_{61} + a_{12}\cos\alpha_{61}\sin\theta_1)$$
$$q_{00} = -a_{23}(h_6\bar{Z}_1 - h_5\bar{Y}_1)\operatorname{cosec}\alpha_{23} + S_{33}\sin\alpha_{34}\sin\theta_{44} + S_{22}\bar{X}_1$$
$$+a_{34}\cos\theta_{44} + a_{12}\cos\theta_1 + (a_{45} + a_{56} + a_{61})$$

where:-

$$h_1 = \cos\alpha_{56} Z_4 - \sin\alpha_{56} Y_4$$
$$h_2 = \sin\alpha_{56} Z_4 + \cos\alpha_{56} Y_4$$
$$h_3 = \sin\alpha_{56} \bar{Z}_1 + \cos\alpha_{56} \bar{Y}_1$$
$$h_4 = \sin\alpha_{56} \bar{Z}_1 - \cos\alpha_{56} \bar{Y}_1$$
$$h_5 = \cos\alpha_{56} Z_4 + \sin\alpha_{56} Y_4$$
$$h_6 = \sin\alpha_{56} Z_4 - \cos\alpha_{56} Y_4$$

and where:-

$$X_4 = \sin\alpha_{34}\sin\theta_{44}$$
$$Y_4 = -(\cos\alpha_{34}\sin\alpha_{45} + \sin\alpha_{34}\cos\alpha_{45}\cos\theta_{44})$$
$$Z_4 = (\cos\alpha_{34}\cos\alpha_{45} - \sin\alpha_{34}\sin\alpha_{45}\cos\theta_{44})$$

In the above coefficients, the terms $\bar{X}_1$, $\bar{Y}_1$, $\bar{Z}_1$ are given in Appendix III. (see also Chapter 4).

APPENDIX VIII

LIST OF COEFFICIENTS

FOR

5R-C MECHANISMS

A. Coefficients for the RRRRCR Mechanism.

The coefficients for the system of equations, (10.30), for the RRRRCR six-link spatial mechanism (see Chapter 10) may be listed as follows:-

$$\begin{aligned} a_1 = \; & Z_{61}(S_{33}\sin\alpha_{23}\sin\alpha_{34} - S_{44}\cos\alpha_{23}) \\ & + Y_{61}(S_{22}\sin\alpha_{34} + S_{33}\cos\alpha_{23}\sin\alpha_{34} + S_{44}\sin\alpha_{23}) \\ & + X_{61}a_{45}\cot\alpha_{45}\sin\alpha_{34} \\ & - \sin\alpha_{34}X_{061} \\ & + S_{44}\cos\alpha_{34}\cos\alpha_{45} \end{aligned}$$

$$\begin{aligned} a_2 = \; & Z_{61}[a_{23}\sin(\alpha_{23} - \alpha_{34}) + a_{34}\cos\alpha_{23}\operatorname{cosec}\alpha_{34} \\ & \qquad + a_{45}\cot\alpha_{45}\cos(\alpha_{23} - \alpha_{34})] \\ & + Y_{61}[a_{23}\cos(\alpha_{23} - \alpha_{34}) - a_{34}\sin\alpha_{23}\operatorname{cosec}\alpha_{34} \\ & \qquad - a_{45}\cot\alpha_{45}\sin(\alpha_{23} - \alpha_{34})] \\ & + X_{61}[S_{22}\sin(\alpha_{23} - \alpha_{34}) - S_{33}\sin\alpha_{34}] \\ & - [\cos(\alpha_{23} - \alpha_{34})Z_{061} - \sin(\alpha_{23} - \alpha_{34})Y_{061}] \\ & - (a_{45}\operatorname{cosec}\alpha_{45} + a_{34}\cos\alpha_{45}\cot\alpha_{34}) \end{aligned}$$

$$a_3 = S_{44}[- Z_{61}\cos(\alpha_{23} - \alpha_{34}) + Y_{61}\sin(\alpha_{23} - \alpha_{34}) + \cos\alpha_{45}]$$

$$\begin{aligned} a_4 = & - Z_{61}[a_{23}\sin\alpha_{23} + (a_{45}\cot\alpha_{45} + a_{34}\cot\alpha_{34})\cos\alpha_{23}] \\ & - Y_{61}[a_{23}\cos\alpha_{23} - (a_{45}\cot\alpha_{45} + a_{34}\cot\alpha_{34})\sin\alpha_{23}] \\ & - X_{61}(S_{22}\sin\alpha_{23} + S_{44}\sin\alpha_{34}) \\ & + (\cos\alpha_{23}Z_{061} - \sin\alpha_{23}Y_{061}) \\ & + (a_{45}\cos\alpha_{34}\operatorname{cosec}\alpha_{45} + a_{34}\cos\alpha_{45}\operatorname{cosec}\alpha_{34}) \end{aligned}$$

$$\begin{aligned} b_1 = 2.[& Y_{61}a_{45}\sin\alpha_{34}\cot\alpha_{45} \\ & - X_{61}(S_{22}\sin\alpha_{34} + S_{33}\cos\alpha_{23}\sin\alpha_{34} + S_{44}\sin\alpha_{23}) \\ & - \sin\alpha_{34}Y_{061}] \end{aligned}$$

$$b_2 = 2.\Big[+ Y_{61}[S_{22}\sin(\alpha_{23} - \alpha_{34}) - S_{33}\sin\alpha_{34}]$$
$$- X_{61}[a_{23}\cos(\alpha_{23} - \alpha_{34}) - a_{34}\sin\alpha_{23}\,\mathrm{cosec}\alpha_{34}$$
$$- a_{45}\cot\alpha_{45}\sin(\alpha_{23} - \alpha_{34})]$$
$$- \sin(\alpha_{23} - \alpha_{34})X_{061}\Big]$$

$$b_3 = -2.S_{44}\sin(\alpha_{23} - \alpha_{34})X_{61}$$

$$b_4 = 2.\Big[- Y_{61}(S_{22}\sin\alpha_{23} + S_{44}\sin\alpha_{34})$$
$$+ X_{61}[a_{23}\cos\alpha_{23} - (a_{45}\cot\alpha_{45} + a_{34}\cot\alpha_{34})\sin\alpha_{23}]$$
$$+ \sin\alpha_{23}X_{061}\Big]$$

$$c_1 = Z_{61}(S_{33}\sin\alpha_{23}\sin\alpha_{34} - S_{44}\cos\alpha_{23})$$
$$- Y_{61}(S_{22}\sin\alpha_{34} + S_{33}\cos\alpha_{23}\sin\alpha_{34} + S_{44}\sin\alpha_{23})$$
$$- X_{61}a_{45}\cot\alpha_{45}\sin\alpha_{34}$$
$$+ \sin\alpha_{34}X_{061}$$
$$+ S_{44}\cos\alpha_{34}\cos\alpha_{45}$$

$$c_2 = Z_{61}[a_{23}\sin(\alpha_{23} - \alpha_{34}) + a_{34}\cos\alpha_{23}\,\mathrm{cosec}\alpha_{34}$$
$$+ a_{45}\cot\alpha_{45}\cos(\alpha_{23} - \alpha_{34})]$$
$$- Y_{61}[a_{23}\cos(\alpha_{23} - \alpha_{34}) - a_{34}\sin\alpha_{23}\,\mathrm{cosec}\alpha_{34}$$
$$- a_{45}\cot\alpha_{45}\sin(\alpha_{23} - \alpha_{34})]$$
$$- X_{61}[S_{22}\sin(\alpha_{23} - \alpha_{34}) - S_{33}\sin\alpha_{34}]$$
$$- [\cos(\alpha_{23} - \alpha_{34})Z_{061} + \sin(\alpha_{23} - \alpha_{34})Y_{061}]$$
$$- (a_{45}\,\mathrm{cosec}\alpha_{45} + a_{34}\cos\alpha_{45}\cot\alpha_{34})$$

$$c_3 = S_{44}[- Z_{61}\cos(\alpha_{23} - \alpha_{34}) - Y_{61}\sin(\alpha_{23} - \alpha_{34}) + \cos\alpha_{45}]$$

$$c_4 = - Z_{61}[a_{23}\sin\alpha_{23} + (a_{45}\cot\alpha_{45} + a_{34}\cot\alpha_{34})\cos\alpha_{23}]$$
$$+ Y_{61}[a_{23}\cos\alpha_{23} - (a_{45}\cot\alpha_{45} + a_{34}\cot\alpha_{34})\sin\alpha_{23}]$$
$$+ X_{61}(S_{22}\sin\alpha_{23} + S_{44}\sin\alpha_{34})$$
$$+ (\cos\alpha_{23}Z_{061} + \sin\alpha_{23}Y_{061})$$
$$+ (a_{45}\cos\alpha_{34}\,\mathrm{cosec}\alpha_{45} + a_{34}\cos\alpha_{45}\,\mathrm{cosec}\alpha_{34})$$

$$d_1 = -Z_{61}[a_{23}\sin(\alpha_{23}+\alpha_{34}) + a_{34}\cos\alpha_{23}\,\mathrm{cosec}\alpha_{34} + a_{45}\cot\alpha_{45}\cos(\alpha_{23}+\alpha_{34})]$$
$$- Y_{61}[a_{23}\cos(\alpha_{23}+\alpha_{34}) - a_{34}\sin\alpha_{23}\,\mathrm{cosec}\alpha_{34} - a_{45}\cot\alpha_{45}\sin(\alpha_{23}+\alpha_{34})]$$
$$- X_{61}[S_{22}\sin(\alpha_{23}+\alpha_{34}) + S_{33}\sin\alpha_{34}]$$
$$+ [\cos(\alpha_{23}+\alpha_{34})Z_{061} - \sin(\alpha_{23}+\alpha_{34})Y_{061}]$$
$$+ (a_{45}\,\mathrm{cosec}\alpha_{45} + a_{34}\cos\alpha_{45}\cot\alpha_{34})$$

$$d_2 = -Z_{61}(S_{33}\sin\alpha_{23}\sin\alpha_{34} + S_{44}\cos\alpha_{23})$$
$$- Y_{61}(S_{22}\sin\alpha_{34} + S_{33}\cos\alpha_{23}\sin\alpha_{34} - S_{44}\sin\alpha_{23})$$
$$- X_{61}a_{45}\cot\alpha_{45}\sin\alpha_{34}$$
$$+ \sin\alpha_{34}X_{061}$$
$$+ S_{44}\cos\alpha_{34}\cos\alpha_{45}$$

$$d_3 = -Z_{61}[a_{23}\sin\alpha_{23} + (a_{45}\cot\alpha_{45} + a_{34}\cot\alpha_{34})\cos\alpha_{23}]$$
$$- Y_{61}[a_{23}\cos\alpha_{23} - (a_{45}\cot\alpha_{45} + a_{34}\cot\alpha_{34})\sin\alpha_{23}]$$
$$- X_{61}(S_{22}\sin\alpha_{23} - S_{44}\sin\alpha_{34})$$
$$+ (\cos\alpha_{23}Z_{061} - \sin\alpha_{23}Y_{061})$$
$$+ (a_{45}\cos\alpha_{34}\,\mathrm{cosec}\alpha_{45} + a_{34}\cos\alpha_{45}\,\mathrm{cosec}\alpha_{34})$$

$$d_4 = S_{44}[Z_{61}\cos(\alpha_{23}+\alpha_{34}) - Y_{61}\sin(\alpha_{23}+\alpha_{34}) - \cos\alpha_{45}]$$

$$e_1 = 2.\Big[-Y_{61}[S_{22}\sin(\alpha_{23}+\alpha_{34}) + S_{33}\sin\alpha_{34}]$$
$$+ X_{61}[a_{23}\cos(\alpha_{23}+\alpha_{34}) - a_{34}\sin\alpha_{23}\,\mathrm{cosec}\alpha_{34} - a_{45}\cot\alpha_{45}\sin(\alpha_{23}+\alpha_{34})]$$
$$+ \sin(\alpha_{23}+\alpha_{34})X_{061}\Big]$$

$$e_2 = 2.[-Y_{61}a_{45}\sin\alpha_{34}\cot\alpha_{45}$$
$$+ X_{61}(S_{22}\sin\alpha_{34} + S_{33}\cos\alpha_{23}\sin\alpha_{34} - S_{44}\sin\alpha_{23})$$
$$+ \sin\alpha_{34}Y_{061}]$$

$$e_3 = 2.\Big[- Y_{61}(S_{22}\sin\alpha_{23} - S_{44}\sin\alpha_{34})$$
$$+ X_{61}[a_{23}\cos\alpha_{23} - (a_{45}\cot\alpha_{45} + a_{34}\cot\alpha_{34})\sin\alpha_{23}]$$
$$+ \sin\alpha_{23} X_{061}\Big]$$

$$e_4 = 2.S_{44}\sin(\alpha_{23} + \alpha_{34}) X_{61}$$

$$f_1 = - Z_{61}[a_{23}\sin(\alpha_{23} + \alpha_{34}) + a_{34}\cos\alpha_{23}\,\mathrm{cosec}\,\alpha_{34}$$
$$+ a_{45}\cot\alpha_{45}\cos(\alpha_{23} + \alpha_{34})]$$
$$+ Y_{61}[a_{23}\cos(\alpha_{23} + \alpha_{34}) - a_{34}\sin\alpha_{23}\,\mathrm{cosec}\,\alpha_{34}$$
$$- a_{45}\cot\alpha_{45}\sin(\alpha_{23} + \alpha_{34})]$$
$$+ X_{61}[S_{22}\sin(\alpha_{23} + \alpha_{34}) + S_{33}\sin\alpha_{34}]$$
$$+ [\cos(\alpha_{23} + \alpha_{34}) Z_{061} + \sin(\alpha_{23} + \alpha_{34}) Y_{061}]$$
$$+ (a_{45}\,\mathrm{cosec}\,\alpha_{45} + a_{34}\cos\alpha_{45}\cot\alpha_{34})$$

$$f_2 = - Z_{61}(S_{33}\sin\alpha_{23}\sin\alpha_{34} + S_{44}\cos\alpha_{23})$$
$$+ Y_{61}(S_{22}\sin\alpha_{34} + S_{33}\cos\alpha_{23}\sin\alpha_{34} - S_{44}\sin\alpha_{23})$$
$$+ X_{61}a_{45}\cot\alpha_{45}\sin\alpha_{34}$$
$$- \sin\alpha_{34} X_{061}$$
$$+ S_{44}\cos\alpha_{34}\cos\alpha_{45}$$

$$f_3 = - Z_{61}[a_{23}\sin\alpha_{23} + (a_{45}\cot\alpha_{45} + a_{34}\cot\alpha_{34})\cos\alpha_{23}]$$
$$+ Y_{61}[a_{23}\cos\alpha_{23} - (a_{45}\cot\alpha_{45} + a_{34}\cot\alpha_{34})\sin\alpha_{23}]$$
$$+ X_{61}(S_{22}\sin\alpha_{23} - S_{44}\sin\alpha_{34})$$
$$+ (\cos\alpha_{23} Z_{061} + \sin\alpha_{23} Y_{061})$$
$$+ (a_{45}\cos\alpha_{34}\,\mathrm{cosec}\,\alpha_{45} + a_{34}\cos\alpha_{45}\,\mathrm{cosec}\,\alpha_{34})$$

$$f_4 = S_{44}[Z_{61}\cos(\alpha_{23} + \alpha_{34}) + Y_{61}\sin(\alpha_{23} + \alpha_{34}) - \cos\alpha_{45}]$$

B. Coefficients for the RRCRRR Mechanism.

The coefficients for the system of equations, (10.32), for the RRCRRR six-link spatial mechanism (see Chapter 10) may be listed as follows:-

$$
\begin{aligned}
a_1' = \; & a_{12}\sin\alpha_{34} R_1 \sin\theta_2 \\
& + a_{45}\sin\alpha_{34}\cot\alpha_{45}(P_1\cos\theta_2 - Q_1\sin\theta_2) \\
& - (a_{56} - a_{61})\sin\alpha_{34}[\sin(\alpha_{56} - \alpha_{61})U_{21} - \cos(\alpha_{56} - \alpha_{61})V_{21}] \\
& - [S_{11}\sin(\alpha_{56} - \alpha_{61}) + S_{66}\sin\alpha_{56}]\sin\alpha_{34} W_{12} \\
& + S_{22}\sin\alpha_{34}(P_1\sin\theta_2 + Q_1\cos\theta_2) \\
& + S_{33}\sin\alpha_{34}[\cos\alpha_{23}(P_1\sin\theta_2 + Q_1\cos\theta_2) + \sin\alpha_{23} R_1] \\
& + S_{44}[\sin\alpha_{23}(P_1\sin\theta_2 + Q_1\cos\theta_2) - \cos\alpha_{23} R_1] \\
& + S_{44}\cos\alpha_{34}\cos\alpha_{45}
\end{aligned}
$$

$$
\begin{aligned}
a_2' = & - a_{12}[\cos(\alpha_{56} - \alpha_{61})Q_2 + \sin(\alpha_{56} - \alpha_{61})R_2\cos\theta_1] \\
& + a_{23}[\cos(\alpha_{23} - \alpha_{34})(P_1\sin\theta_2 + Q_1\cos\theta_2) + \sin(\alpha_{23} - \alpha_{34})R_1] \\
& - a_{34}\operatorname{cosec}\alpha_{34}[\sin(\alpha_{56} - \alpha_{61})(\bar{X}_2\sin\theta_1 + \bar{Y}_2\cos\theta_1) - \cos(\alpha_{56} - \alpha_{61})\bar{Z}_2] \\
& - a_{45}\cot\alpha_{45}[\sin(\alpha_{56} - \alpha_{61})(P_2\sin\theta_1 + Q_2\cos\theta_1) - \cos(\alpha_{56} - \alpha_{61})R_2] \\
& + (a_{56} - a_{61})[\cos(\alpha_{56} - \alpha_{61})(P_2\sin\theta_1 + Q_2\cos\theta_1) + \sin(\alpha_{56} - \alpha_{61})R_2] \\
& + [S_{11}\sin(\alpha_{56} - \alpha_{61}) + S_{66}\sin\alpha_{56}](P_2\cos\theta_1 - Q_2\sin\theta_1) \\
& + [S_{22}\sin(\alpha_{23} - \alpha_{34}) - S_{33}\sin\alpha_{34}](P_1\cos\theta_2 - Q_1\sin\theta_2) \\
& - (a_{34}\cot\alpha_{34}\cos\alpha_{45} + a_{45}\operatorname{cosec}\alpha_{45})
\end{aligned}
$$

$$
a_3' = S_{44}[\sin(\alpha_{56} - \alpha_{61})(P_2\sin\theta_1 + Q_2\cos\theta_1) - \cos(\alpha_{56} - \alpha_{61})R_2 + \cos\alpha_{45}]
$$

$$
\begin{aligned}
a_4' = \; & a_{12}[\cos(\alpha_{56} - \alpha_{61})\bar{Y}_2 + \sin(\alpha_{56} - \alpha_{61})\bar{Z}_2\cos\theta_1] \\
& - a_{23}[\cos\alpha_{23}(P_1\sin\theta_2 + Q_1\cos\theta_2) + \sin\alpha_{23} R_1] \\
& + (a_{34}\cot\alpha_{34} + a_{45}\cot\alpha_{45})[\sin\alpha_{23}.(P_1\sin\theta_2 + Q_1\cos\theta_2) - \cos\alpha_{23} R_1] \\
& - (a_{56} - a_{61})[\cos(\alpha_{56} - \alpha_{61})(\bar{X}_2\sin\theta_1 + \bar{Y}_2\cos\theta_1) + \sin(\alpha_{56} - \alpha_{61})\bar{Z}_2] \\
& - [S_{11}\sin(\alpha_{56} - \alpha_{61}) + S_{66}\sin\alpha_{56}](\bar{X}_2\cos\theta_1 - \bar{Y}_2\sin\theta_1) \\
& - (S_{22}\sin\alpha_{23} + S_{44}\sin\alpha_{34})(P_1\cos\theta_2 - Q_1\sin\theta_2) \\
& + (a_{34}\operatorname{cosec}\alpha_{34}\cos\alpha_{45} + a_{45}\operatorname{cosec}\alpha_{45}\cos\alpha_{34})
\end{aligned}
$$

$$b_1' = 2.\Big[a_{12}\sin\alpha_{34}\sin\alpha_{56}U_{21}\sin\theta_1$$
$$- (a_{45}\cot\alpha_{45}\sin\alpha_{56} - a_{56}\cos\alpha_{56})\sin\alpha_{34}W_{12}$$
$$+ \sin\alpha_{34}\sin\alpha_{56}(S_{11}V_{21} + S_{22}V_{12})$$
$$+ S_{33}\sin\alpha_{34}\sin\alpha_{56}(\sin\alpha_{23}U_{12} + \cos\alpha_{23}V_{12})$$
$$- S_{44}\sin\alpha_{56}(\bar{X}_2\cos\theta_1 - \bar{Y}_2\sin\theta_1)$$
$$+ S_{66}\sin\alpha_{34}\sin\alpha_{56}(\sin\alpha_{61}U_{21} + \cos\alpha_{61}V_{21})\Big]$$

$$b_2' = 2.\Big[- a_{12}\sin\alpha_{56}R_2\sin\theta_1$$
$$+ a_{23}\sin\alpha_{56}[\sin(\alpha_{23} - \alpha_{34})U_{12} + \cos(\alpha_{23} - \alpha_{34})V_{12}]$$
$$+ a_{34}\sin\alpha_{56}\,\mathrm{cosec}\,\alpha_{34}(\bar{X}_2\cos\theta_1 - \bar{Y}_2\sin\theta_1)$$
$$+ (a_{45}\sin\alpha_{56}\cot\alpha_{45} - a_{56}\cos\alpha_{56})(P_2\cos\theta_1 - Q_2\sin\theta_1)$$
$$+ S_{11}\sin\alpha_{56}(P_2\sin\theta_1 + P_2\cos\theta_1)$$
$$- [S_{22}\sin(\alpha_{23} - \alpha_{34}) - S_{33}\sin\alpha_{34}]\sin\alpha_{56}W_{12}$$
$$+ S_{66}\sin\alpha_{56}[\cos\alpha_{61}(P_2\sin\theta_1 + Q_2\cos\theta_1) - \sin\alpha_{61}R_2]\Big]$$

$$b_3' = -2.S_{44}\sin\alpha_{56}(P_2\cos\theta_1 - Q_2\sin\theta_1)$$

$$b_4' = 2.\Big[a_{12}\sin\alpha_{56}\bar{Z}_2\sin\theta_1$$
$$- a_{23}\sin\alpha_{56}(\sin\alpha_{23}U_{12} + \cos\alpha_{23}V_{12})$$
$$- [(a_{34}\cot\alpha_{34} + a_{45}\cot\alpha_{45})\sin\alpha_{56} - a_{56}\cos\alpha_{56}](\bar{X}_2\cos\theta_1 - \bar{Y}_2\sin\theta_1)$$
$$- S_{11}\sin\alpha_{56}(\bar{X}_2\sin\theta_1 + \bar{Y}_2\cos\theta_1)$$
$$+ (S_{22}\sin\alpha_{23} + S_{44}\sin\alpha_{34})\sin\alpha_{56}W_{12}$$
$$- S_{66}\sin\alpha_{56}[\cos\alpha_{61}(\bar{X}_2\sin\theta_1 + \bar{Y}_2\cos\theta_1) - \sin\alpha_{61}\bar{Z}_2]\Big]$$

$$
\begin{aligned}
c_1' = \; & a_{12}\sin\alpha_{34} N_1 \sin\theta_2 \\
& - a_{45}\sin\alpha_{34} \cot\alpha_{45} (L_1\cos\theta_2 - M_1\sin\theta_2) \\
& - (a_{56} + a_{61})\sin\alpha_{34}[\sin(\alpha_{56}+\alpha_{61}) U_{21} + \cos(\alpha_{56}+\alpha_{61}) V_{21}] \\
& + [S_{11}\sin(\alpha_{56}+\alpha_{61}) + S_{66}\sin\alpha_{56}]\sin\alpha_{34} W_{12} \\
& - S_{22}\sin\alpha_{34}(L_1\sin\theta_2 + M_1\cos\theta_2) \\
& - S_{33}\sin\alpha_{34}[\cos\alpha_{23}(L_1\sin\theta_2 + M_1\cos\theta_2) - \sin\alpha_{23} N_1] \\
& - S_{44}[\sin\alpha_{23}(L_1\sin\theta_2 + M_1\cos\theta_2) + \cos\alpha_{23} N_1] \\
& + S_{44}\cos\alpha_{34}\cos\alpha_{45}
\end{aligned}
$$

$$
\begin{aligned}
c_2' = & - a_{12}[\cos(\alpha_{56}+\alpha_{61}) Q_2 - \sin(\alpha_{56}+\alpha_{61}) R_2\cos\theta_1] \\
& - a_{23}[\cos(\alpha_{23}-\alpha_{34})(L_1\sin\theta_2 + M_1\cos\theta_2) - \sin(\alpha_{23}-\alpha_{34}) N_1] \\
& + a_{34}\operatorname{cosec}\alpha_{34}[\sin(\alpha_{56}+\alpha_{61})(\bar{X}_2\sin\theta_1 + \bar{Y}_2\cos\theta_1) + \cos(\alpha_{56}+\alpha_{61})\bar{Z}_2] \\
& + a_{45}\cot\alpha_{45}[\sin(\alpha_{56}+\alpha_{61})(P_2\sin\theta_1 + Q_2\cos\theta_1) + \cos(\alpha_{56}+\alpha_{61}) R_2] \\
& - (a_{56} + a_{61})[\cos(\alpha_{56}+\alpha_{61})(P_2\sin\theta_1 + Q_2\cos\theta_1) - \sin(\alpha_{56}+\alpha_{61}) R_2] \\
& - [S_{11}\sin(\alpha_{56}+\alpha_{61}) + S_{66}\sin\alpha_{56}](P_2\cos\theta_1 - Q_2\sin\theta_1) \\
& - [S_{22}\sin(\alpha_{23}-\alpha_{34}) - S_{33}\sin\alpha_{34}](L_1\cos\theta_2 - M_1\sin\theta_2) \\
& - (a_{34}\cot\alpha_{34}\cos\alpha_{45} + a_{45}\operatorname{cosec}\alpha_{45})
\end{aligned}
$$

$$
c_3' = - S_{44}[\sin(\alpha_{56}+\alpha_{61})(P_2\sin\theta_1 + Q_2\cos\theta_1) + \cos(\alpha_{56}+\alpha_{61}) R_2 - \cos\alpha_{45}]
$$

$$
\begin{aligned}
c_4' = \; & a_{12}[\cos(\alpha_{56}+\alpha_{61})\bar{Y}_2 - \sin(\alpha_{56}+\alpha_{61})\bar{Z}_2\cos\theta_1] \\
& + a_{23}[\cos\alpha_{23}(L_1\sin\theta_2 + M_1\cos\theta_2) - \sin\alpha_{23} N_1] \\
& - (a_{34}\cot\alpha_{34} + a_{45}\cot\alpha_{45})[\sin\alpha_{23}(L_1\sin\theta_2 + M_1\cos\theta_2) + \cos\alpha_{23} N_1] \\
& + (a_{56} + a_{61})[\cos(\alpha_{56}+\alpha_{61})(\bar{X}_2\sin\theta_1 + \bar{Y}_2\cos\theta_1) - \sin(\alpha_{56}+\alpha_{61})\bar{Z}_2] \\
& + [S_{11}\sin(\alpha_{56}+\alpha_{61}) + S_{66}\sin\alpha_{56}](\bar{X}_2\cos\theta_1 - \bar{Y}_2\sin\theta_1) \\
& + (S_{22}\sin\alpha_{23} + S_{44}\sin\alpha_{34})(L_1\cos\theta_2 - M_1\sin\theta_2) \\
& + (a_{34}\operatorname{cosec}\alpha_{34}\cos\alpha_{45} + a_{45}\operatorname{cosec}\alpha_{45}\cos\alpha_{34})
\end{aligned}
$$

$$
\begin{aligned}
d'_1 = \; & a_{12}[\cos(\alpha_{56} - \alpha_{61}) M_2 + \sin(\alpha_{56} - \alpha_{61}) N_2 \cos\theta_1] \\
& - a_{23}[\cos(\alpha_{23} + \alpha_{34})(P_1 \sin\theta_2 + Q_1 \cos\theta_2) + \sin(\alpha_{23} + \alpha_{34}) R_1] \\
& + a_{34} \operatorname{cosec}\alpha_{34} [\sin(\alpha_{56} - \alpha_{61})(\bar{X}_2 \sin\theta_1 + \bar{Y}_2 \cos\theta_1) - \cos(\alpha_{56} - \alpha_{61}) \bar{Z}_2] \\
& + a_{45} \cot\alpha_{45} [\sin(\alpha_{56} - \alpha_{61})(L_2 \sin\theta_1 + M_2 \cos\theta_1) - \cos(\alpha_{56} - \alpha_{61}) N_2] \\
& - (a_{56} - a_{61}) [\cos(\alpha_{56} - \alpha_{61})(L_2 \sin\theta_1 + M_2 \cos\theta_1) + \sin(\alpha_{56} - \alpha_{61}) N_2] \\
& - [S_{11} \sin(\alpha_{56} - \alpha_{61}) + S_{66} \sin\alpha_{56}] (L_2 \cos\theta_1 - M_2 \sin\theta_1) \\
& - [S_{22} \sin(\alpha_{23} + \alpha_{34}) + S_{33} \sin\alpha_{34}] (P_1 \cos\theta_2 - Q_1 \sin\theta_2) \\
& + (a_{34} \cot\alpha_{34} \cos\alpha_{45} + a_{45} \operatorname{cosec}\alpha_{45})
\end{aligned}
$$

$$
\begin{aligned}
d'_2 = & - a_{12} \sin\alpha_{34} R_1 \sin\theta_2 \\
& - a_{45} \sin\alpha_{34} \cot\alpha_{45} (P_1 \cos\theta_2 - Q_1 \sin\theta_2) \\
& + (a_{56} - a_{61}) \sin\alpha_{34} [\sin(\alpha_{56} - \alpha_{61}) U_{21} - \cos(\alpha_{56} - \alpha_{61}) V_{21}] \\
& + [S_{11} \sin(\alpha_{56} - \alpha_{61}) + S_{66} \sin\alpha_{56}] \sin\alpha_{34} W_{12} \\
& - S_{22} \sin\alpha_{34} (P_1 \sin\theta_2 + Q_1 \cos\theta_2) \\
& - S_{33} \sin\alpha_{34} [\cos\alpha_{23} (P_1 \sin\theta_2 + Q_1 \cos\theta_2) + \sin\alpha_{23} R_1] \\
& + S_{44} [\sin\alpha_{23} (P_1 \sin\theta_2 + Q_1 \cos\theta_2) - \cos\alpha_{23} R_1] \\
& + S_{44} (\cos\alpha_{34} \cos\alpha_{45})
\end{aligned}
$$

$$
\begin{aligned}
d'_3 = \; & a_{12}[\cos(\alpha_{56} - \alpha_{61}) \bar{Y}_2 + \sin(\alpha_{56} - \alpha_{61}) \bar{Z}_2 \cos\theta_1] \\
& - a_{23}[\cos\alpha_{23} (P_1 \sin\theta_2 + Q_1 \cos\theta_2) + \sin\alpha_{23} R_1] \\
& + (a_{34} \cot\alpha_{34} + a_{45} \cot\alpha_{45}) [\sin\alpha_{23} (P_1 \sin\theta_2 + Q_1 \cos\theta_2) - \cos\alpha_{23} R_1] \\
& - (a_{56} - a_{61}) [\cos(\alpha_{56} - \alpha_{61})(\bar{X}_2 \sin\theta_1 + \bar{Y}_2 \cos\theta_1) + \sin(\alpha_{56} - \alpha_{61}) \bar{Z}_2] \\
& - [S_{11} \sin(\alpha_{56} - \alpha_{61}) + S_{66} \sin\alpha_{56}] (\bar{X}_2 \cos\theta_1 - \bar{Y}_2 \sin\theta_1) \\
& - (S_{22} \sin\alpha_{23} - S_{44} \sin\alpha_{34})(P_1 \cos\theta_2 - Q_1 \sin\theta_2) \\
& + (a_{34} \operatorname{cosec}\alpha_{34} \cos\alpha_{45} + a_{45} \operatorname{cosec}\alpha_{45} \cos\alpha_{34})
\end{aligned}
$$

$$
d'_4 = - S_{44}[\sin(\alpha_{56} - \alpha_{61})(L_2 \sin\theta_1 + M_2 \cos\theta_1) - \cos(\alpha_{56} - \alpha_{61}) N_2 + \cos\alpha_{45}]
$$

$$e'_1 = 2.\Big[a_{12}\sin\alpha_{56}\,N_2\sin\theta_1$$
$$- a_{23}\sin\alpha_{56}\left[\sin(\alpha_{23}+\alpha_{34})\,U_{12} + \cos(\alpha_{23}+\alpha_{34})\,V_{12}\right]$$
$$- a_{34}\sin\alpha_{56}\,\mathrm{cosec}\,\alpha_{34}\,(\bar{X}_2\cos\theta_1 - \bar{Y}_2\sin\theta_1)$$
$$- (a_{45}\sin\alpha_{56}\cot\alpha_{45} - a_{56}\cos\alpha_{56})\,(L_2\cos\theta_1 - M_2\sin\theta_1)$$
$$- S_{11}\sin\alpha_{56}\,(L_2\sin\theta_1 + M_2\cos\theta_1)$$
$$+ \left[S_{22}\sin(\alpha_{23}+\alpha_{34}) + S_{33}\sin\alpha_{34}\right]\sin\alpha_{56}\,W_{12}$$
$$- S_{66}\sin\alpha_{56}\left[\cos\alpha_{61}\,(L_2\sin\theta_1 + M_2\cos\theta_1) - \sin\alpha_{61}\,N_2\right]\Big]$$

$$e'_2 = 2.\Big[- a_{12}\sin\alpha_{34}\,\sin\alpha_{56}\,U_{21}\sin\theta_1$$
$$+ (a_{45}\cot\alpha_{45}\,\sin\alpha_{56} - a_{56}\cos\alpha_{56})\,\sin\alpha_{34}\,W_{12}$$
$$- \sin\alpha_{34}\,\sin\alpha_{56}\,(S_{11}V_{21} + S_{22}V_{12})$$
$$- S_{33}\sin\alpha_{34}\,\sin\alpha_{56}\,(\sin\alpha_{23}\,U_{12} + \cos\alpha_{23}\,V_{12})$$
$$- S_{44}\sin\alpha_{56}\,(\bar{X}_2\cos\theta_1 - \bar{Y}_2\sin\theta_1)$$
$$- S_{66}\sin\alpha_{34}\,\sin\alpha_{56}\,(\sin\alpha_{61}\,U_{21} + \cos\alpha_{61}\,V_{21})\Big]$$

$$e'_3 = 2.\Big[a_{12}\sin\alpha_{56}\,\bar{Z}_2\sin\theta_1$$
$$- a_{23}\sin\alpha_{56}\,(\sin\alpha_{23}\,U_{12} + \cos\alpha_{23}\,V_{12})$$
$$- \left[(a_{34}\cot\alpha_{34} + a_{45}\cot\alpha_{45})\,\sin\alpha_{56} - a_{56}\cos\alpha_{56}\right]\,(\bar{X}_2\cos\theta_1 - \bar{Y}_2\sin\theta_1)$$
$$- S_{11}\sin\alpha_{56}\,(\bar{X}_2\sin\theta_1 + \bar{Y}_2\cos\theta_1)$$
$$+ (S_{22}\sin\alpha_{23} - S_{44}\sin\alpha_{34})\,\sin\alpha_{56}\,W_{12}$$
$$- S_{66}\sin\alpha_{56}\left[\cos\alpha_{61}\,(\bar{X}_2\sin\theta_1 + \bar{Y}_2\cos\theta_1) - \sin\alpha_{61}\,\bar{Z}_2\right]\Big]$$

$$e'_4 = 2.S_{44}\sin\alpha_{56}\,(L_2\cos\theta_1 - M_2\sin\theta_1)$$

$$
\begin{aligned}
f_1' = {} & a_{12}[\cos(\alpha_{56}+\alpha_{61})M_2 - \sin(\alpha_{56}+\alpha_{61})N_2\cos\theta_1] \\
& + a_{23}[\cos(\alpha_{23}+\alpha_{34})(L_1\sin\theta_2 + M_1\cos\theta_2) - \sin(\alpha_{23}+\alpha_{34})N_1] \\
& - a_{34}\operatorname{cosec}\alpha_{34}[\sin(\alpha_{56}+\alpha_{61})(\bar{X}_2\sin\theta_1 + \bar{Y}_2\cos\theta_1) + \cos(\alpha_{56}+\alpha_{61})\bar{Z}_2] \\
& - a_{45}\cot\alpha_{45}[\sin(\alpha_{56}+\alpha_{61})(L_2\sin\theta_1 + M_2\cos\theta_1) + \cos(\alpha_{56}+\alpha_{61})N_2] \\
& + (a_{56} + a_{61})[\cos(\alpha_{56}+\alpha_{61})(L_2\sin\theta_1 + M_2\cos\theta_1) - \sin(\alpha_{56}+\alpha_{61})N_2] \\
& + [S_{11}\sin(\alpha_{56}+\alpha_{61}) + S_{66}\sin\alpha_{56}](L_2\cos\theta_1 - M_2\sin\theta_1) \\
& + [S_{22}\sin(\alpha_{23}+\alpha_{34}) + S_{33}\sin\alpha_{34}](L_1\cos\theta_2 - M_1\sin\theta_2) \\
& + (a_{34}\cot\alpha_{34}\cos\alpha_{45} + a_{45}\operatorname{cosec}\alpha_{45})
\end{aligned}
$$

$$
\begin{aligned}
f_2' = {} & - a_{12}\sin\alpha_{34}N_1\sin\theta_2 \\
& + a_{45}\sin\alpha_{34}\cot\alpha_{45}(L_1\cos\theta_2 - M_1\sin\theta_2) \\
& + (a_{56} + a_{61})\sin\alpha_{34}[\sin(\alpha_{56}+\alpha_{61})U_{21} + \cos(\alpha_{56}+\alpha_{61})V_{21}] \\
& - [S_{11}\sin(\alpha_{56}+\alpha_{61}) + S_{66}\sin\alpha_{56}]\sin\alpha_{34}W_{12} \\
& + S_{22}\sin\alpha_{34}(L_1\sin\theta_2 + M_1\cos\theta_2) \\
& + S_{33}\sin\alpha_{34}[\cos\alpha_{23}(L_1\sin\theta_2 + M_1\cos\theta_2) - \sin\alpha_{23}N_1] \\
& - S_{44}[\sin\alpha_{23}(L_1\sin\theta_2 + M_1\cos\theta_2) + \cos\alpha_{23}N_1] \\
& + S_{44}\cos\alpha_{34}\cos\alpha_{45}
\end{aligned}
$$

$$
\begin{aligned}
f_3' = {} & a_{12}[\cos(\alpha_{56}+\alpha_{61})\bar{Y}_2 - \sin(\alpha_{56}+\alpha_{61})\bar{Z}_2\cos\theta_1] \\
& + a_{23}[\cos\alpha_{23}(L_1\sin\theta_2 + M_1\cos\theta_2) - \sin\alpha_{23}N_1] \\
& - (a_{34}\cot\alpha_{34} + a_{45}\cot\alpha_{45})[\sin\alpha_{23}(L_1\sin\theta_2 + M_1\cos\theta_2) + \cos\alpha_{23}N_1] \\
& + (a_{56} + a_{61})[\cos(\alpha_{56}+\alpha_{61})(\bar{X}_2\sin\theta_1 + \bar{Y}_2\cos\theta_1) - \sin(\alpha_{56}+\alpha_{61})\bar{Z}_2] \\
& + [S_{11}\sin(\alpha_{56}+\alpha_{61}) + S_{66}\sin\alpha_{56}](\bar{X}_2\cos\theta_1 - \bar{Y}_2\sin\theta_1) \\
& + (S_{22}\sin\alpha_{23} - S_{44}\sin\alpha_{34})(L_1\cos\theta_2 - M_1\sin\theta_2) \\
& + (a_{34}\operatorname{cosec}\alpha_{34}\cos\alpha_{45} + a_{45}\operatorname{cosec}\alpha_{45}\cos\alpha_{34})
\end{aligned}
$$

$$
f_4' = S_{44}[\sin(\alpha_{56}+\alpha_{61})(L_2\sin\theta_1 + M_2\cos\theta_1) + \cos(\alpha_{56}+\alpha_{61})N_2 - \cos\alpha_{45}]
$$

C. Terms Used in the Above Coefficients.

In the above coefficients, $a_i, \ldots, f_i$, $a_i', \ldots, f_i'$ $(i = 1, \ldots, 4)$ the terms P_1, P_2, etc., are defined as follows:-

$$P_1 = \sin(\alpha_{56} - \alpha_{61}) \sin\theta_1$$
$$Q_1 = [\cos(\alpha_{56} - \alpha_{61}) \sin\alpha_{12} - \sin(\alpha_{56} - \alpha_{61}) \cos\alpha_{12} \cos\theta_1]$$
$$R_1 = [\cos(\alpha_{56} - \alpha_{61}) \cos\alpha_{12} + \sin(\alpha_{56} - \alpha_{61}) \sin\alpha_{12} \cos\theta_1]$$

$$P_2 = \sin(\alpha_{23} - \alpha_{34}) \sin\theta_2$$
$$Q_2 = -[\cos(\alpha_{23} - \alpha_{34}) \sin\alpha_{12} + \sin(\alpha_{23} - \alpha_{34}) \cos\alpha_{12} \cos\theta_2]$$
$$R_2 = [\cos(\alpha_{23} - \alpha_{34}) \cos\alpha_{12} - \sin(\alpha_{23} - \alpha_{34}) \sin\alpha_{12} \cos\theta_2]$$

$$L_1 = \sin(\alpha_{56} + \alpha_{61}) \sin\theta_1$$
$$M_1 = -[\cos(\alpha_{56} + \alpha_{61}) \sin\alpha_{12} + \sin(\alpha_{56} + \alpha_{61}) \cos\alpha_{12} \cos\theta_1]$$
$$N_1 = [\cos(\alpha_{56} + \alpha_{61}) \cos\alpha_{12} - \sin(\alpha_{56} + \alpha_{61}) \sin\alpha_{12} \cos\theta_1]$$

$$L_2 = \sin(\alpha_{23} + \alpha_{34}) \sin\theta_2$$
$$M_2 = -[\cos(\alpha_{23} + \alpha_{34}) \sin\alpha_{12} + \sin(\alpha_{23} + \alpha_{34}) \cos\alpha_{12} \cos\theta_2]$$
$$N_2 = [\cos(\alpha_{23} + \alpha_{34}) \cos\alpha_{12} - \sin(\alpha_{23} + \alpha_{34}) \sin\alpha_{12} \cos\theta_2]$$

$$U_{12} = \sin\alpha_{12} \sin\theta_1$$
$$V_{12} = -(\cos\theta_1 \sin\theta_2 + \sin\theta_1 \cos\theta_2 \cos\alpha_{12})$$
$$W_{12} = (\cos\theta_1 \cos\theta_2 - \sin\theta_1 \sin\theta_2 \cos\alpha_{12})$$

$$U_{21} = \sin\alpha_{12} \sin\theta_2$$
$$V_{21} = -(\cos\theta_2 \sin\theta_1 + \sin\theta_2 \cos\theta_1 \cos\alpha_{12})$$

$$\bar{X}_2 = \sin\alpha_{23} \sin\theta_2$$
$$\bar{Y}_2 = -(\cos\alpha_{23} \sin\alpha_{12} + \sin\alpha_{23} \cos\alpha_{12} \cos\theta_2)$$
$$\bar{Z}_2 = (\cos\alpha_{23} \cos\alpha_{12} - \sin\alpha_{23} \sin\alpha_{12} \cos\theta_2)$$

$$X_{0612} = (X_{061}\cos\theta_2 - Y_{061}\sin\theta_2) - S_{22}(X_{61}\sin\theta_2 + Y_{61}\cos\theta_2)$$

$$Y_{0612} = - a_{23}Z_{612} + S_{22}\cos\alpha_{23}X_{612}$$
$$+ [\cos\alpha_{23}(X_{061}\sin\theta_2 + Y_{061}\cos\theta_2) - \sin\alpha_{23}Z_{061}]$$

$$Z_{0612} = a_{23}Y_{612} + S_{22}\sin\alpha_{23}X_{612}$$
$$+ [\sin\alpha_{23}(X_{061}\sin\theta_2 + Y_{061}\cos\theta_2) + \cos\alpha_{23}Z_{061}]$$

$$X_{061} = (X_{06}\cos\theta_1 - Y_{06}\sin\theta_1) - S_{11}(X_6\sin\theta_1 + Y_6\cos\theta_1)$$

$$Y_{061} = - a_{12}Z_{61} + S_{11}\cos\alpha_{12}X_{61}$$
$$+ [\cos\alpha_{12}(X_{06}\sin\theta_1 + Y_{06}\cos\theta_1) - \sin\alpha_{12}Z_{06}]$$

$$Z_{061} = a_{12}Y_{61} + S_{11}\sin\alpha_{12}X_{61}$$
$$+ [\sin\alpha_{12}(X_{06}\sin\theta_1 + Y_{06}\cos\theta_1) + \cos\alpha_{12}Z_{06}]$$

$$X_{06} = a_{56}\cos\alpha_{56}\sin\theta_6 + S_{66}\sin\alpha_{56}\cos\theta_6$$

$$Y_{06} = -a_{61}Z_6 + S_{66}\cos\alpha_{61}X_6$$
$$+a_{56}(\sin\alpha_{61}\sin\alpha_{56} - \cos\alpha_{61}\cos\alpha_{56}\cos\theta_6)$$

$$Z_{06} = a_{61}Y_6 + S_{66}\sin\alpha_{61}X_6$$
$$-a_{56}(\cos\alpha_{61}\sin\alpha_{56} + \sin\alpha_{61}\cos\alpha_{56}\cos\theta_6)$$

where the terms X_{612}, Y_{612}, Z_{612}, X_{61}, Y_{61}, Z_{61}, X_6, Y_6 and Z_6 are defined in Appendix III (see also Chapter 4).

REFERENCES

1. ARCHBOLD, J. W., "Algebra", Sir Isaac Pitman and Sons Ltd., London, 1970.

2. BÔCHER, M., "Introduction to Higher Algebra", Dover Publications, Inc., New York, 1964.

3. BRAND, L., "Vector Analysis", J. Wiley and Sons, Inc., New York, 1957.

4. BRAND, L., "Vector and Tensor Analysis", J. Wiley and Sons, Inc., New York, 1947.

5. CLIFFORD, W. K., "Preliminary Sketch of Bi-Quaternions", Proc. Lon. Math. Soc., v. 4, Nos. 64, 65, 1873.

6. DENAVIT, J., "Displacement Analysis of Mechanisms Based on (2 x 2) Matrices of Dual Numbers", V. D. I. Berichte, v. 29, 1958.

7. DIMENTBERG, F. M., "A General Method for the Investigation of Finite Displacements of Spatial Mechanisms and Certain Cases of Passive Constraints", (Russian) Purdue Translation No. 436, Purdue University Libraries.

8. DOBROVOLSKI, V. V., "Methods of Spherical Representation in the Theory of Spatial Mechanisms", (Russian). Trudi Seminara po Teorii Mashin i Mechanizmov, SSSR, v. 3, No. 11, 1947.

9. DUFFY, J., "A Derivation of Dual Displacement Equations for Five, Six and Seven-Link Spatial Mechanisms Using Spherical Trigonometry", Submitted to Roumanian Journal of Technical Sciences.

10. DUFFY, J., "An Analysis of Five, Six and Seven-Link Spatial Mechanisms", Papers C-7, C-8, Proceedings of Third World Congress, IFToMM, Kupari, Yugoslavia, 1971.

11. DUFFY, J., and HABIB-OLAHI, H. Y., "A Displacement Analysis of Spatial Five-Link 3R-2C Mechanisms - 1. On the Closures of the RCRCR Mechanism", Journal of Mechanisms, vol. 6, 1971.

12. DUFFY, J., and HABIB-OLAHI, H. Y., "A Displacement Analysis of Spatial Five-Link 3R-2C Mechanisms - 2. Analysis of the RRCRC Mechanism", Journal of Mechanisms, vol. 6, 1971.

13. DUFFY, J., and HABIB-OLAHI, H. Y., "A Displacement Analysis of Spatial Five-Link 3R-2C Mechanisms - 3. Analysis of the RCRRC Mechanism", Journal of Mechanisms, vol. 7, 1972.

14. DUFFY, J., and HABIB-OLAHI, H. Y., "Displacement Analysis of Spatial Five-Link Mechanisms - 1. Analysis of RRCRC Mechanism, - 2. Analysis of RRRCC Mechanism", To appear in Feinwertechnik, 1972.

15. DUFFY, J., and KEEN, D., "A Displacement Analysis of the Spatial Five-Link RERRR Mechanism", Journal of Mechanisms, vol. 7, 1972, pp 211-230.

16. DUKKIPATI, R.V., and SONI, A. H., "Closed Form Displacement Relationships of Single and Multi-Loop Six-Link Spatial Mechanisms", Paper No. 72-Mech-50 12th ASME Mechanism Conference, San Francisco, U.S.A., October, 1972.

17. DUKKIPATI, R. V., and SONI, A. H., "Closed Form Displacement Analysis of the RRRCRR Six-Link Spatial Mechanism", Paper No. B-17, IFToMM International Symposium on Linkages and Computer Design Methods, Bucharest, Romania, June 7-13, 1973.

18. GRÜBLER, M., "Getriebelehre", Springer-Verlag, Berlin, 1917.

19. HABIB-OLAHI, H. Y., "A Displacement Analysis of Five-Link Spatial Mechanisms", Doctoral Dissertation, Liverpool Polytechnic, Liverpool, 1973.

20. HERSTEIN, I. N., "Topics in Algebra", Blaisdell Publishing Company, 1964.

21. HUNT, K. H., "Screw Systems in Spatial Kinematics", Report MMERS 3, Monash University, Victoria, Australia, June, 1970.

22. KEEN, D. R., "A Unified Theory for the Analysis of Spatial Mechanisms Using a Vector-Loop Method", Doctoral Dissertation, Liverpool Polytechnic, To be submitted early 1974.

23. KEEN, D., and DUFFY, J., "Displacement Analysis of Spatial Five-Link RSRRR Mechanism", Presented at Roumanian Conference 1973.

24. KELER, M., "Die Verwendung dual-komplexer Grössen in Geometrie, Kinematik und Mechanik", Feinwerktechnik, vol. 74, No. 1, 1970.

25. KELER, M., "Analyse und Synthese räumlicher, sphärischer und ebener Getriebe in dual-komplexer Darstellung", Feinwerktechnic, vol. 74, No. 8, 1970.

26. KOTELNIKOV, A. P., "Screw Calculus and Some Applications to Geometry and Mechanics", Annals of the Imperial University of Kazan, 1895.

27. KUTZBACH, K., "Bewegliche Verbindungen", Z. V. D. I., v. 77, 1933.

28. MUIR, T., "A Treatise on the Theory of Determinants", Dover Publications, Inc., New York, 1960.

29. OPEN UNIVERSITY, "Linear Algebra I", Maths Foundation Course Unit 22, The Open University Press, 1971.

30. REULEAUX, F., "The Kinematics of Machinery", (Translated by A. B. W. Kennedy). Dover Publications Inc., New York, 1963.

31. SALMON, G., "Lessons Introductory to the Modern Higher Algebra", Third Edition, Cambridge University Press, 1876.

32. SEMPLE, J. G., and ROTH, L., "Introduction to Algebraic Geometry", Clarendon Press, Oxford, 1949.

33. SONI, A. H., and PAMIDI, P. R., "Closed Form Relations of a Five-Link RRCCR Spatial Mechanism", Journal of Engineering for Industry, Transactions of ASME, vol. 93, Series B, 1971.

34. SPIEGEL, M. R., "Theory and Problems of Complex Variables", Schaum's Outline Series, McGraw-Hill Book Company, 1964.

35. STUDY, E., "Geometrie der Dynamen", Verlag Teubner, Leipzig, 1901.

36. TODHUNTER, I., and LEATHEM, J. G., "Spherical Trigonometry", Macmillan and Co., Limited, London, 1932.

37. TORFASON, L. E., and CROSSLEY, F. R. E., "The Intersection of Solids Shown by Electronic Analog for Mechanism Simulation", Trans. ASME, Jl. Engineering for Industry, February, 1971.

38. TURNBULL, H. W., "Theory of Equations", University Mathematical Texts. Oliver and Boyd Ltd.

39. WALDRON, K. J., "The Constraint Analysis of Mechanisms", J. Mechanisms, v. 1, No. 2, 1966, pp. 101-114.

40. WALLACE, D. M., "Displacement Analysis of Spatial Mechanisms with More than Four Links", Doctoral Dissertation, Columbia University, New York, 1968. University Microfilms, Ann Arbor, Mich.

41. WALLACE, D. M., and FREUDENSTEIN, F., "The Displacement Analysis of the Generalised Tracta Coupling", Journal of Applied Mechanics, vol. 37, Trans. ASME, vol. 92, Series E, 1970.

42. WOO, L., and FREUDENSTEIN, F., "Application of Line Geometry to Theoretical Kinematics and the Kinematic Analysis of Mechanical Systems", IBM Corporation, New York Scientific Center Technical Report No. 320-2982, 1969.

43. WÖRLE, H., "Getriebeanalytische und Getriebesynthetische unterlagen für den Entwurf Zwanglaufiger und Ubergeschlossener Viergelenkiger Raumkurbelgetriebe", Buletinul Institutului Politechnic Din Iasi, Serie Nouă, vol. 6, No. 10, 1960.

44. YANG, A. T., "Application of Quaternion Algebra and Dual Numbers to the Analysis of Spatial Mechanisms", Doctoral Dissertation, Columbia University, New York, 1963, University Microfilms, Ann Arbor, Michigan.

45. YANG, A. T., "Displacement Analysis of Spatial Five-Link Mechanisms Using (3 x 3) Matrices with Dual Number Elements", Journal of Engineering for Industry, Transactions of ASME, vol. 91, Series B, 1969.

46. YANG, A. T., and FREUDENSTEIN, F., "Application of Dual Quaternion Algebra to the Analysis of Spatial Mechanisms", Journal of Applied Mechanics, Transactions of ASME, vol. 86E, 1964.

47. YUAN, M. S. C., "Displacement Analysis of the RCRCR Five-Link Spatial Mechanism", Journal of Mechanisms, vol. 6, 1971.

48. YUAN, M. S. C., "Displacement Analysis of the RRCCR Five-Link Spatial Mechanism", Journal of Applied Mechanics, Transactions of ASME, vol. 37, Series E, 1970.

49. YUAN, M. S. C., "Displacement Analysis of the RPRCRR Mechanism", Journal of Applied Mechanics, Transactions of ASME, vol. 38, Series E, 1971.

ADDITIONAL REFERENCE.

50. GILMARTIN, M. J., "An Investigation of the Motion of Spatial Mechanisms", Doctoral Dissertation, Liverpool Polytechnic, Liverpool, 1971.