DXN074791

THE USE OF COMPUTER TECHNOLOGY AND CONSTRUCTIVISM TO ENHANCE VISUALISATION SKILLS IN MATHEMATICS EDUCATION

IAN MALABAR

A thesis submitted in partial fulfilment of the requirements of Liverpool John Moores University for the degree of Doctor of Philosophy

August 2003

BEST COPY

AVAILABLE

Variable print quality

Acknowledgements

The author would like to thank the following people:

Dr David Pountney: Special thanks go to my Director of Studies, not only for providing excellent supervision throughout this research, but for offering constant motivation, encouragement, good humour, and friendship.

Mr Stewart Townend and Professor John Berry (University of Plymouth): Many thanks go to my supervisors, for their valuable advice and guidance in terms of content and direction.

Without the three members of my supervisory team, completion of this work would not have been possible.

Mr Ken McKelvie and Ms Mary Rouncefield: Various lengthy discussions regarding the statistical analysis of the data were interesting, enjoyable, and proved to be a source of enlightenment.

Mr Richard Davies, Mr John Wohlers, and Ms Linda Burthem: I am indebted to the Heads of Mathematics at The Liverpool Blue Coat School, King George V College, and West Kirby Girls Grammar School respectively, who were helpful, understanding, and accommodating throughout the project. Without the co-operation of the three schools and their students during the case-study, the key aims of the research could not have been satisfied.

Many thanks also go to my wife, Amanda, and my mother, Anne, for their patience, constant encouragement, and support.

Abstract

Computer technology continues to provide alternative, innovative approaches to the teaching and learning of mathematics at all levels, and this is encouraging the promotion of 'learning by doing', where interactivity replaces passivity. This research endeavours to show how/why constructivism (building knowledge through exploration) is a preferable methodology to instructivism (passive information transfer) when considering the effective use of technology to enhance visualisation skills.

Students can generally demonstrate the ability to follow routines, but many find it difficult to visualise in mathematics. A key aim of this research is to evaluate enhanced student learning of mathematical concepts and to assess the extent of any skills development, via the constructivist use of computer-based visualisation.

After a review and examination of the effectiveness of previous work, consideration is given to the best way to employ constructivism in teaching and learning with visualisation. This leads to the design of a piece of interactive software that motivates students to explore the relationship between visual and symbolic functional forms, and promotes flexible switching between representations. A case-study provides empirical evidence that quantifies the benefits of an approach integrating both constructivism and visualisation in terms of the development of visualisation and other higher order mathematical skills.

As a direct result of the positive and practical outcomes of the case-study, a generic theoretical framework is formulated, which builds on existing theories of teaching and learning. The framework is applied to the process of problem solving and links subject and skills development together. Future mathematics education developments as a result of the findings of this thesis are suggested, both specific to the case-study and for mathematics education in general.

Contents

CHAPTER 1

VISUALISATION IN MATHEMATICS EDUCATION:

THE ISSUES 1

1.1. Introduction 2

1.2. Problem Definition 5

1.3. Factors Involved 6

1.3.1. Constructivism: Teaching and Learning Styles 6

1.3.2. Students' Skills 9

1.3.2.1. Fundamental Problems 9

1.3.2.2. 'More Able' and 'Less Able' Students 12

1.3.3. Visualisation and Conceptual Understanding 14

1.3.3.1. Enhancement of Visualisation Skills 15

1.3.3.2. Improved Understanding of Concepts 17

1.3.3.3. Evaluation of Enhanced Learning 18

1.3.4. Software Design Issues Incorporating Imagery 19

1.3.4.1. Cognitive Technologies 20

1.3.4.2. Advantages of Multimedia 21

1.3.4.3. Future Developments 22

1.4. Aims of the Research 24

1.4.1. Key Aims 25

1.4.2. Secondary Aims 26

1.5. Summary 27

CHAPTER 2

VISUALISATION IN MATHEMATICS EDUCATION:

A REVIEW OF PREVIOUS WORK 29

2.1. Introduction 30

2.2. Constructivism: Teaching and Learning Styles 31

2.2.1. A Constructivist Approach to Teaching and Learning 31

2.2.2. Classroom Experiences with Technology 37

2.3. Students' Skills 41

2.3.1. Related Skills Taxonomies 41
2.3.1.1. NCVQ (National Council for Vocational Qualifications) 42
2.3.1.2. Mathskills Discipline Network 43
2.3.1.3. MATH Taxonomy 45
2.3.1.4. Skills Studies by Galbraith and Haines 46
2.3.2. Students' Difficulties in Developing Visualisation Skills 47
2.4. Visualisation and Conceptual Understanding 56
2.4.1. Visualisation to Enhance Mental Image Making 56
2.4.2. Creating Mental Images for Students 65
2.4.3. Theories of Teaching and Learning in Mathematics 73
2.4.3.1. White-Box/Black-Box (WBBB) Principle [Buchberger] 73
2.4.3.2. Communication Model [Laurillard] 75
2.4.3.3. Construction of Conceptual Understanding [Hitt] 75
2.4.3.4. Mental Processes [Dreyfus] 76
2.4.3.5. Concept Definition and Concept Image [Vinner] 78
2.4.3.6. Conceptual Entities [Harel and Kaput] 83
2.4.3.7. Reflective Abstraction [Dubinsky] 84
2.4.3.8. Relational and Instrumental Understanding [Skemp] 86
2.5. Software Design Issues Incorporating Imagery 87
2.5.1. Increasing Motivation 87
2.5.1.1. An Alternative Approach to Learning 88
2.5.1.2. Making Learning Fun 90
2.5.1.3. Maintaining Interest 94
2.5.2. Increasing Retention Level 96
2.5.3. Case-Study of Functions and Graphs 98
2.6. Summary 101

CHAPTER 3

CONSTRUCTIVISM: A THEORY OF LEARNING 107

3.1. Introduction 108
3.2. Instructivism versus Constructivism 109
3.3. Piagetian and Vygotskian Perspectives on Constructivism 122
3.4. Constructivism in Relation to Educational Technology 132
3.5. Summary 137

CHAPTER 4

MATHEMATICAL SKILLS AND THE ROLE OF VISUALISATION......140

4.1. Introduction 141
4.2. DEVISe Skills Classification 142
4.3. Linkages Between the Taxonomies 143
4.4. Visualisation as a Developing Skill 143
4.5. Visualisation to Enhance Skills 145
4.6. Preliminary Investigation of Mathematical Skills in a Computer-Based Environment 151
4.6.1. Background to the Investigation 152
4.6.2. Testing Mathematical Knowledge and Skills 153
4.6.3. Skills Investigation 154
4.6.4. Outcomes 156
4.6.5. Conclusions 164
4.6.6. Discussion 166
4.7. Summary 169

CHAPTER 5

STUDENTS' ABILITY TO VISUALISE: A CASE-STUDY 171

5.1. Introduction 172
5.2. Aims 174
5.2.1. Key Aims 174
5.2.2. Secondary Aims 174
5.3. Methods 175
5.3.1. Rationale for the Chosen Topic 175
5.3.3. The Interactive Software 178
5.3.3.1. Factors Affecting the Design of the Software 181
5.3.3.2. Theoretical Rationale for the Design of the Software 184
5.3.3.3. Dialogue to Illustrate the Learning Process 191
5.3.4. Evaluation Process 203
5.3.4.1. Experimental Design 204
5.3.4.2. Test Questions 205
5.4. Results 216
5.4.1. Statistical Analysis of the Data 217
5.4.2. Student Feedback 235

5.5. Conclusions 238
5.6. Summary 242

CHAPTER 6
VISUALISATION AND CONSTRUCTIVISM:
THEIR ROLES IN MATHEMATICAL COGNITION 247

6.1. Introduction 248
6.2. The Role of Visualisation in Mathematical Cognition 249
6.3. Problem Solving and the Role of Visualisation 251
6.4. Mathematical Skills and Visualisation Revisited 256
6.5. Visualisation Skills and Constructivism 260
6.6. Teaching, Learning and Assessment 'Blockage' Issues 262
 6.6.1. Teaching 262
 6.6.2. Learning 263
 6.6.3. Assessment 265
 6.6.4. The Role of Computer Technology 266
6.7. Other Opportunities for Visual Skills Development 267
6.8. Summary 269

CHAPTER 7
FUTURE DEVELOPMENTS 270

7.1. Introduction 271
7.2. Further Developments Concerning the Case-Study 271
 7.2.1. Software Enhancements 272
 7.2.2. Issues Concerning the Experimental Design 283
 7.2.3. Experimental Techniques 284
 7.2.4. Alternative Topics for Treatment 286
 7.2.5. Measurement of Retention Level 287
 7.2.6. Summary of Issues Concerning a Constructivist Approach 288
7.3. More General Future Mathematics Education Research 289
 7.3.1. Holistic and Serialistic Learning Styles 289
 7.3.2. Mental Imagery 291
 7.3.3. Integration into the School Curriculum 293
  7.3.3.1. Teaching and Learning Issues 293

5.5. Conclusions 238
5.6. Summary 242

CHAPTER 6
VISUALISATION AND CONSTRUCTIVISM:
THEIR ROLES IN MATHEMATICAL COGNITION 247

6.1. Introduction 248
6.2. The Role of Visualisation in Mathematical Cognition 249
6.3. Problem Solving and the Role of Visualisation 251
6.4. Mathematical Skills and Visualisation Revisited 256
6.5. Visualisation Skills and Constructivism 260
6.6. Teaching, Learning and Assessment 'Blockage' Issues 262
6.6.1. Teaching 262
6.6.2. Learning 263
6.6.3. Assessment 265
6.6.4. The Role of Computer Technology 266
6.7. Other Opportunities for Visual Skills Development 267
6.8. Summary 269

CHAPTER 7
FUTURE DEVELOPMENTS 270

7.1. Introduction 271
7.2. Further Developments Concerning the Case-Study 271
7.2.1. Software Enhancements 272
7.2.2. Issues Concerning the Experimental Design 283
7.2.3. Experimental Techniques 284
7.2.4. Alternative Topics for Treatment 286
7.2.5. Measurement of Retention Level 287
7.2.6. Summary of Issues Concerning a Constructivist Approach 288
7.3. More General Future Mathematics Education Research 289
7.3.1. Holistic and Serialistic Learning Styles 289
7.3.2. Mental Imagery 291
7.3.3. Integration into the School Curriculum 293
7.3.3.1. Teaching and Learning Issues 293

7.3.3.2. Assessment Issues 295
7.3.4. Application of Outcomes to Developing Computer Technologies 296
7.4. Summary 298

REFERENCES 302

APPENDICES 328
Appendix A – Software Screens 329
Appendix B – Summary Sheet 340
Appendix C – Feedback Questionnaire 343
Appendix D – Analysis of Test Questions 345
Appendix E – Experimental Data 348
Appendix F – Publications by the Author 355

CHAPTER 1

Visualisation in Mathematics Education: The Issues

For those, like me, who are not mathematicians, the computer can be a powerful friend to the imagination. Like mathematics it doesn't only stretch the imagination, it also disciplines and controls it.

RICHARD DAWKINS

1.1. Introduction

Currently within mathematics education, continually advancing computer technology is providing alternative, innovative approaches to the teaching and learning of mathematics at all levels. There have been significant changes in the last two decades in the way in which mathematics has been taught, particularly concerning the involvement of the computer, with more emphasis based on practical and investigational work leading to a 'deeper' knowledge of the topic under study (Houston, 1998; Tall, 2000a, 2000b, 2000c; Habre, 2001; Ahlander, 2002; Challis et al., 2002; Connor, 2002; Leinbach et al., 2002).

The ease with which the computer can now display fast, accurate, colourful images has led to a more visual approach to the teaching of mathematics, especially in visually attractive subject areas such as graph plotting, transformations, rotations, enlargements, etc. (Elliott et al., 2000; Duatepe and Ersoy, 2002; Kautschitsch, 2002; Stehlikova, 2002). One of the most challenging issues for educators, however, is that of determining the most effective time to introduce such technologies, and how to use them, in the overall teaching and learning process which will maximise both educational and motivational value.

Mathematical software and Computer Algebra Systems (CAS) such as AUTOGRAPH, CABRI-GEOMETRE, DERIVE, MAPLE, MATHEMATICA, etc. (see References for Internet addresses), coupled with advances in multimedia technology, have impacted significantly on both symbolic and visual aspects of mathematics teaching (Challis and Gretton, 2000; Mogetta, 2000; Ardahan and Ersoy, 2002; Blyth et al., 2002; Garcia et al., 2002), and attempts have been made to measure the benefits to the learning process (Noguera, 2001; Kidron and Zehavi, 2002; Berry, 2002; Smith and Berry, 2002). Mathematics packages can be used either as tools during problem solving, or as vehicles for learning mathematics. Computer-based learning (CBL) software, such as CALMAT and MATHWISE (see References for Internet addresses), provides the opportunity for students to work at their own pace.

It is not only mathematical software packages, such as those mentioned above, that have had an impact on mathematics education, but also advanced graphics calculators such as the TI-89 and TI-92 (see References for Internet address), which are readily available and considerably more affordable than a computer. The use of handheld technology in schools is now slowly becoming more widespread, and can provide an approach to teaching and learning, at all levels, based upon investigative work incorporating both pictorial and symbolic forms (however graphic calculators are banned in much of current A-level assessment). This teaching mode would appear to be particularly advantageous for the 16-19 years age range (A-level in the UK), as there is an apparent difference between the skills base that is expected, and required, for undergraduate study in mathematics, and that which the students actually possess on entry to university (London Mathematical Society (LMS), Institute of Mathematics and its Applications (IMA), and Royal Statistical Society (RSS), 1995; Learning and Teaching Support Network (MSOR), Institute of Mathematics and its Applications (IMA), London Mathematical Society (LMS), and Engineering Council, 2000; Daskalogianni and Simpson, 2002; Hooper, 2002). Despite this reported shortfall, teaching methods incorporating technology of this nature have seldom been employed in schools (Ruthven, 1997). Ruthven's report for the UK Government contained findings on institutional availability, use and value of technology, and student access and ownership, and pointed to logistic issues of using technology in assessment. Additional to these issues, there is current debate concerning controversial 'non-calculator' assessment papers which do little to promote the advantages of technology in teaching.

Teaching and learning using CAS at A-level can be regarded as an important case-study having strong implications for the preparation of students entering higher education (French, 1998). This type of technology can enable both algebraic and graphical manipulation, not simply as a means of finding correct answers, but more importantly as a tool for exploring mathematical concepts. However, the importance of procedural expertise, for example the ability to differentiate, integrate, solve systems of equations, manipulate symbolic expressions, etc., must not be forgotten, but rather incorporated into a teaching, learning and assessment strategy. Mathematics educators need to be able to understand what amount of procedural facility is required by a student in order to understand concepts at a hierarchically higher level, and what can be left to the machine (the issues here are similar to those of the White-Box/Black-Box Principle (Buchberger, 1989) which is described in detail in Section 2.4.3.1 of Chapter 2). A CAS needs to be used in a considered way, i.e. students need to know what they want it to do for them, whereas some keystroke capture work (Berry, 2002; Smith and Berry, 2002) suggests that usage is sometimes just random button pressing.

Electronic learning and teaching materials are continually becoming more popular in the classroom, at both school and university level, although a number of teachers remain apprehensive. Many mathematics teachers, however, are trying to move away from the traditional 'handle turning' exercises of the past, to more enlightening exercises leading to conceptual understanding (Cochrane, 1996; Tall, 1991, 2000a, 2000b; Habre, 2001; Mackie, 2002). Multimedia software and visualisation technology, employing the use of video and sound in order to enhance and enliven the presentation of mathematical ideas (Bishop, 1997; Al-Jumeily, 2002; Pappas et al., 2002), provide the opportunity of transforming complex mathematical ideas into understandable, pictorially aesthetic forms. The following quotation encapsulates this belief: "*Probably the single most powerful tool available for teaching mathematics is visualisation*" (Cochrane, 1996).

1.2. Problem Definition

This section describes the numerous problems that enhanced visualisation might overcome. The thesis attempts to address such problems.

Teaching and learning styles adopted in schools do not necessarily lend themselves to the development of a deep understanding of mathematics. Students are not encouraged to be active participants in the learning process. Instead of experiencing passivity in the classroom, they need to be actively engaged, thus learning by building up knowledge structures based on their own experiences.

Students can demonstrate the ability to follow routines and regurgitate factual information, but often struggle when required to utilise higher order skills such as the ability to interpret, conjecture and evaluate, or to apply existing knowledge in an alternative context. Students do not have a natural inclination to employ visualisation in their learning or application of mathematics. They have difficulty appreciating the link between symbolic and pictorial representations, as the representational norm that has predominated throughout their experiences is that of symbolism. The use of visualisation, coupled with the rigorous algebraic approaches with which they have become familiar, can help to integrate the different representational forms. It is conjectured by this author that the gap between what is expected at GCSE and that which is expected at A-level, and likewise between school and university study, can be bridged by incorporating more visual approaches into the teaching, learning and assessment of mathematics.

Students lack conceptual understanding of mathematics, having spent much of their mathematical studies at school concentrating on procedural tasks and activities of a manipulative nature. The use of pictorial representations to enhance and reinforce the understanding of mathematical concepts is used to a certain extent at university level, but is not widespread in schools.

Lack of motivation can be a barrier for many in studying mathematics. It is conjectured here that a more student-centred approach, coupled with materials designed for use in visually compelling computer-based environments, can not only help to motivate individuals, but can actually serve to enhance the process of learning mathematics.

1.3. Factors Involved

This section serves to highlight the problems outlined above, and attempts to illustrate how visualisation ability is linked to the teaching, learning and assessment of mathematics. Visualisation is central to the four major factors discussed below, and is a key ingredient in the relationship between them. Consideration is given to the factors that influence the use of visualisation in student learning, and how visualisation can be employed to enhance conceptual understanding. Teaching and learning styles, in particular constructivism, are discussed (Section 1.3.1), and the skills that students possess (Section 1.3.2), together with those which are considered desirable for the understanding and application of mathematics, are taken into account. These two factors provide the foundation for considering how conceptual understanding can be developed by employing visualisation (Section 1.3.3), and how this influences software design incorporating imagery (Section 1.3.4).

1.3.1. Constructivism: Teaching and Learning Styles

Consideration is given here to the role of advanced technologies in active learning. It has been described (Schank, 1994) how, as a child, we did not learn to walk by taking a walking class, but instead we learnt by 'doing'. From birth to becoming a small child we do not participate in any formal classes or lessons, but the progress that we make during these years is enormous. Schank believes that teachers must therefore consider why pupils should endure a passive approach to learning in school. Well-designed educational software should support active participation, with the student being more in control of the learning process.

It is currently a particularly exciting time in mathematics education. Continually advancing technology is providing educators with new opportunities to enhance the teaching and learning processes. Interactive technology can allow students to be more involved in the learning process, so that they are not merely subjected to passive page-turning. However, even though all this technological capability is readily available, it is not a trivial matter to utilise it most effectively for educational benefit.

In recent years, there has been considerable evidence of a change in learning activities, as described above (Brown, 1994a; Clements and Battista, 1994; Kutzler, 1996; Bishop, 1997; Abboud, 2002; Challis et al., 2002). However, the educational benefits associated with this innovative change have been the subject of much debate (Jackson, 1997a; Jackson, 1997b; Cretchley, 1998; Hannah, 1998), and this remains a contentious issue in mathematics education (Gil-Pérez et al., 2002).

Educators, especially in the United States, are encouraging a constructivist, as opposed to an instructivist, approach to learning (Liao, 1993). In its simplest terms, a constructivist approach is one in which the student explores and experiments in order to learn, whereas an instructivist approach is one in which the student is taught in a traditional manner (a detailed discussion of instructivism and constructivism in the mathematics classroom is provided in Chapter 3).

It has been reported (Bailey and Chambers, 1996) that there was a real need for change in science and mathematics education in the United States, and the focus of change was on the roles of interactive learning and technology. Bailey and

Chambers reported a significant decline of educational standards in science and mathematics in the United States in spite of evidence that they had spent more on education in these areas than other countries such as England, Wales, Ireland, France, Spain and Canada.

It is now believed that traditional teaching methods involving the delivery of lectures and the use of textbooks do not have the desired motivating effect on students, but instead have a dampening effect (Bailey and Chambers, 1996). It is important, therefore, to increase motivation by utilising imaginative multimedia materials to capture attention, and increase interactivity by involving students directly in the learning process. It is conjectured that this active participation in understanding mathematical ideas and concepts, instead of rote learning, facilitates the long-term retention of knowledge.

Piaget stated: "*I'm convinced that one could develop a marvellous method of participatory education giving a child the apparatus to do experiments and thus discover a lot of things by himself. For me, education means making creators*" (Liao, 1993). This is the basis of constructivist thinking. It is an educational philosophy in which the students learn to think for themselves. Liao believes that the teacher needs to be more of a 'guide on the side', than a 'sage on the stage'. This approach to learning allows the students to re-think their initial ideas through interaction with the learning environment and fellow students.

Students' learning experiences can be enriched by helping them to construct knowledge for themselves. Visually stimulating interactive software can provide students with the appropriate environment in which they can become immersed in their own knowledge construction (Pea, 1987; Malone and Lepper, 1987; Phillips et al., 1995; Stevenson, 2000; Chae and Tall, 2001; Olivero, 2001; Kawski, 2002; McDill and Rash, 2002).

1.3.2. Students' Skills

This section provides a discussion of the apparent lack of students' mathematical skills acquired at school, together with implications for further study in mathematics. The reasoning skills of students with differing abilities are also addressed.

1.3.2.1. Fundamental Problems

One of the reasons for this research is that there is a real problem that needs addressing. Many students are failing, or even avoiding taking, mathematics at A/S-level, A-level and undergraduate level (London Mathematical Society (LMS), Institute of Mathematics and its Applications (IMA), and Royal Statistical Society (RSS), 1995; Edwards, 1996; Ahmad et al., 2000; Learning and Teaching Support Network (MSOR), Institute of Mathematics and its Applications (IMA), London Mathematical Society (LMS), and Engineering Council, 2000; Sam and Ernest, 2000; Middleton, 2001). Graduates with mathematical skills are in great demand by employers (MathSkills Discipline Network, see References for Internet address; Kopp and Higgins, 1997), and so positive steps need to be taken to rectify this disturbing situation. This research endeavours to illustrate how effective use of computer visualisation in teaching can increase the appeal of mathematics to those who have traditionally viewed the subject as an unattractive proposition.

It is always possible to improve the quality of teaching, and this can be achieved more easily if teachers and lecturers are willing to try new teaching methods. Teacher education programmes cannot provide potential teachers with the desired qualities (Harel, 1994). The mathematics knowledge that they acquire is far from satisfactory for school mathematics, let alone university level. Harel is convinced that there is a distinct lack of attention to the three crucial components of teachers' knowledge: mathematics content, epistemology, and pedagogy, and that these components need to be effectively integrated into teacher education programmes in order to provide the grounding necessary to produce teachers of a suitable quality. Attempts are being made, however, to address this deficiency (Llinares, 2000; Bloch, 2002; Noguera, 2002; Poblete and Diaz, 2002). Teacher education programmes also need to train the potential teacher in the use of sophisticated technology (Baldin, 2002). It is no use handing new teachers a TI-92, for example, and expecting them to utilise it in their lessons. They need instruction and guidance with respect to its potential.

School leavers are increasingly ill-prepared for further study (London Mathematical Society (LMS), Institute of Mathematics and its Applications (IMA), and Royal Statistical Society (RSS), 1995; Learning and Teaching Support Network (MSOR), Institute of Mathematics and its Applications (IMA), London Mathematical Society (LMS), and Engineering Council, 2000; Daskalogianni and Simpson, 2002). Virtually all science subjects include some mathematics, and the above reports highlight the fact that students do not have the necessary preparation for such a mathematical content. Various diagnostic and observational studies have highlighted undergraduate weaknesses in mathematics (Porkess, 1996; Ihejieto and Emenalo, 1996; Edwards, 1997; Hooper, 2002), and it is suggested that new undergraduates are uncomfortable with the different style of learning (Steyn and Maree, 2002), i.e. more interaction as opposed to transmission, hence learning being more constructivist as opposed to instructivist.

There is a noticeable 'gap' between the mathematical skills acquired at school, and those which are expected at university (Gill, 1998). School curricula on the whole concentrate on *instrumental* tasks, i.e. the routine use of procedures, whereas universities like to provide students with a variety of experiences in order to enhance their *relational* skills (Skemp, 1976), i.e. the ability to generalise globally (Anderson, 1996; Smith et al., 1996; Hershkowitz et al., 2001). As a result, new undergraduates, even the bright ones, insist on carrying out procedural tasks in a robot-like fashion. One solution is to make mathematics more practical (Ormell, 1997; Pappas et al., 2002), and highlight its usefulness in the world. This is the philosophy of the new A/S-level 'Use of Mathematics' course of the Assessment and Qualifications Alliance (AQA, see References for Internet address), introduced from 2003. The course has more emphasis on applying, understanding, reasoning and communication. It is practically oriented and promotes the application of mathematical principles to analyse real world problems, using information technology and real data sets. However, with this approach, there can be problems concerned with diluting both content and skills (see discussion on page 129 of Chapter 3).

Due to market forces, some schools switch to 'less demanding' examination boards so that their pupils can attain higher grades. This will result in the school featuring more prominently on national league tables, but yet perhaps may mask any issue of underdeveloped mathematical skills.

Another problem is that a trend away from combining mathematics and science A-levels as a coherent course of study has developed. Students are now often encouraged to choose a broader course of study, containing subjects from both arts and sciences. However, seeking a broader education can create problems, for example the mutual support mathematics and physics traditionally provided for each other in terms of consolidation can no longer be assumed. Many students also enter Higher Education from different routes, for example from Access courses or with vocational qualifications, and as a result may not possess the rigorous mathematical background required for many Mathematics, Physics, or Engineering undergraduate degree courses. Mathematical skills and knowledge therefore vary considerably depending on the route taken (A-level or Access course) and other chosen subjects (sciences or arts).

The fact that mathematics is considered by many students to be a traditionally difficult subject has contributed to a continuous decline in the number of students studying mathematics (Fujita et al., 1996), which has continued into the twenty-first century (Porkess, 2002). Students need to be attracted to study more mathematics by motivating them during their studies. Mathematical skills are in demand in a world increasingly dominated by sophisticated knowledge and technology. Computer technology can help to deliver these skills (Kahn, 1998). Another negative aspect (common to all subjects, not just mathematics) is the modular model, and its 'teach-test-teach-test' philosophy with little attention paid to synthesis of the material over different modules.

To reverse the apparent decline in students' mathematical skills and numbers studying mathematics, educators must therefore strive toward the creation of alternative, more attractive approaches to the teaching and learning of mathematics, which will make studying mathematics at all levels an altogether more inviting prospect. It is the contention here that the appropriate use of computer visualisation will not only assist in this development, but will also increase student motivation to learn.

1.3.2.2. 'More Able' and 'Less Able' Students

The outcomes of this research are applicable to students with a wide range of perceived mathematical abilities, but one of the problems faced is that there is a difference in qualitative thinking between 'more able' and 'less able' mathematics students (Tall and Razali, 1993). It is of interest how 'more able' students approach mathematics, and the problems that 'less able' students face. Tall and Razali state that the more gifted student will think at a higher level, i.e. will be able to manipulate concepts, and will adopt a more flexible way of thinking. Those who are 'less able' try to co-ordinate procedures rather than manipulate concepts.

As an illustrative example, Tall and Razali consider the expression $2+3x$. This represents both the process 'add two to the product of three times x' and also the result of the calculation, the expression '$2+3x$'. It can thus be seen how the dynamic process (the former) can be crystallised into a static concept (the latter). The flexible, crystallised concept can now be mentally manipulated and used for higher level thinking. For instance, $2+3x$ can be part of a more complex expression such as $(2+3x)^2-15x(2+3x)$. The 'more able', faced with the problem of factorisation, may 'chunk' this sub-expression as a single entity and see the factorisation $(2+3x-15x)(2+3x)=(2-12x)(2+3x)$. The 'less able', however, may only follow rules such as multiply out brackets, collect together like terms, etc. Therefore, the 'more able' are manipulating mental objects whilst the 'less able' are having to co-ordinate the processes of algebra. Thus the 'more able' succeed because they are performing qualitatively easier tasks, manipulating symbols, whilst the 'less able' are attempting to perform harder tasks or co-ordinate processes. The relationship between the process and the concept, with the notion of their dual existence, is known as a 'procept' (Gray and Tall, 1994, 2001).

Any teaching strategies that involve the integrated use of computing technology, and focus on a constructivist rather than an instructivist approach to learning, must be applicable to all mathematical ability ranges. It is a requirement of the mathematical software developed and used in this research, as described in Chapter 5, that it is to be useful for both 'more able' and 'less able' students (confirmation of this is provided). The software is concerned with the relationship between the symbolic and pictorial representations of functions. It uses visualisation in an attempt to enhance students' conceptual understanding of functions and graphs. The software can assist, via meaningful visualisations, 'more able' students who are comfortable with the manipulation of symbols, but do not necessarily conceptually understand the consequences of these manipulations on graphical forms. The software is equally beneficial to 'less able' students who are not comfortable symbolically. It invites them to conjecture the symbolic form of certain graphs, and offers a supporting mechanism by illustrating the effect of any symbolic manipulation on the graph. In this way, they can become more confident with algebraic manipulation and, at the same time, acquire a greater conceptual understanding of the relationship between alternative forms of functional representation. The software also requires students to progress through various stages of difficulty (with as much or as little help as is necessary), and this successful completion of tasks inspires greater confidence. The experiment described in Chapter 5 attempts to prove, amongst other things, that a constructivist approach to teaching and learning which incorporates computer-based visualisation is appropriate for both 'more able' and 'less able' students. Further details and outcomes of the experiment are discussed in Chapter 5.

1.3.3. Visualisation and Conceptual Understanding

Visualisation is a very powerful educational tool for the understanding of mathematical concepts (Jones and Bills, 1998). Computer visualisation is vital for exploring some complex subject areas, for example chaos and fractals, and it holds the key to creating a generalised understanding of non-linear mathematics (Cochrane, 1996).

As previously discussed, the school mathematics curriculum predominantly assesses procedural skills. Visualisation is not used significantly in either the teaching process or the assessment process, and so students can therefore be high achievers by simply being able to accurately perform the required processes (Etchells and Monaghan, 1994). Due to the manner in which mathematics is currently taught and assessed, good visualisers are seriously under-represented amongst high mathematical achievers (Presmeg, 1986; Malabar and Pountney, 2000). The inclusion of visualisation in the teaching, learning and assessment process would help to develop more 'complete' mathematicians with a greater conceptual understanding, i.e. those that possess more interpretive and constructive, as well as procedural, skills.

Current computing technology, whether it be handheld calculators, computer algebra systems (CAS) or bespoke teaching software, should enable more emphasis to be placed on the use of visualisation in teaching to enhance student understanding of mathematical concepts (as opposed to merely processes).

Visualisation is very much a cognitive skill which can act as a vehicle for conceptual understanding. It is the ability to interpret symbols visually, and not simply a method of pictorial representation of something habitually symbolic. However, the shift from a more conventional pattern of teaching mathematical processes with perhaps the use of visualisation at the end of the process, to one where the visual aspects of a concept are integrated into the teaching as a whole is not always straightforward. In the UK for example, pioneering work over a period of time on understanding via visualisation (Tall, 1986a, 1995, 2000a) has led to the production of visualisation software available for the teaching of mathematical concepts in 16-19 year A-level mathematics and beyond, but software of this nature does not seem to be in widespread use in schools (the dynamic software AUTOGRAPH is used in some schools). Similar efforts to integrate symbolic and visual understanding using a CAS have been reported. For example (Leinbach et al., 1997, 2002), a CAS can assist students to become active participants in their learning of mathematics. Students can work with the CAS by experimenting and making conjectures.

1.3.3.1. Enhancement of Visualisation Skills

Educators need to attempt to enrich thinking and visualising whilst students are doing mathematics, as the presentation of mathematical material is becoming increasingly visual. For example, mathematics educators can take advantage of modern technologies to enable the graphs of functions to be plotted with relative ease (Mason and Heal, 1995; Elliott et al., 2000; Pesonen, 2002; and see Chapter

5). This takes away the tedium of plotting, and in turn allows the student to concentrate on the interpretation and analysis of functions through their graphs.

Mason and Heal (1995) state that "*the pictorial enables aesthetic appreciation and invokes holistic visual processing directly, whereas written symbols must be processed sequentially, and require construction of a rich inner imagery to achieve the same effect*".

The computer can help in the building of more stable mental representations. After an experiment (Moreno and Sacristán, 1995) in which students experimented with mathematical software of a dynamic nature, one of the students responded that, "I feel that each time I am in contact with the computer, my mind becomes a blackboard on which I am doing something - in this case drawing, and I awake to a world where, although I cannot touch a square, I can see it and construct it, even if only graphically". It is this mental image-making that needs to be developed via computer visualisations.

The use of animation has shown itself to be a great motivator for often dreary subject matter (Milheim, 1993). It is not just a motivator however, but can be a significant factor within educational packages. It can be particularly useful when considering the mathematics of motion, for example using the CBL (Computer Based Laboratory) for interactive data capture (Gretton and Challis, 1996, 1999). Consider, for example, the throwing of a ball. An animated sequence could show the path of the ball through the air and the resultant shape of the curve. This dynamic pictorial build-up could be alternatively presented as a still image with an accompanying textual description, but the dynamic data capture is more likely to lead to students making hypotheses and conjectures about the information that they 'own'.

Given the earlier debate in Section 1.3.2 regarding students' skills in general, it was of interest to discover the extent of any visual skills that students possessed on entry into undergraduate mathematics degree programmes, especially where these used a significant amount of computer technology. Results highlighted the fact that students are generally poor at visualising, and so the strong graphical capabilities of the computer can be employed to enhance this skill. More details of this initial survey are provided in Chapter 4.

1.3.3.2. Improved Understanding of Concepts

In mathematics, visualisation can be utilised in order to achieve conceptual understanding. To visualise a concept means to understand the concept in terms of a diagram or visual image (Zimmermann and Cunningham, 1991). Mathematical visualisation is the process of forming images, and using such images effectively for mathematical discovery and understanding.

Mathematical visualisation gives depth and meaning to understanding, and can inspire creative discoveries. To achieve this kind of understanding, visualisation cannot be isolated, and must be linked to other forms of representation. Students must learn how ideas can be represented symbolically, numerically, and graphically, and to move back and forth among these different representations (Zimmermann and Cunningham, 1991). This way of thinking is stressed in Chapter 5, where the bespoke software links algebraic and graphical forms.

The computer incorporates a visual dimension into mathematics in ways that were not previously possible (Moreno and Sacristán, 1995; Tall, 2000b; Cha and Kent, 2002). If we again consider the plotting of graphs, one great asset of the computer is that it can show how a process is actually generated over time, for example the path of a projectile. We can therefore appreciate the dynamic nature and the behaviour of the process as opposed to merely the result.

Although most mathematics teachers would agree that the understanding of mathematical concepts is ideally the key aim, student experiences in schools, particularly at A-level, are far too procedure-based (French, 1991; Hubbard, 1997; Malabar and Pountney, 2000; Daskalogianni and Simpson, 2002). For example, it is quite common for students to be able to differentiate complicated functions, but the same students will not necessarily be able to look at a fairly simple graph and explain the key features (e.g. turning points, asymptotes, etc.). Thinking, rather than just doing, needs to be encouraged.

This research has attempted to determine the truth or falsity of the hypothesis that the use of computer visualisation improves the understanding of mathematical *concepts* (as opposed to merely processes). This was achieved through the design and implementation of an experiment. The dynamic educational package concerned with the graphical representation of functions, developed for the experiment, is aimed at students studying A-level and first year undergraduate mathematics topics. This chosen domain of the graphical representation of functions, found in upper secondary and Higher Education syllabuses, has been selected as it is this type of subject matter on which our dynamic approach should have a more considerable impact. It is anticipated, however, that the conclusions reached relating to this approach to teaching and learning will show the approach to be beneficial and applicable to all age groups and levels of mathematical ability. A description of the teaching software and full details of the experiment are presented in Chapter 5.

1.3.3.3. Evaluation of Enhanced Learning

The literature suggests that some work has been done to evaluate any educational benefits of a computer-based, visual approach to learning, however much of the evidence is anecdotal. It has been reported that there are educational benefits of handheld technology in the classroom (Short, 1998; Gardiner et al., 2000; Pope, 2002), that computers facilitate the construction of mathematical concepts (Clements and Battista, 1994; Tall, 2000a, 2000b; Habre, 2001; Mackie, 2002), that the learning experience is enhanced using interactive software (Dhillon, 1997; Stevenson, 2000; Olivero, 2001; Kawski, 2002; McDill and Rash, 2002), and that learning is enhanced using a CAS to perform visualisations (Amrhein et al., 1997; Challis and Gretton, 2000; Blyth et al., 2002), but very few studies actually attempt to measure any benefits in a large controlled experiment. This thesis describes such an experiment (245 students).

Feedback is an important part of the evaluation process, as it can help assess the quality and usefulness of a particular piece of educational material. Academic content, logical structure, and any limitations of the software are other evaluation considerations (Levin, 1986). A questionnaire was given to all students who used the dedicated software in order to receive feedback on the learning experience and the usefulness of the software. An examination of the feedback is provided in Chapter 5, and future work as a result of the feedback is discussed in Chapter 7.

Once the experiment had been designed, it then had to be implemented. This implementation stage included the statistical analysis and interpretation of results, leading to the re-examination of various theories of teaching and learning. A theoretical underpinning of the constructivist use of computer-based visualisation, as a result of the outcomes of the experiment, is presented in Chapter 6.

1.3.4. Software Design Issues Incorporating Imagery

This section discusses factors influencing the design of software incorporating imagery, and how these factors affect motivation. Consideration is given to students' cognitive development via the use of technology, the motivational benefits of multimedia, and the application of outcomes to future technological developments.

1.3.4.1. Cognitive Technologies

Cognitive technologies are simply tools that help understanding, as they assist in the organisation of thinking outside the physical confines of the brain, and computers have the potential to be the most extraordinary and influential cognitive technology to date (Pea, 1987; Mogetta, 2000; Tall, 2000c).

Pea (1987) describes how the dynamic nature of activities carried out with computer technology makes gaining an intuitive understanding of the relationship between pictorial and symbolic representations more accessible to the user. It is Pea's belief that some students will always be good mathematical thinkers, whereas others will not thrive without a richer environment for nurturing mathematical thinking. This statement is true, to a certain extent, however it is not just a particular category (the 'less able') that can benefit from a richer learning environment. Technology can act as a tool which all students, of all abilities, can utilise in order to achieve greater conceptual understanding. Even students who are always good mathematical thinkers can still benefit from innovative computing activities, as this can lead to the confirmation of existing ideas and the reinforcement of mathematical knowledge.

The experiment described in Chapter 5 illustrates how students used the dedicated teaching software as a cognitive technology in their quest for knowledge pertaining to the relationship between symbolic and pictorial representations of functions. The student learning process can thus be analysed by examining the various steps involved in the solution of a problem, in terms of both thought processes and functionality of the software. It is the combination of technological capability, innovative teaching software, and an investigative approach, that assists in the students' cognitive development.

1.3.4.2. Advantages of Multimedia

Whilst using creative educational packages, the student also experiences 'incidental learning' (Schank, 1994; Panoutsopoulos and Potari, 1995). This is the gaining of information whilst having fun and carrying out interesting tasks. The information is sub-consciously digested whilst the student revels in the change in learning activities. It is considered important (Lepper et al., 1993) for students to control the environment in which they study, as this will bring out their natural inclination to learn.

The use of multimedia is a complementary process (Kozma, 1991); activities are carried out sometimes by the user and sometimes by the medium. The powerful presentation capabilities can influence the structure of mental representations and cognitive processes (Kozma, 1991). Moving pictures should not just be seen as motivating factors (although this is important) but as additional instructional stimuli in their own right (Christel, 1994). Multimedia not only offers moving pictures, but also high quality sound reproduction. The presence of supposedly extraneous 'bells and whistles' in interfaces can result in the subject matter being considered as more active and more powerful (Christel, 1994) - visual attention may wander, but this attention can be recaptured by the use of appropriate audio cues. In many situations, simple diagrams are used for clarification purposes, and so the combination of moving pictures, audio accompaniment, and computing capability, could have huge potential in the understanding of a particular concept.

There are a number of features concerning multimedia that could be of considerable educational importance (Phillips and Pead, 1994). These features include:

- Much of mathematics is highly generalisable. Multimedia presentations focus on particular instances that are often vivid and entertaining. Afterwards, these can be developed into more general cases.

- The facility to show real people (as opposed to graphic/cartoon figures) appears to be important in offering role models, as students imitate those that they have seen on screen.

- Superior audio-visual presentations generate interest and involvement, which is helpful as compulsory mathematics in schools is often disliked by many 'less able' students of all ages (the issue of motivation is considered in greater detail in Section 2.5 of Chapter 2).

Distinction between multimedia and computers has blurred over the past few years as multimedia facilities have become standard on many computers (this was predicted by Phillips and Pead (1994)).

As multimedia technology continues to become more commonplace, it is important that developers of mathematical software, such as Computer Algebra Systems (CAS), attempt to incorporate features of multimedia into their design. The educational benefits of CAS have been reported (Townend, 1994; Kutzler, 1996, 1999; Malabar et al., 1998; Malabar and Pountney, 2000; Challis and Gretton, 2000; Ahlander, 2002; Leinbach et al., 2002; Zorn, 2002), and it is conjectured here that if they continue to be thoughtfully developed in terms of mathematical content and operational facility, and yet at the same time introduce elements of multimedia such as moving images and audio enhancements, then there will be an even greater impact on the student learning experience.

1.3.4.3. Future Developments

Educational practices will continue to adapt to incorporate both existing technologies, such as the Internet, and developing technologies, for example intelligent tutors and virtual reality. Successful implementation of highly intelligent mathematics education packages or the use of virtual reality to assist in the teaching of mathematics are both perhaps a little unrealistic at the present time, however teaching and assessment via the Internet (on the World Wide Web) are already well established (Booth et al., 2001; Gage et al., 2002; Waldock et al., 2002). It enables us to move away from stand-alone educational packages to interactive modules on the World Wide Web (Miller and Abramson, 2002; Murphy et al., 2002; Woods, 2002), which can provide a framework for the inclusion of text, images, and video (Marshall et al., 1994). Advantages of this approach include the following:

- Students can make mistakes without feeling embarrassed, as they might do in a classroom situation. The lack of human interaction could be overcome by an intelligent tutoring system which can detect a student's lack of understanding and can thus consider remedial action.

- Students can progress at their own pace. A structure is needed, however, so that they can be given guidance if necessary (an intelligent tutoring system could help here).

- Good practices can be shared.

- It offers virtually unlimited access - students are not restricted to library opening hours or the number of available copies of a particular text. Students can work from wherever they choose - this is an obvious advantage for distance learning.

Chapter 7 considers future developments in terms of specific experiment issues as a consequence of the outcomes reported in Chapter 5, and also generic mathematics education issues as a result of outcomes of this research.

1.4. Aims of the Research

The introduction to this chapter has discussed generally how the use of computer technology can provide alternative approaches to teaching and learning, and has considered the impact of Computer Algebra Systems (CAS), bespoke teaching software, and handheld technology on mathematics teaching.

Many educators are promoting a change in student learning activities, as discussed in Section 1.3.1, from an instructivist (passive) approach to a constructivist (learning by doing) approach involving investigative work with computers, which has both educational and motivational value.

The A-level mathematics curriculum is not providing students with the necessary knowledge and skills in order to be prepared for further study in mathematics in Higher Education, as explained in Section 1.3.2. It is important that an alternative method of teaching centred around the dedicated use of computers is applicable to both 'more able' and 'less able' students alike. There is also the danger with respect to assessment of removing the procedural element ('comfort blanket') on which the 'less able' students might rely.

It is conjectured in Section 1.3.3 that the shortage of desirable skills can be overcome by the thoughtful use of visualisation in aesthetically stimulating computer-based teaching scenarios.

Cognitive technologies, which are specifically designed to assist in the acquisition of conceptual knowledge, can act as tools which all students, of all abilities, can utilise in order to achieve greater understanding. Section 1.3.4 discussed how the computer is potentially the most influential cognitive technology to date.

This research attempts to show how the constructivist use of computer-based visualisation can be adopted in order to address, and overcome, the problems outlined in Section 1.2. The thesis is adding to existing knowledge in relation to the key issues raised in this chapter by attempting to achieve the following aims:

1.4.1. Key Aims

A main aim of this research is "*to evaluate enhanced student learning of mathematical concepts, and to assess the extent of any skills development, via the constructivist use of computer-based visualisation*". This is examined by means of a case-study (Chapter 5) which endeavours to explore the impact of the constructivist use of technology on the cognitive aspects of visualisations created by learners and the motivational effects of such approaches to learning for students of all abilities. The case-study illustrates how greater conceptual understanding can be achieved via the enhancement of students' visualisation skills. The case-study aims to compare the performance of students after learning via either an instructivist or constructivist (incorporating computer-based visualisation) approach, in terms of both procedural and visual skills, as well as other higher order skills. It was conjectured a priori that the constructivist approach would lead to the development of a broader skills base, and would enhance problem solving skills in general.

For use in the case-study, a piece of bespoke software was designed and developed linking visual and symbolic representations of functions (Chapter 5). Unlike a graphic calculator, the software forces the student to translate from graphic to symbolic forms, and therefore allows for the conjecture and rehearsal of more general relationships between representational forms. The software aims to enhance visualisation skills, which in turn help in the development of conceptual understanding, together with other desirable higher order skills. It aims to develop a more holistic view of mathematics, and to provide students with better strategies for problem solving. It aims to develop an understanding that is independent of specific examples used, so that the conceptual knowledge acquired can be applied to any function, i.e. the knowledge gleaned from local tasks can be applied globally.

A final key aim is to consider the most effective use of symbols, pictures, and the computer in the move from procedural to conceptual knowledge by providing a theoretical framework for the constructivist use of computer-based visualisation in mathematics education (Chapter 6) as a result of the outcomes of the case-study.

1.4.2. Secondary Aims

Secondary aims of the research include:

- Investigation of the effective use of computers and handheld technology in the teaching and learning of mathematics (Chapter 2).

- Consideration of the effect of constructivism, promoting strategic questionning and social interaction, on the learning process (Chapter 3).

- Examination of which skills students demonstrate when assessment allows the use of a CAS (Chapter 4).

- Analysis of specific performance comparisons in the case-study (Chapter 5):

    - Schools (Schools 1, 2 and 3).
    - Sex (Male and Female).
    - Subject (Mechanics and Statistics).
    - GCSE Grade (A/A* and B/C).

- Assessment of the motivational effects, in terms of usefulness and enjoyment, of a constructivist computer-based visual approach to learning (Chapter 5).

- Exploration of the impact of a constructivist computer-based visual teaching approach on existing teaching methods (Chapter 7).

- Identification of further research as a result of the outcomes of this thesis (Chapter 7).

1.5. Summary

This thesis considers students' apparent inability to visualise, and it is of particular interest as to whether or not dedicated visual software can enhance conceptual understanding through constructivist visualisation exercises. It provides empirical evidence of the benefits of the dedicated use of such software, together with a theoretical underpinning, which could be useful to teachers when considering integration of such software into mathematics classrooms.

In order to satisfy the aims outlined above, the thesis comprises the following:

Chapter 1 has described the role of continually advancing technology in mathematics education, has defined problems that enhanced visualisation might overcome, has discussed the major factors of interest, and has stated the aims that attempt to address the problems.

Chapter 2 provides a review of the effectiveness of previous work in relation to the major factors described earlier, and describes in detail the remaining chapters in terms of how they make a significant contribution to the literature.

Chapter 3 discusses the merits of constructivism, as opposed to instructivism, when considering the use of visualisation in mathematics education, and illustrates how educational technology can act as a vehicle for supporting the constructivist philosophy.

Chapter 4 proposes a skills classification appropriate for the incorporation of technology in mathematics education, and shows how visualisation is key to bringing together various skills taxonomies. It illustrates how the constructivist use of visualisation can enhance higher order skills, and discusses the extent to which students demonstrate certain skills when technology is at their disposal in assessment.

Chapter 5 describes in detail the case-study which attempts to satisfy the key aims, a detailed rationale for the design of the bespoke software, and the experimental design for the case-study. An analysis of data collected from 245 students, on a variety of variables, leads to conclusions as to the effectiveness of the approach in terms of conceptual understanding, the development of skills, and motivation.

Chapter 6 builds on existing theories of teaching and learning by providing a theoretical framework for the constructivist use of computer-based visualisation in mathematics education as a direct result of the outcomes of the case-study.

Chapter 7 suggests possible future research as a result of the findings of this thesis, specific to the case-study and for mathematics education in general.

CHAPTER 2

Visualisation in Mathematics Education: A Review of Previous Work

Mathematics, rightly viewed, possesses not only truth, but supreme beauty - a beauty cold and austere, like that of sculpture.

BERTRAND RUSSELL

2.1. Introduction

This chapter addresses the key issues highlighted in Chapter 1, upon which this thesis builds. Consideration is given to previous work in the integrated use of visualisation techniques in the mathematics classroom in order to aid conceptual understanding, together with the associated difficulties involved. It looks at the properties of visual information that influence image making, and the mental images created by learners as a result of both good and bad use of imagery, and how this relates to the teaching of mathematics. Features of learning environments combining the computer with visual effects are discussed. It can be seen how the creative potential of the computer can make learning fun and maintain interest, and how dynamic environments and interactivity can increase retention levels.

A review of the effectiveness of previous work in relation to the four major factors highlighted in Section 1.3 of Chapter 1 is provided as follows:

Firstly, classroom experiences employing exploratory teaching and learning approaches which aim to show how the constructivist use of technology, such as CAS and handheld technology, can develop desirable transferable skills are examined (transferable skills are those higher order skills which can be exploited in other subject domains and problem solving scenarios).

Secondly, consideration is given to students' skills. Various skills taxonomies are discussed, and the difficulties associated with the use of visualisation from a student perspective are addressed.

Thirdly, examples are provided which highlight the motivational and educational benefits of the effective use of visualisation, together with a discussion of the types of mental images created by learners as a result of the use of visualisation. Various theories of teaching and learning of mathematics are discussed in terms of the development of conceptual understanding.

Finally, software design issues are addressed in terms of the educational and motivational benefits of the use of computer-based visualisation. This is aided by a case-study concerned with graphical software which shows how it can enable students to concentrate more on the visual and holistic, and can change the emphasis from the physical creation of graphs to the appreciation of their global features. The specific case-study concerned with functions and their graphs was chosen as an example of understanding concepts via visualisation.

The summary of this chapter highlights the key issues raised which are addressed in the remainder of the thesis. A description of the remaining chapters is provided in terms of how they make a significant contribution to the literature by building upon the previous work discussed throughout this chapter.

2.2. Constructivism: Teaching and Learning Styles

Teaching and learning approaches supporting the constructivist philosophy in order to develop desirable higher order skills are reported here, together with a discussion of experiences with technology in the mathematics classroom.

2.2.1. A Constructivist Approach to Teaching and Learning

Computer environments can encourage and facilitate a constructivist approach. Anecdotal evidence suggests that an area in which the constructivist use of technology can help develop students' thinking is that of geometry (Clements and

Battista, 1994). The objects on the computer screen become manipulable representations of the students' thinking. The students can thus make conjectures, evaluate visualisations of those conjectures, and reformulate their thinking. Clements and Battista believe this to be essential for developing reasoning skills in geometry. Educators therefore need to appreciate the implications of constructivism when designing and using mathematical software.

Peer support can also be valuable whilst carrying out constructivist problem-solving activities (Brown, 1994a). Brown describes how computer-supported workshops for first year undergraduates studying solid mechanics provided a rich learning environment in which a variety of learning activities could take place. Student interactions were an important component of learning processes in the workshops, and working in pairs led to very positive outcomes. The key findings were as follows:

- There was a tremendous richness in the variety of tasks that could be offered via IT, with great value put on visual representations. Visualisation offers the opportunity for different teaching and learning methods.

- It was seen by the students as requiring much more involvement as well as being much more interesting than lectures.

- There was considerable evidence of peer support both within and between groups.

Peer support was one of the most impressive aspects of how the workshops operated. This computer-based constructivist approach promotes active learning, which in turn encourages discussion and reflection. Collaboration can help students become more skilful in the thinking required for problem solving.

Ruthven (1989) reported that there was little use of more exploratory approaches in the teaching and learning of advanced (upper secondary level) mathematics, and this still appears to be the case over a decade later. A project was described that implemented and evaluated an exploratory teaching model (as opposed to a conventional approach) in which students explored a visual approach to the factorisation of polynomials. Despite positive findings in terms of student learning and motivation, there are still few signs of the exploratory model being adopted in schools. The major difference between the conventional and exploratory approaches can best be understood in terms of two very simple models of the sequencing of classroom processes, shown in Fig. 2.01 below. In the conventional approach, new ideas are introduced through teacher exposition, whereas in the exploratory approach teacher exposition is deferred, and draws on prior student investigation. The exploratory model replaces teacher exposition with student exploration and codification. In the codification phase, the teacher draws on evidence gathered by students in developing mathematical ideas, leading to a deeper understanding in the consolidation phase.

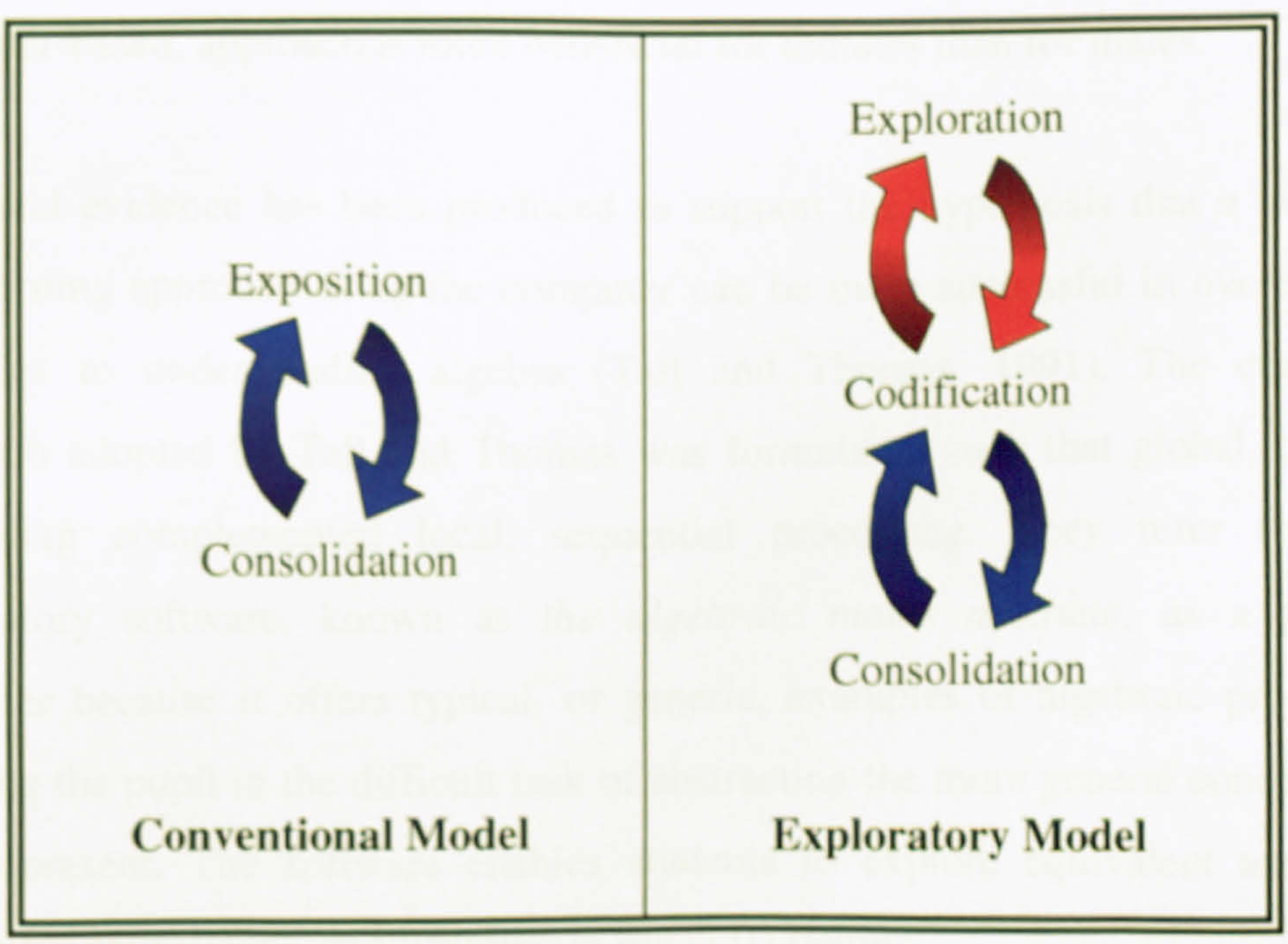


Fig. 2.01.

These approaches map nicely onto the methodologies already discussed, whereby the conventional and exploratory models reflect instructivism and constructivism respectively.

Ruthven was concerned that due to the nature of assessment, the adoption of an exploratory style may prejudice examination attainment, that is students taught in an exploratory style may underachieve in those aspects of mathematics which present examinations are designed to assess. The results, however, proved this not to be the case. Rigorous statistical evaluation showed that there was no difference in the attainments of students taught in exploratory and conventional styles, for all gender and previous experience groups, even though the assessment is heavily routine-based. Another finding of interest, from interviews and questionnaires, was that females preferred routine activity and greater teacher direction, which it is suggested can be explained in terms of gender stereotypes which emphasise the desirability of rule-following, rather than rule-challenging, behaviour in girls. This is unusual as there is evidence in the literature (Ferrini-Mundy, 1987; Ruthven, 1990; Tall, 1993), and in Chapter 5 of this thesis, that an exploratory, albeit computer-based, approach is more beneficial for females than for males.

Empirical evidence has been produced to support the hypothesis that a teaching and learning approach using the computer can be more successful in overcoming obstacles to understanding algebra (Tall and Thomas, 1991). The computer approach adopted by Tall and Thomas was formulated such that global, holistic processing complemented local, sequential processing. They refer to their exploratory software, known as *the algebraic maths machine*, as a generic organiser because it offers typical, or generic, examples of algebraic processes, assisting the pupil in the difficult task of abstracting the more general concept that they represent. The software enables students to explore equivalent and non-equivalent expressions, as illustrated in Fig. 2.02 below:

2. There was an improvement in encapsulation of processes, shown by the ability of the experimental group to discuss processes, without having to carry them out first.

3. The experimental group pupils showed the ability to take a global view of a problem, rather than being pressed into processes implied by the operations present in the notation.

These observations are typical of those concerned with similar exploratory approaches described throughout this chapter.

Tall and Thomas argue that the emphasis of their software is on conceptualisation and use of mental images rather than on skill acquisition, and that an environment encouraging the formation and manipulation of cognitive structures containing both images and symbols should be more likely to produce versatile thinking (thinking in which concepts can be represented by imagery as well as symbolically and verbally). This thesis argues, however, that *the algebraic maths machine* seems to concentrate solely on symbolic manipulation, with the pictorial aspect of the software merely being a display of the algebraic notation. A more beneficial use of visualisation, as with the software described in Chapter 5, would be to integrate pictorial representations into the software that serve to provide meaning for, or reinforce, the symbols. Clearly, much depends upon the nature of the chosen topic, and the extent to which it lends itself to a delivery incorporating visualisation. The software in Chapter 5 acts as a generic organiser in the same way as *the algebraic maths machine*, by developing the abstraction of concepts via student experiences, but significantly enhances the learning process by forcing a constructivist approach in which students are required to conjecture the symbolic notation for given pictorial forms. In this manner, students develop versatile thinking skills, together with a deeper understanding of the relationship between different representations.

2.2.2. Classroom Experiences with Technology

CAS are now being used more extensively at undergraduate level. A CAS can be used to perform visualisations, and to provide an important link between symbolic, numerical and pictorial information. Students can discover mathematical concepts by themselves, i.e. adopting a constructivist approach, which allows them to have a more active role in the learning process and lets them take control of the development of their own mathematical knowledge. An interactive learning environment, written with the aid of Mathematica, with integrated, powerful graphic capabilities was developed (Amrhein et al., 1997), which enabled students to devise their own examples of increasing complexity. The students may then learn the mathematical concepts through experiments, in which visualisation plays a crucial role. Amrhein et al. provide anecdotal evidence of enhanced learning via this approach. A 'student plus manipulation tool' can be more successful in conceptual and computational tasks than a student working in a traditional manner.

Investigative work integrating algebra and visualisation has been carried out with the computer algebra system DERIVE (Mathews, 1994; Malabar et al., 1998; Challis and Gretton, 2000). It has been reported how the dedicated use of DERIVE can provide students with conceptual understanding, and can lead students to the discovery of mathematical results which illustrate real world applications of mathematics. Exercises exploiting the power of visualisation can be constructed to actively engage students in their own learning, and in turn, students will find themselves *doing* and *understanding* mathematics.

Townend describes the delivery of a mathematics workshop module using DERIVE (Townend, 1994). The module was designed to test students' ability to analyse, solve and experiment using DERIVE. Formal (by questionnaire) and informal evaluations of the module indicated that it was well received by all students. The main advantages were found to be as follows:

- removes mathematical 'drudgery'.
- encourages a 'what if ...?' approach.
- permits a ready checking of answers.
- offers easy visualisation of functions.

The final bullet point, above, can be interpreted in two equally valid ways. DERIVE can be used to find out what a specific function looks like graphically, but can also be used to generate a number of related graphs in order to visualise families of functions in a generic sense.

Townend believes that this approach to teaching mathematics has bolstered the confidence of the weaker students, and has encouraged the more able to be more mathematically investigative.

In 1987, Berry et al. described the enormous potential for using computer algebra in teaching and learning mathematics (Berry et al., 1987). They found a remarkable change in attitude and motivation in students after using DERIVE to introduce the concepts and skills of differentiation of polynomials. Student learning was undertaken in an investigative manner, resulting in the following observations:

- Students are better motivated towards mathematics.
- Students approach mathematical tasks more confidently.
- Students are able to do mathematics as well, if not better than before, because of their added confidence.
- Students can solve more realistic problems using the power of DERIVE.

- The integrated use of graphics and algebra provides a solid conceptual base on which to build.

The potential foreseen by Berry et al. has indeed been realised. These observations have proven to be accurate, examples of which have been reported in recent years (Kutzler, 1999; Leinbach et al., 1997, 2002; Ahlander, 2002).

Studies have shown how the TI-92 can be an assistant in mathematical problem solving (Candlish, 1997). Candlish describes how the student has to do all the thinking, and set up the necessary calculations, and then the TI-92 can process them. It simply carries out the 'number crunching' whilst the student concentrates on solving the problem. The value of the emphasis on conceptual as opposed to procedural understanding has also been highlighted at Napier University, where the TI-92 has been used as an integral part of a first year engineering course (Short, 1998). Some of the essential points arising from the experience from the teacher's perspective are described. Short explains that it was encouraging that students who were normally not well motivated were concerned about the mathematics they were receiving, which was a novel reaction when compared to the usual 'resigned acceptance' of whatever they were taught. Comments such as, "I know the mathematical manipulations are correct, and I can concentrate on the problem itself", and "I never realised equations had graphical meanings. It's much easier this way", were indications that this integrated use of technology was having a positive affect on student learning.

A further example of students acquiring hands-on experience of exploring concepts for themselves is that of introducing differentiation through the gradient function, using the TI-92 (Francis, 1996). Francis explains how students can derive the gradient function for simple cubics, quartics, etc., and hence lead up to 'the rules' for differentiating polynomials. A certain graph is drawn, followed by a tangent to the curve. Each time a tangent is drawn, its gradient is plotted as a dot on the screen, and when all tangents have been plotted, the dots will be joined up to reveal the graph of the gradient function. The gradient function graph can then be inspected, and its equation deduced (this is, in itself, not a trivial task). Collected results enable patterns to be spotted and the 'rules' discovered by the students (it is this kind of pattern spotting and ensuing formalisation that the software described in Chapter 5 employs).

Handheld technologies such as graphical and symbolic calculators can both promote the acquisition of conventional mathematical expertise, and support mathematical innovation on the part of students (Ruthven, 1991; French, 1998). The performance of A-level students using a TI-92, the experimental group, was compared with that of a similar group of students without access to any technology (handheld or otherwise), the control group (Ruthven, 1990, 1991). The attainment of the experimental group was substantially and significantly higher than that of the control group in recognising graphic forms and relating them to appropriate symbolic forms. The performance of the experimental group was superior in two respects. Firstly, they were more proficient at recognising a graph as being of a particular type (belonging to a certain family of graphs). Secondly, they were more successful at building up a precise symbolic description of the graph, by exploiting salient information, such as its orientation, and the position of its extreme values, zeroes and asymptotes, and knowledge of relationships between these features of a graph and its symbolisation. There was also a significant difference in the influence of the technology on gender. In the experimental group, female students outperformed male, while the reverse pattern was evident in the control group (an outcome which is also apparent in this thesis, with the dedicated software being more beneficial to female students). This would suggest that, without any additional visual stimuli, males are better visualisers, but females respond better to technology-based visual training.

The interaction of graphical, algebraic and programming features of the TI-92 increases its usefulness enormously (Short, 1998). Regular use of a graphic calculator is likely to reinforce specific relationships between particular symbolic and graphical forms, as it is through such relationships that the calculator itself is operated (Ruthven, 1990). Despite the reported advantages, handheld technology is not used extensively in A-level mathematics classes. One of the few pieces of mathematical software that seems to be used (or at least available) in many schools is AUTOGRAPH, which supports the dynamic exploration of mathematics. It is not clear, however, whether it is used in an exploratory manner, or merely as a plotting tool.

2.3. Students' Skills

This section considers various related skills taxonomies, and discusses visualisation in terms of student difficulties in interpreting visual information (linkages between the taxonomies and the prevalence of visualisation amongst them are discussed in greater detail in Chapter 4).

2.3.1. Related Skills Taxonomies

Skills requirements and their classification are considered, concentrating particularly at the A-level/University interface. Broadly speaking, the A-level curriculum focuses on candidates being able to demonstrate a high level of 'technical expertise' with much less attention paid to the application of mathematics to the solution of real world problems (there are of course exceptions to this generalisation, notably Mathematics in Education and Industry (MEI), Assessment and Qualifications Alliance (AQA), and International Baccalaureate (IB), which contain more modelling and applied mathematics than other syllabuses). Students can often show success in one area but fail when the context or style of the problem is changed. There is evidence of the learning of mathematics as a set of disconnected skills in order to pass examinations (Gill, 1998). This produces cohorts of students who are good at passing certain styles of examination regardless of content.

The widening of the school mathematics curriculum has given students less opportunity to practice mathematical skills required for university mathematics courses (see the Group C skills of the MATH taxonomy in Section 2.3.1.3 of this chapter), because of the broad range of mathematical study required. At the same time, this increase in content may have left less time for the development of conceptual understanding (Etchells and Monaghan, 1994; Monaghan et al., 1998; Porkess, 2002). High performance at school level does not necessarily constitute an index for projecting into performance at university level (Ihejieto and Emenalo, 1996; Porkess, 1996; Steyn and Maree, 2002). The modular approach is also an issue, as the 'teach-test-teach-test' philosophy does not give time for reflection, which is when conceptual understanding might develop.

2.3.1.1. NCVQ (National Council for Vocational Qualifications)

Designers of undergraduate and postgraduate degree courses in Higher Education establishments, including those involving mathematics, are well aware that graduate employability is becoming increasingly important. Apart from the appropriate 'technical expertise', employers additionally expect new graduates to have the confidence and ability to undertake a number of varied tasks. In short, there is an expectation that graduates will possess skills such as the six highlighted by the NCVQ (see References for Internet address):

- Communication
- Numeracy
- Information Technology
- Working as part of a team

- Improving own learning performance (linked to Continual Professional Development, CPD)
- Problem solving

In order for GNVQ (General National Vocational Qualification) candidates to be successful, they *must* demonstrate an ability to apply mathematics to the solution of real, vocational problems - success is not measured solely in terms of the ability to 'do' mathematics (Corporate Author School Mathematics Project, 1996; Molyneux-Hodgson and Sutherland, 1996, 2002; Molyneux-Hodgson et al., 1998; Butterfield et al., 2000). A further, incidental, benefit of this approach of applying mathematics is the enhancement of students' communication and other transferable skills. This approach is also exemplified by the AQA A/S-level 'Use of Mathematics' course described in Section 1.3.2.1 of Chapter 1.

2.3.1.2. Mathskills Discipline Network

A survey to find out which mathematical skills are required by employers who employ mathematics graduates was carried out (Kopp and Higgins, 1997). The survey, conducted for the Mathskills Discipline Network (see References for Internet address), showed that while there was a small demand for highly qualified graduates with a deep and specialised knowledge (to provide tomorrow's researchers and university academic staff), the major demand was for graduates who are assumed to be technically competent in their discipline but who also possessed problem solving skills, communication skills, team skills, etc. - in short a mathematics graduate in tune with the GNVQ philosophy.

The challenge to the university sector remains to develop these skills in students while at the same time not diminishing the knowledge content of the degree courses. As Kahn (1998) says: "*Undergraduate mathematics degrees need to offer students a distinctive and relevant profile of skills. The study of mathematics* *enables students to formulate and solve complex logical and quantitative problems by thinking logically, modelling, analysing, synthesising, and communicating the solutions to others. The degree of complexity in undergraduate mathematics ensures that these skills can be highly developed in mathematics graduates, equipping them with skills that are in demand by employers*".

A look at the seven 'themes' of the Mathskills Discipline Network, below, indicates the Network's belief that the right calibre of graduate, with the appropriate skills, can be produced by introducing innovations in teaching and assessment, fully exploiting the use of IT in the presentation of courses and embedding problem solving activities (mathematical modelling) in undergraduate mathematics programmes.

Theme 1	What employers want.
Theme 2	Directory of expert mathematical modellers.
Theme 3	Teaching innovations.
Theme 4	Assessment methods.
Theme 5	Methods for embedding core skills in the mathematics curriculum.
Theme 6	Group work and peer tutoring.
Theme 7	Use of technology, multimedia and distance learning in undergraduate mathematics courses.

N.B. Themes 3, 4, 5 and 7 have scope for the use of visualisation explicitly.

The Mathskills Discipline Network, and indeed the NCVQ, are primarily concerned with wider 'graduate skills'. Mathematical skills that particularly relate to many of the above seven themes can now be examined more closely.

2.3.1.3. MATH Taxonomy

Whilst the Mathskills Discipline Network attempts to link the skills of mathematics graduates with generic skills sought by employers, a further taxonomy known as the MATH taxonomy (Mathematical Assessment Task Hierarchy) has been proposed (Smith et al., 1996; Ball et al., 1998) as a development of Bloom's taxonomy (Bloom et al., 1956), as a means of structuring ***assessment tasks*** of students in terms of a skills set required to complete them.

Mathematics questions differ considerably in terms of the skills required in order to complete them, even for questions relating to the same mathematical topic domain. Many questions require students to carry out certain steps (routine use of procedures) in order to arrive at a desired outcome, but less common are those that challenge students to use their interpretive and constructive skills in order to achieve success. Smith et al. have developed this taxonomy which classifies mathematics questions by the nature of the activity required for successful completion of the task, rather than in terms of difficulty. The MATH taxonomy describes a hierarchy of skills ranging from lower order skills (Group A/B), such as factual knowledge and the ability to follow procedures, to higher order skills (Group B/C), such as the ability to interpret, conjecture and evaluate, as represented in Fig. 2.03 below:

Group A	Group B	Group C
Factual knowledge	Information transfer	Justifying and interpreting
Comprehension	Application in new situations	Implications, conjectures and comparisons
Routine use of procedures		Evaluation

Fig. 2.03.

N.B. Mathematics A-level questions typically assess Group A and B skills only, whereas an undergraduate mathematics programme should also assess Group C skills significantly.

2.3.1.4. Skills Studies by Galbraith and Haines

In a similar fashion to the above, a test framework for developmental skills required in the application of mathematics has been considered (Galbraith and Haines, 1995, 2000a, 2000b), which classifies questions into three distinct categories as follows:

- ***Mechanical questions*** cue the students to respond with an answer that involves the systematic application of basic knowledge or procedures.

- ***Interpretive questions*** require the students to select and put together information in order to reach a conceptually based conclusion. Concepts rather than procedures are the basis here.

- ***Constructive questions*** involve the creation of links between concepts and procedures that must be generated by the student as part of the solution process.

Galbraith and Haines argue that such a classification provides a means of establishing whether these important skills improve as students progress through their course of study. The three categories here map closely onto the three groups of the MATH taxonomy (linkages between the taxonomies are discussed in greater detail in Section 4.3 of Chapter 4).

2.3.2. Students' Difficulties in Developing Visualisation Skills

Many students find it difficult to visualise in mathematics, which leads to lack of meaning (Tall, 1991), and one of the key aims of this research, as outlined in Chapter 1, is to attempt to enhance this skill. Only if educators can understand the difficulties that students experience, can they then enhance the teaching and learning processes in order to address the problem.

A study of visualisation skills was carried out (Mariotti, 1995) which emphasised this shortcoming. The study involved 333 first year undergraduate Mathematics, Physics and Computer Science students. One of the questions asked was, "A plane intersects a sphere - what is the shape of the cross section?". Just over half of the students answered incorrectly. This is a very large percentage of students that are just not capable of visualising a geometrical situation. School children also took part in the study. They were shown a cube, and then the cube was hidden. They were then asked to count the number of faces, edges and vertices of that hidden cube. Again, poor answers were given as they found it very difficult to visualise mentally the cube and its properties.

Students also find it difficult to appreciate the effective use of different types of diagrams. For example, there is an important difference between the role of diagrams in proofs and the role of diagrams in problem-solving (Zimmermann, 1991). A problem diagram describes the special conditions of the problem, whereas a diagram associated with a proof must describe the general case under consideration. For example, the diagram in Fig. 2.04 below could be used as a problem diagram to find angle ACB. As AO and CO are both radii of the circle, and angle AOC is a right angle, then angle ACO must be 45 degrees. Similarly, angle BCO must also be 45 degrees, making angle ACB a right angle.

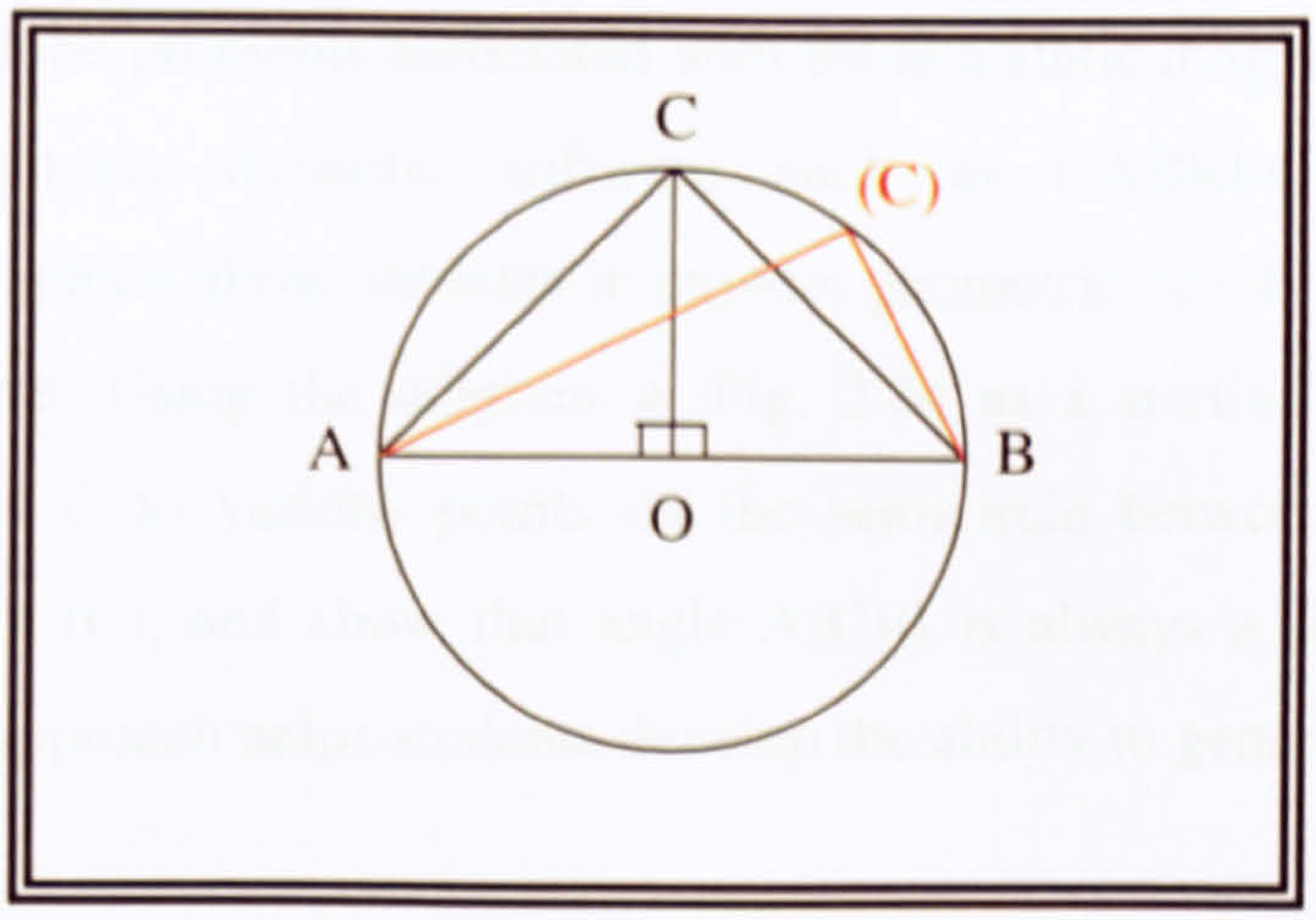


Fig. 2.04.

On the other hand, if the purpose of the diagram is to prove the general case that an angle inscribed in a semicircle is a right angle, then the diagram in Fig. 2.04 above would clearly not suffice. Using a diagram to prove a general case presents an obvious difficulty because any diagram, being a concrete entity, has its own special characteristics. Students find it difficult to understand which features of the diagram are incidental and which features are essential. Some of the difficulties experienced by visualisers (Presmeg, 1986) in a situation such as this are as follows:

- The one-case concreteness of an image or diagram may tie thought to irrelevant details.
- An image of a standard figure may introduce inflexible thinking, which prevents the recognition of a concept in a non-standard diagram.
- An uncontrollable image may persist, thereby preventing the opening up of more fruitful avenues of thought.
- Especially if it is vague, imagery which is not coupled with rigorous analytical thought processes may be unhelpful.

One way around the problems associated with using a static diagram to illustrate a proof is to employ dynamic software such as CABRI-GEOMETRE or AUTOGRAPH, which allow students to explore geometric proofs such as the one highlighted above. Using the diagram in Fig. 2.04 as a starting point, students could drag point C to various points on the semicircle between A and B, for example to point (C), and show that angle A(C)B is always a right angle. This dynamic visual approach helps students develop the ability to generalise.

It appears that many students are reluctant to visualise when doing mathematics. Students exhibit a definite bias toward an algebraic approach instead, and tend to shy away from visual thinking (Eisenberg and Dreyfus, 1991). Three possible reasons for this reluctance are:

- cognitive (visual is more difficult).
- sociological (visual is harder to teach).
- mathematical beliefs (visual is not mathematical).

The understanding of these reasons has informed the development of the teaching and learning strategies that promote visual thinking adopted in the case-study in Chapter 5.

Empirical evidence shows that there is algebraic bias and an avoidance of visual considerations among calculus students (Vinner, 1989). Among those who succeed in mathematics, the algebraic mode is more common when solving routine problems. Vinner gives two possible reasons for algebraic bias or visual avoidance:

- An algebraic proof is believed to be more mathematical and more general, and hence in an examination it is preferable to be 'on the safe side' by taking the algebraic option, despite the clarity, simplicity, immediacy, etc. of a visual 'proof' (examples of visual 'proofs' are discussed in Section 2.4.2 of this chapter).

- Preparation for an examination is very often by rote learning. Students give up meaningful learning and prefer to memorise formulae and algebraic techniques, an effective prescription for success in standard tests.

The second point, above, is possibly only true if the examination is procedure-biased (which most are). Thus it does not necessarily reflect students' desires, but rather what they perceive to be necessary to succeed.

Presmeg and Bergsten argue, however, that while there is research evidence that in classes of 'non-visual' teachers even 'visual' students will suppress their preferred visual modes in favour of non-visual methods used by their teachers, data show that it is simplistic to claim that students are reluctant to visualise (Presmeg and Bergsten, 1995). They are not necessarily reluctant to visualise; they are merely following the example set by their teacher. This is known as 'teacher privileging' (Kendal and Stacey, 1999), although Kendal and Stacey discussed this in the context of the use of technology in the classroom.

Currently within mathematics education, success is essentially measured by routine problems which do not require visual ability. The need for visual ability is usually minimal. Visual thinking is an important part of higher mathematical thinking, and so it should be taught and tested (Vinner, 1989). Graphical interpretations and considerations have a crucial role in understanding, and provide reinforcement of algebraic methods. Vinner explains how visual considerations are always illuminating even if not taken as a mathematical proof, and are indispensable in courses such as calculus. He emphasised, however, that changing cognitive structures and attitudes would be extremely difficult, and radical steps would need to be taken before a meaningful change could be expected.

Visualisation is a key part of understanding, and students who operate with very few mental pictures are not really learning mathematics (Hughes-Hallett, 1991). They work with a vast series of algorithms and a complicated cataloguing system which tells them which procedure to use, and when. Hughes-Hallett believes that such forms of teaching and learning merely perpetuate the idea that mathematics involves doing calculations rather than thinking. For example, there are many students who can calculate derivatives of extremely 'messy' functions but who cannot look at a graph and identify where the derivative is positive and where it is negative. Unfortunately, it seems that educators have led students to believe that "*real mathematics consists entirely of the skilful manipulation of x's*" (Hughes-Hallett, 1991).

Zimmermann (1991) stated that most of the concepts of, and many problems in, calculus can be represented graphically, however geometrical reasoning had been used inconsistently at best, and the role of visual thinking had not been seriously addressed. Few of the examples or problems were designed to develop students' ability to represent or solve problems graphically. The growth of interactive computer graphics opened up new realms of possibilities for visualisation in calculus and other fields. Zimmermann described how symbol manipulation had been overemphasised, and in the process the 'spirit' of calculus had been lost.

Several Higher Education institutions in America have since designed a new calculus course emphasising a graphical approach. The designers believed that wherever possible, topics should be taught graphically and numerically, as well as analytically (Hughes-Hallett, 1991; Zimmermann, 1991). The object was to produce a course where these three points of view are balanced, and where students see each major idea from several angles. This attempts to build conceptual understanding rather than just manipulative skill. Creating a web of connections between different representations is an essential part of understanding. The kind of conceptual understanding that develops from seeing the connection between the algebraic and the graphical does not come easily to students, particularly since it has seldom been asked of them in the past. Nevertheless, educators must strive toward providing students with opportunities to develop this kind of understanding (the software employed in the case-study in Chapter 5 achieves this ideal, as it focuses on precisely this development of conceptual understanding by exploring the relationship between graphical and symbolic representations). Hughes-Hallett (1991) stated: "*Getting students to think has to be our top priority. Thinking means being able to look at problems from several points of view, one of which must be visual. Consequently, if we are to be successful in reviving calculus, we must move graphs from their supporting, extra-credit role to centre stage*". This change in the approach to teaching calculus has been the subject of much debate (Jackson, 1997a; Jackson, 1997b), but reports have shown that the adoption of such an approach has indeed been successful (de Alwis, 2002; McDill and Rash, 2002).

Consideration is now given to the types of mental images that are created by learners whilst subjected to pictorial information. The following example illustrates that visual images are interpreted abstractions rather than direct encodings of visual information. Fig. 2.05 below shows pictures presented by Slezak (Dreyfus, 1995), which are ambiguous in the sense that they are recognisable as one object in one orientation but as a different object when rotated by 90 degrees.

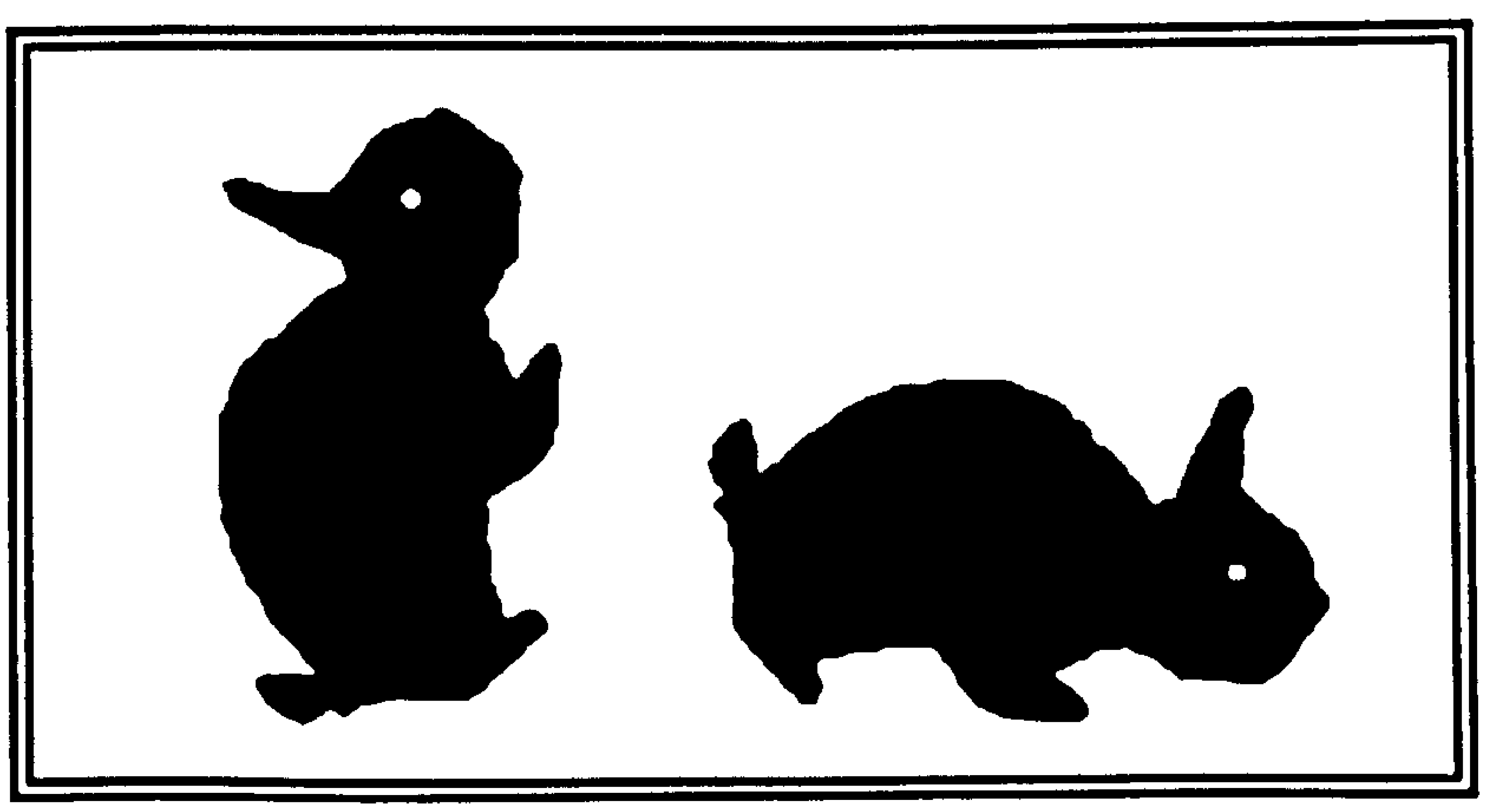
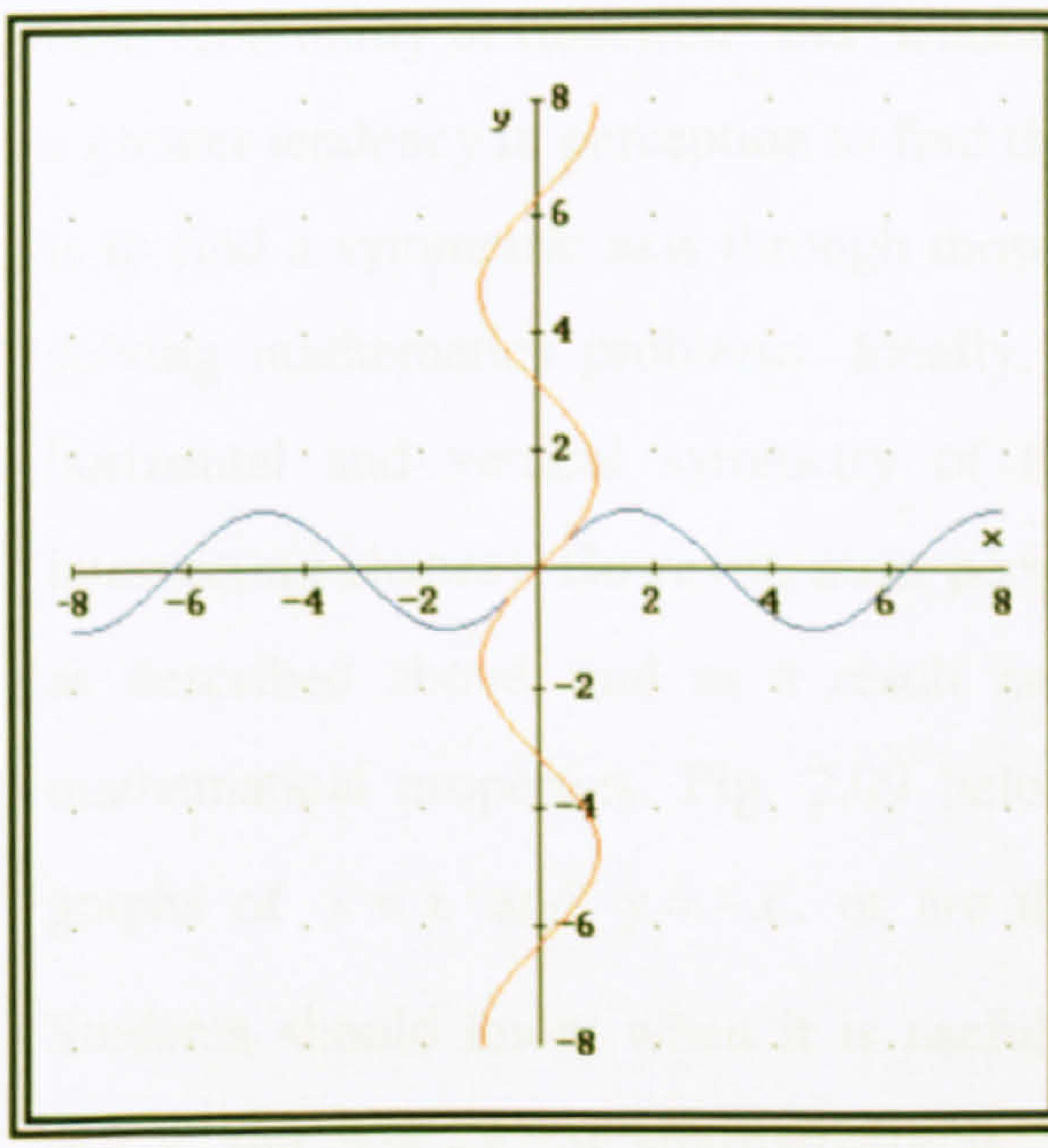


Fig. 2.05.

Students were shown the picture of the duckling and asked to memorise it. The picture was then taken away, and the students were asked to imagine rotating it 90 degrees clockwise. They typically described it as a duckling on its back, whereas the 'actual' rotated picture could clearly be interpreted as a rabbit. This shows us that image reconstruction is extremely difficult, and that images are bound to a certain interpretation. This could be a problem in certain aspects of mathematics, for example the interpretation of graphs of inverse functions. Let us consider the graphs of sin(x) and arcsin(x) (in Figs. 2.06 and 2.07, sin(x) is depicted in blue, and arcsin(x) is depicted in red). The graph of arcsin(x) can be constructed in an instructivist manner by simply finding the corresponding y-values for various x-values. It would be useful, however, for students to appreciate that the inverse function is a rotation and reflection of the original graph of sin(x). Unfortunately, problems arise with this method of construction. A graph would be produced where each x-value ($-1 \leq x \leq 1$) maps to many y-values, i.e. no longer a well-defined function, as in Fig. 2.06. The properties of the original graph of sin(x) therefore do not carry over to the graph of arcsin(x). Instead, a different interpretation must be applied involving the generic definition of a function in order to maintain the original functional properties, thus arriving at the graph in Fig. 2.07 where each x-value maps to one y-value.



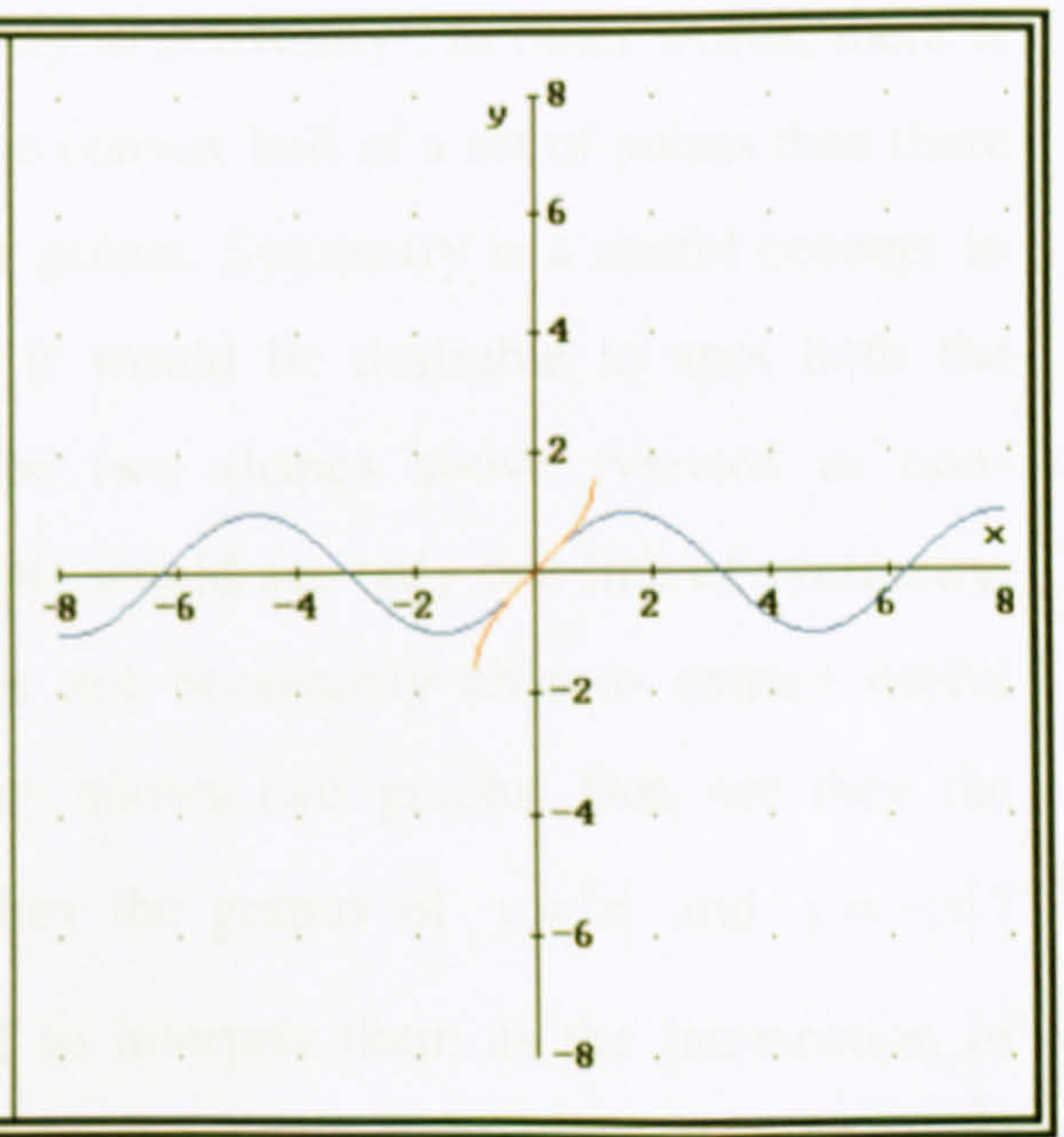


Fig. 2.06. **Fig. 2.07.**

Kanizsa (Shire, 1992) considers some interesting diagrams that require careful visual interpretation. The first diagram is Fig. 2.08 below:

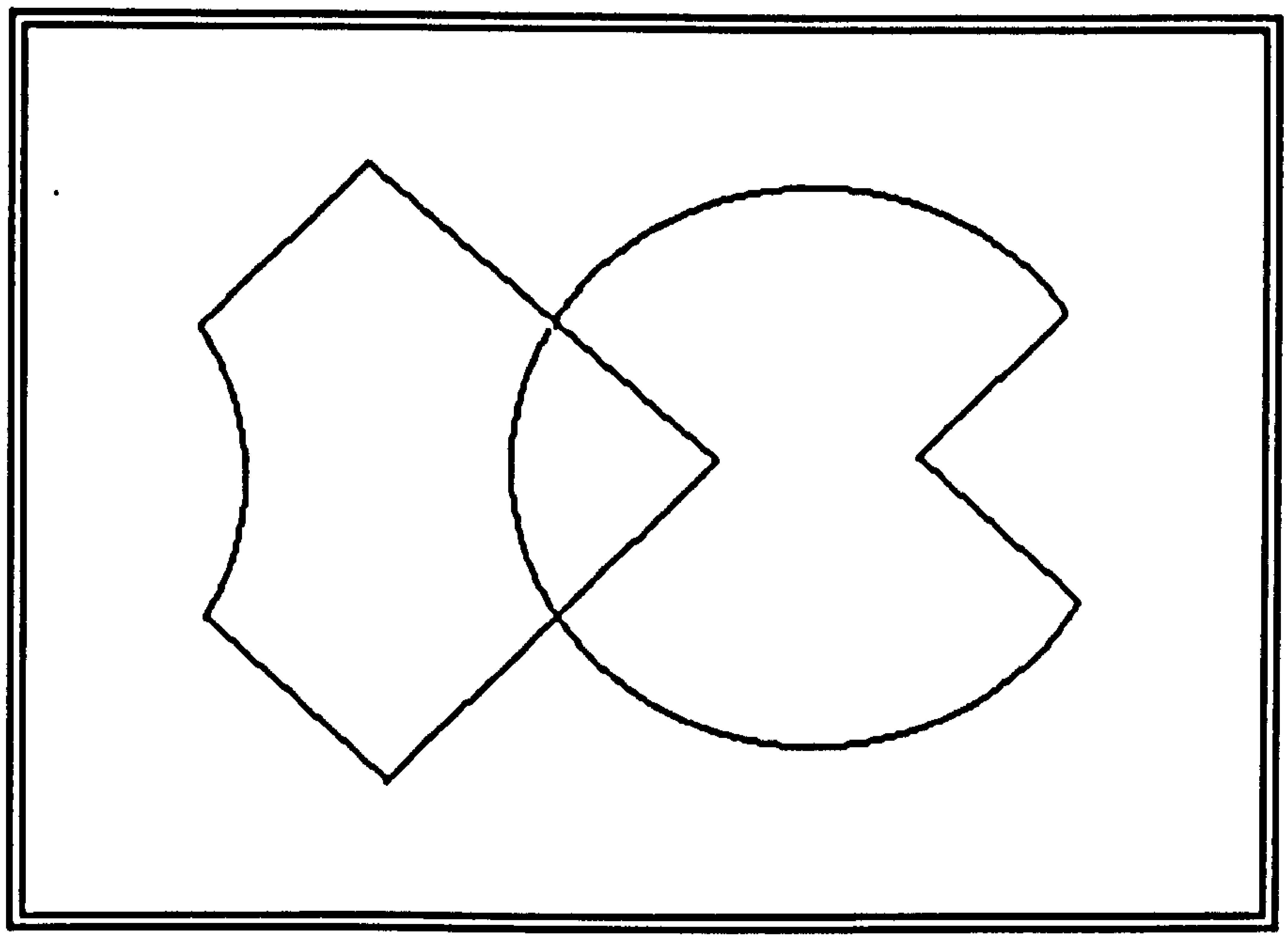


Fig. 2.08.

Why is the overlap of two convex shapes seen, rather than the adjacency of two symmetric shapes? Shire describes how there is always a strong tendency towards symmetry in thought, however in perception it can be shown to be weaker than both 'continuity of direction' and 'tendency to convexity'. In other words, there is a greater tendency in perception to find the convex hull of a set of points than there is to find a symmetric axis through those points. Symmetry is a useful concept in solving mathematics problems. Ideally, it would be desirable to spot both the horizontal and vertical symmetry of the two shapes above (viewed as non-intersecting shapes). However, most people would see only one line of symmetry, as described above, and as a result are not necessarily able to extract useful mathematical properties. Fig. 2.09 below shows two graphs. But, are they the graphs of $y=x$ and $y=-x$, or are they the graphs of $y=|x|$ and $y=-|x|$? Students should know when it is useful to interpret them as the intersection of $y=x$ and $y=-x$, or alternatively as $y=|x|$ and its reflection in the x-axis (a Group C skill, as defined in the MATH taxonomy earlier).

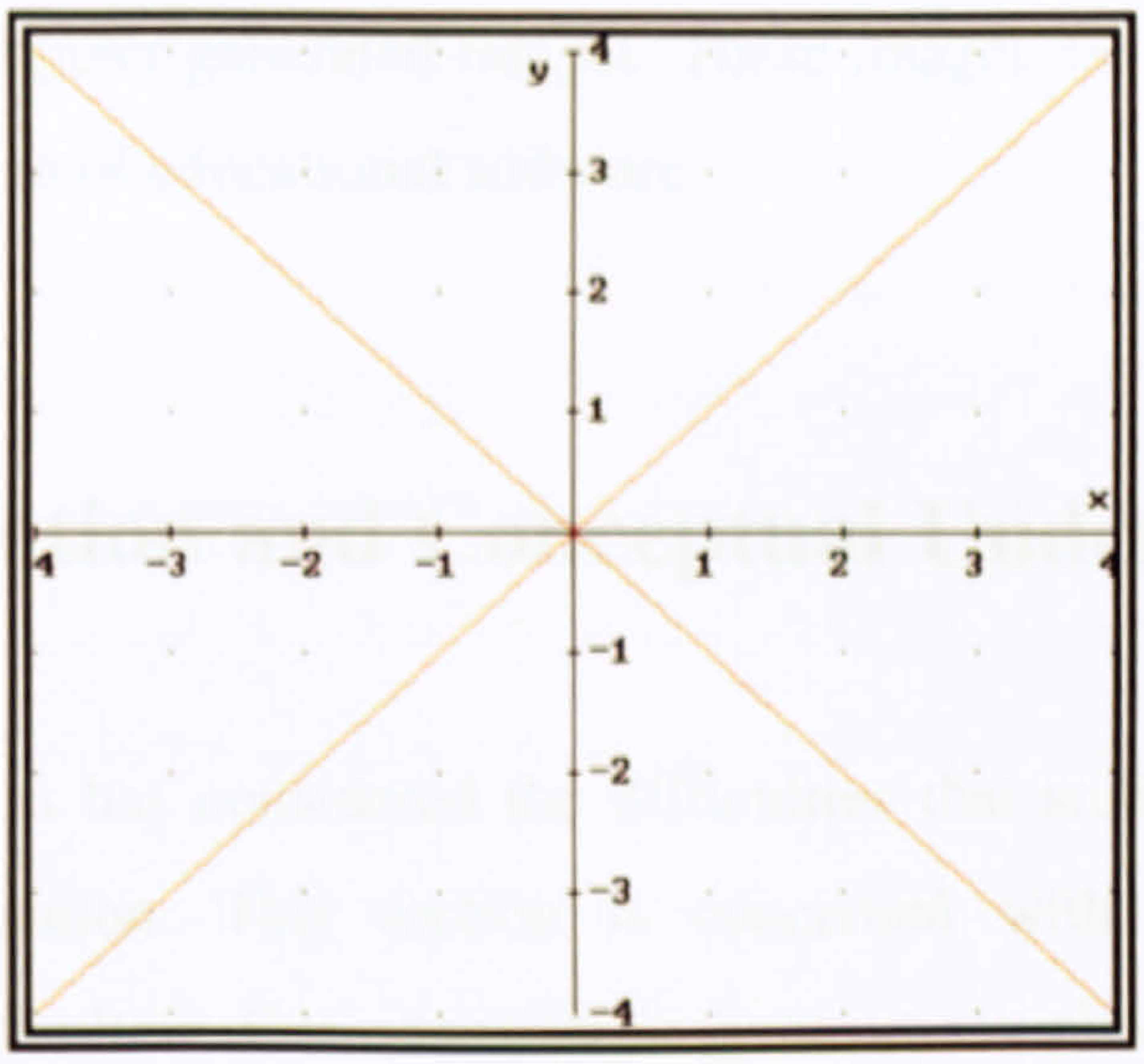


Fig. 2.09.

Kanizsa (Shire, 1992) also provides Fig. 2.10 below which asks the viewer if he or she interprets the small figures inside the rectangles as squares or diamonds.

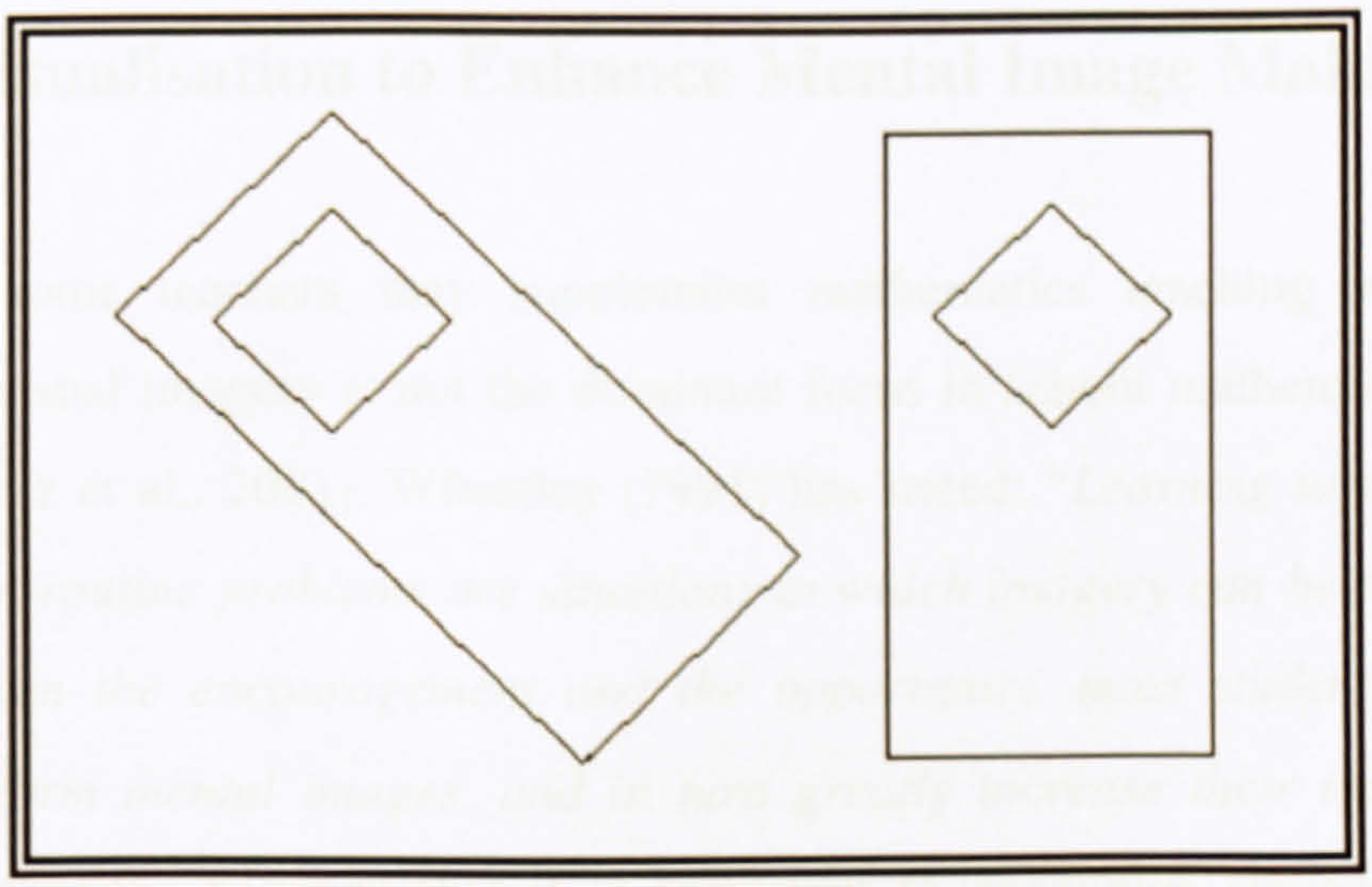


Fig. 2.10.

The appearance of a figure depends upon its orientation with respect to the nearest frame of reference (in this case the rectangles) when determining whether an object will be seen as a square or a diamond.

Shire explains that by looking at how people draw together various perceptual cues we may shed some light upon features and techniques that will assist in the development of computer generated images. These images can then be used in the thoughtful production of educational software.

2.4. Visualisation and Conceptual Understanding

The previous section has considered the difficulties that students can encounter with visual information. This section is concerned with how students can overcome such difficulties. Consideration is given to visualisation as a powerful tool for generating mental images leading to conceptual understanding, and the types of mental images that are created from the use of both appropriate and inappropriate diagrams.

2.4.1. Visualisation to Enhance Mental Image Making

Although some teachers may supplement mathematics teaching with visual activities, visual imagery is not the dominant focus in school mathematics classes (Hershkowitz et al., 2001). Wheatley (1991) has stated: "*Learning new ideas and solving non-routine problems are situations in which imagery can be particularly useful. Given the encouragement and the opportunity, most students have the ability to form mental images, and in turn greatly increase their mathematical power*". Wheatley believes that it is important to encourage students to build mental pictures and communicate their images by building models (drawing pictures and graphs), and design activities that promote the use of visualisation. The manipulation of mathematical concepts is facilitated considerably by the mental construction of appropriate images (Dörfler, 1991).

It is important for students to recognise that a diagram may contain information needed for the solution of a problem and to develop the habit of looking to diagrams as a source of such information. Technology can play an important role here. Computers can allow students to explore and experiment with graphics. However, guidance, feedback, and a synthesis of important results must be built into the process (Zimmermann, 1991). Students must be able to interpret computer-generated graphics with understanding, as they are subject to misinterpretation. For example, on a small enough scale, all smooth functions look linear. Additionally, educators must be aware of whether or not the student has 'seen' the whole picture. For example, a graphic calculator plot of $y = \frac{1}{x(2-x)}$ could well give the graph in Fig. 2.11 below, with vertical lines drawn at $x = 0$ and $x = 2$ as part of the graph. Many students would copy this without hesitation, and would therefore store this mental image in error, instead of more appropriately interpreting the superfluous lines as vertical asymptotes.

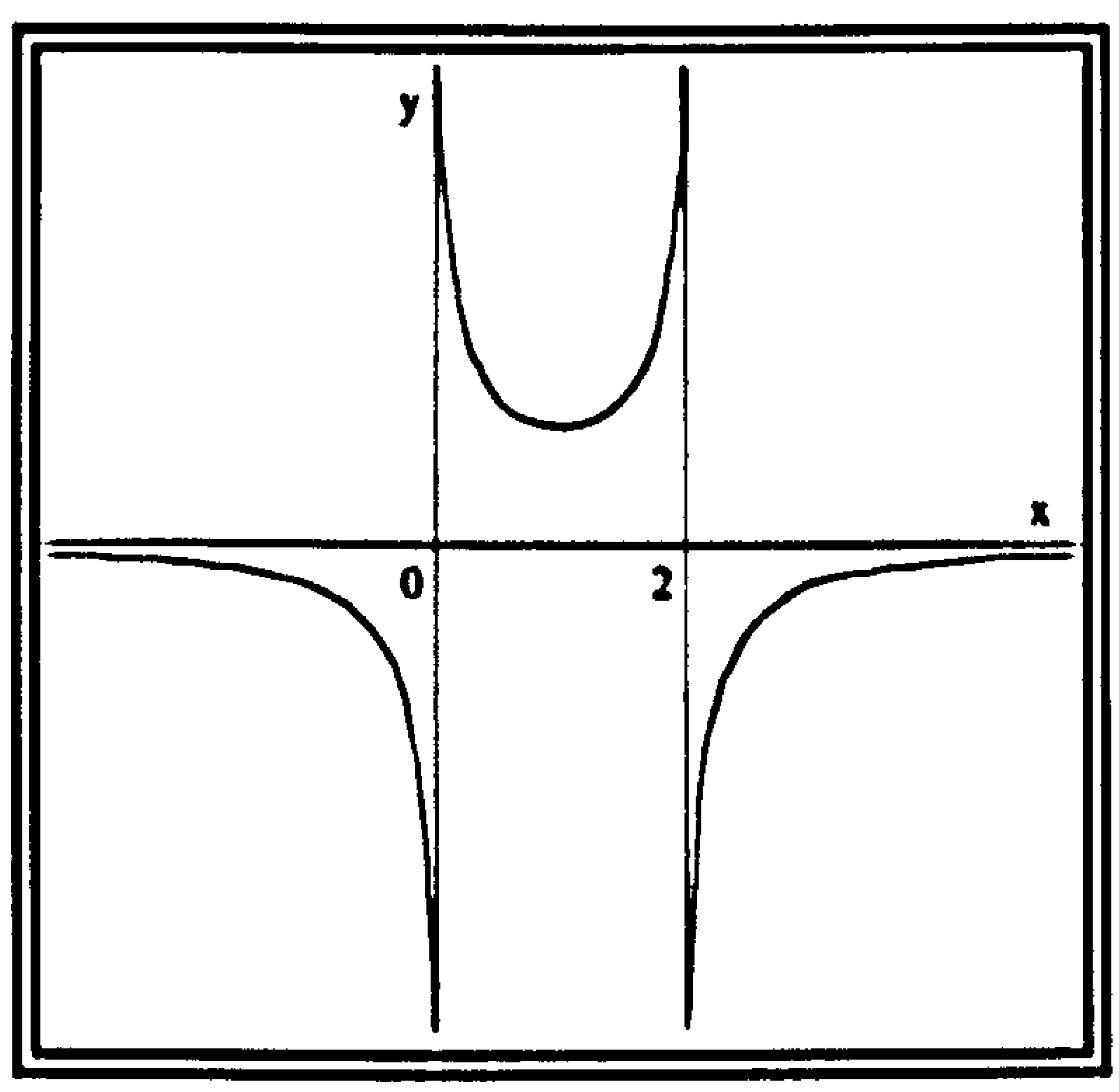


Fig. 2.11.

Zimmermann (1991) explains that an important component of visual thinking is to recognise incorrect or misleading graphics, and to make an appropriate interpretation (Giraldo et al., 2002). Part of the challenge for the educator is to ensure that the students are receiving the intended message.

It is no accident that when we understand something we often say that we 'see' it. The visual side of the learning process is one that complements the symbolic representation, and flexible graphical packages provide us with very powerful tools to enhance this much-neglected aspect (Tall, 1987). As a consequence of the visual capabilities of the computer, educators have developed interactive mathematical software that not only illustrates algebraic forms, but also uses visualisation as a vehicle for mathematical discovery. In this manner, students can conjecture mathematical facts through the process of visualisation prior to any algebraic manipulation, instead of merely following a mathematical proof with an appropriate illustration. Since Tall's reported work in 1987, many educators have taken steps to utilise computing capabilities in mathematics education, but it is still not clear how to optimise their usage with respect to learning.

Pen-and-paper tasks normally require a fixed answer to a fixed question, whereas the computer can activate dynamic visual thinking. Sutherland (1995) presents a powerful example of this, in which she describes how two boys worked on the following problem:

In Fig. 2.12 below, with point A fixed, where should point B be placed so that triangle ABO has the largest possible area?

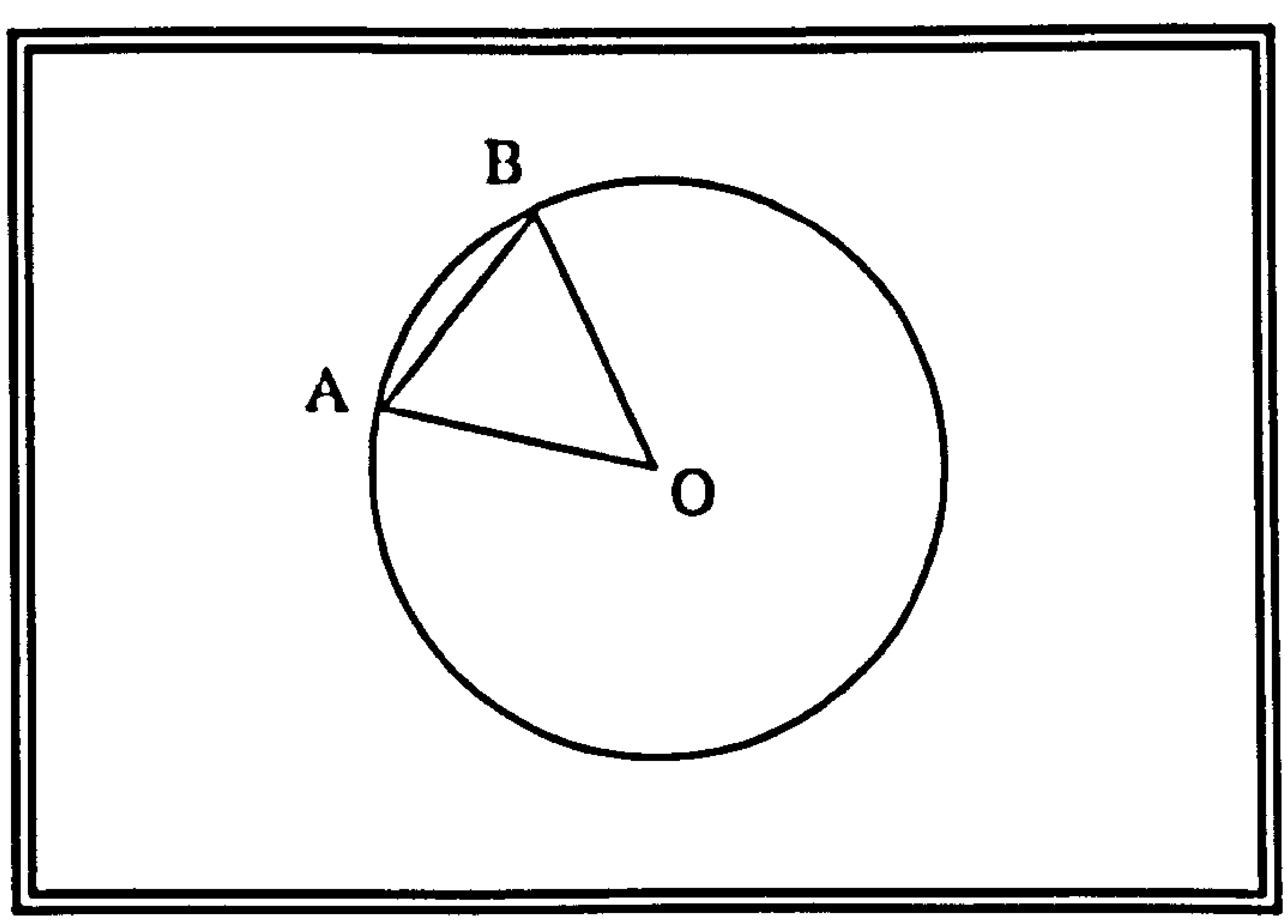


Fig. 2.12.

The boys thought for a short while, and then the teacher suggested that they construct the height BD of triangle ABO, as in Fig. 2.13 below:

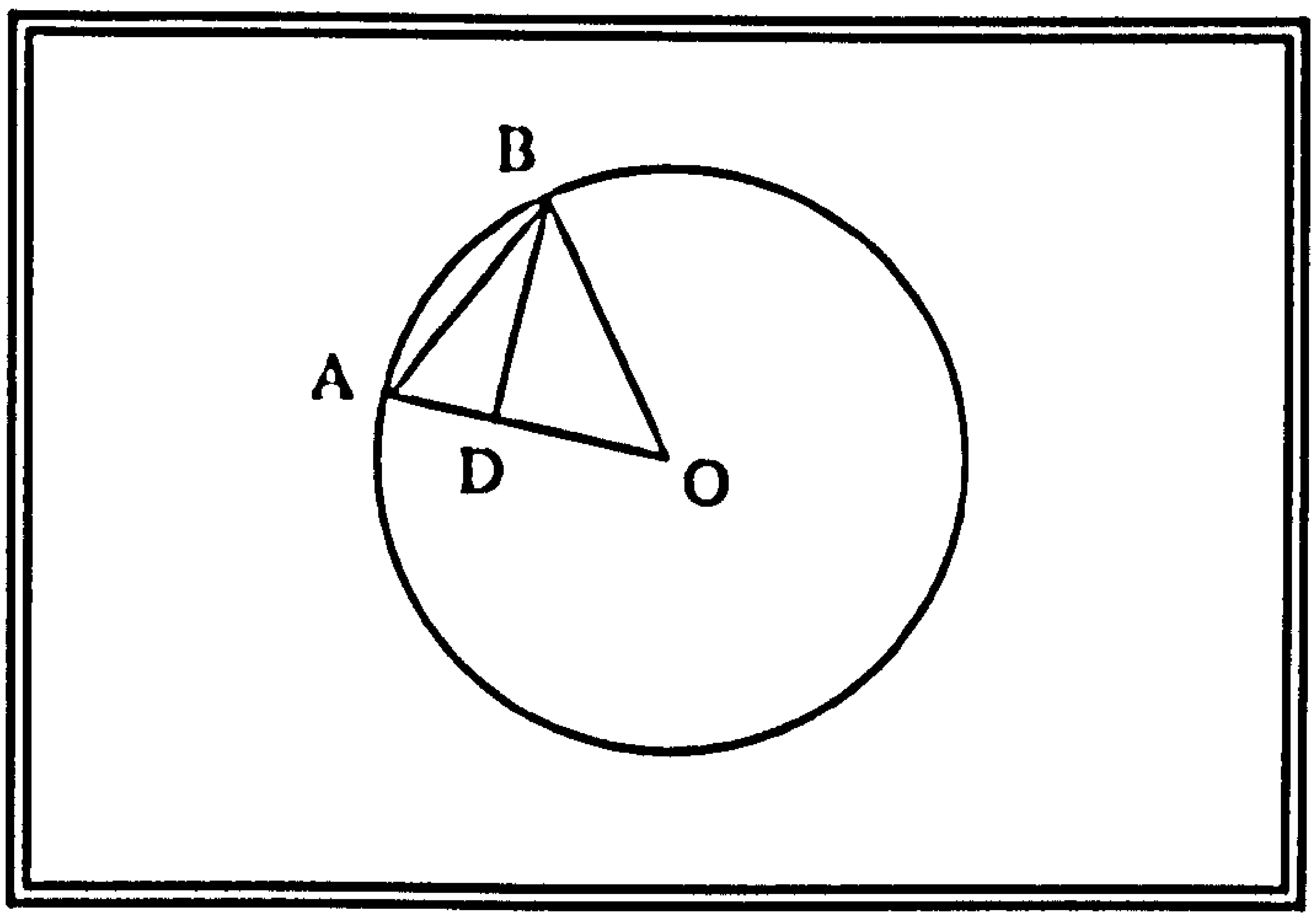


Fig. 2.13.

They then proceeded to drag point B around the circumference of the circle, and very quickly realised the solution (when BO is perpendicular to AO).

The visual aspect here played an important role in the solution process. This is a good example of how ***visualisation and interactive exploration*** can lead to the solution of a problem, and can lead to the realisation of mathematical concepts (in this case, the concept of maximum area given certain constraints). It is precisely this philosophy that this thesis adopts.

The order of teaching via different representations is also of interest when considering the enhancement of mental image making. Traditionally, the learning of graphs of functions usually occurs after a long period of numerical and symbolic manipulations and is normally introduced as a final stage of the subject. Yerushalmy and Schwartz (1993) believe that it is likely that certain difficulties observed in the understanding of functions in various representations (numerical, visual and symbolic) might be grounded in this form of learning. It would seem that this sequential learning of the symbolic followed by the graphical does not promote a tendency on the part of the learner to move between alternate representations. Each of the representations presents a separate system for the learner, with no mutual and constructive interaction between them. It is important that the visual complements the symbolic, as one form in isolation is not enough for students to conceptualise. Educators therefore need to encourage more visual methods of teaching, either in conjunction with symbolic methods or as a precursor to symbolic formalism, in order to improve students' visualisation skills (Tall, 1991), which are of great value in the pursuit of conceptual understanding.

Graphical packages allow for a more innovative strategy whereby new concepts can be approached visually and intuitively *before* the need for symbolic analysis (Tall, 1987). The symbolic structures themselves, and the links to the visual representations still need to be established, but they can now be encountered in a context where the student has some overall idea as to what the new concept represents. The successful construction of concepts thus requires the co-ordination of different representations, and unsuitable construction will impede conceptualisation (Hitt, 1998a, 1998b).

Tall has explained that his quest is to use the computer to visualise mathematical concepts in helpful ways. He carried out a study in which computer-based graphical techniques were employed to enhance students' ability to visualise (Tall, 1995). The participants in the study were mathematics education students who were planning to be teachers. He explains that it wasn't expected that they would obtain greater facility with any formal aspects, but that they were more likely to be able to visualise and discuss the concepts. This proved to be the case, and the level of discussion was mature and insightful. Tall concludes by saying that the discussions indicated a level of visualisation and verbalisation far greater than had been traditionally expected in earlier courses, and this did not just apply to a minority of the students.

A cognitive approach to the calculus has been described in which computer graphics are used to enhance learning (Tall, 1986b, 1995, 2000a). Although the first of these reports was almost twenty years ago, the ideas presented are still important, and there is still little evidence that they have been incorporated into the school curriculum. Through experiences that enable the students to develop suitable mental images of the mathematical concepts, Tall explains how students come to see specific examples (single entities) as generic examples (representatives of a class of examples), which in turn help in the abstraction of the general concept. For example, instead of beginning the theory of differentiation with a discussion on limits (formal approach), one may present the notion of the gradient of the graph (cognitive approach). This can be greatly assisted by using computer graphics to show a tangent moving over a curve, and at the same time plotting the gradient function. The learner can explore examples of mathematical processes and concepts. By manipulating a number of examples, their common characteristics may be abstracted to give the general concept that is embodied in the examples. Concepts can thus be built up from familiar examples. Tall has observed that, via this approach, students are far more positive in their attitudes, and are significantly better at sketching gradients, recognising gradient functions, and discussing concepts.

Materials have been developed for use with the TI-92, which are aimed at promoting students' abilities to visualise the graphs of functions (Elliott, 1998). The study was carried out with a class of thirteen year twelve students (5 male, 8 female). The exercises involved graphing functions, and exploring and identifying the effects of transformations, finding inverse functions, solving equations graphically and algebraically, and investigating trigonometric and logarithmic functions. Elliott explains how technology can assume a very powerful and influential role in stimulating and shaping students' powers of visualisation, and as such may prove to contribute significantly to the depth of students' understanding. A questionnaire revealed that students thought technology provided a quick and accurate means of strengthening their visualisation ability, and reinforcing their understanding of functions. Usage of the TI-92, reported by Elliott, is more limited than the software usage described in Chapter 5, as the direction of information transfer is primarily from the symbolic to the pictorial, whereas the software in Chapter 5 forces the transfer between representations in both directions.

Before the advent of computers, visualisation had been carried out with the use of sketches and drawings on a blackboard, and the static nature of pre-designed graphics in textbooks limits their usefulness in teaching. Animations, however, introduce a new dimension, and bring with them the possibility of presenting processes, such as a transformation, in a dynamic manner. It is also possible to illustrate a varying parameter or the construction of an object. Colour can improve the expressiveness of visualisations, and can also be used to distinguish different objects in the same picture.

Dynamic mental imagery is concerned with how people manipulate images in the mind. It is clear that some people have the ability to move objects about in their heads, and some clearly do not (Goldenberg, 1995). However, Goldenberg believes that the people who do not have this ability could acquire the necessary skills if given appropriate experiences. He explains that some students can perform dynamic experiments mentally without prior experiences of similar experiments with their hands and eyes. Other students, however, might well learn to perform such experiments like these 'mentally' if they had appropriate experiences of performing them first. Modern technology can now allow students to get a 'feel' for the dynamics of experiments. An interactive computer environment, particularly when dynamic visual images (i.e. moving pictures) are employed, can encourage and to some extent develop students' visualisation abilities (Bishop, 1989). Research in the development of visualisation ability is considered to be extremely valuable (Goldenberg, 1995), and it is with this area of enhancement that this research is primarily concerned.

A report of the Board on Mathematical Sciences, USA (National Research Council, 1990), identified a number of research accomplishments and related opportunities in mathematics. One of the areas cited was "Computer Visualisation as a Mathematical Tool." According to this report, "*In recent years computer graphics have played an increasingly important role in both core and applied mathematics, and the opportunities for utilisation are enormous*" (Zimmermann and Cunningham, 1991). However, this report was produced over a decade ago, and although much has been reported throughout the literature on the incorporation of graphical technology in Higher Education, it is still not in widespread use in schools.

Graphic images can also serve as an important link between mathematical models and the phenomena of the real world. The ability to interpret simple physical processes graphically is an important aspect of visual thinking (Zimmermann, 1991). Zimmermann believes that dynamic computer visualisations have the potential to reveal change vividly and directly, and could thus have a major influence on the teaching and learning of calculus concepts for example.

The use of a CAS allows the focus to be shifted from procedural skills to expressing a problem in a mathematical context (Maeder et al., 1995). An area where these possibilities are especially welcome is that of differential equations. Whilst studying differential equations, emphasis is put on sophisticated solution approaches, and in textbooks they are classified by their solution methods instead of their application fields. Reported work with the CAS Mathematica has shown how the focal point can be moved towards the application, whereas the solution and visualisation of the differential equation can be left to the computer (Maeder et al., 1995; Garcia et al., 2002).

The benefits of visualisation include the ability to focus on specific components of very complex problems, to show the dynamics of systems and processes, and to increase intuition and understanding of mathematical problems and processes (Cunningham, 1991). For example, a student could be faced with a complicated function in terms of its symbolic form, and it may be very difficult to locate roots, turning points, etc. algebraically. A computer-generated graph of the function, however, would provide a means of extracting such information. The use of facilities such as the 'cursor position' and 'zooming' in order to obtain approximations of these features could be a precursor to solving an integration problem. The ability of computer algebra systems to combine symbolic and visual representations seems to offer computer-based mathematics the best of both worlds.

Informal studies with students indicate that students respond much more strongly to dynamic images than to static ones (Cunningham, 1991). These may involve showing a display as it is computed, which can be done so that the order of development illustrates the mathematical processes shown. An example of this is the production of displacement-time and temperature-time graphs (Graham, 1996; Gretton and Challis, 1996, 1999) via the CBL (Computer Based Laboratory), a product from TEXAS INSTRUMENTS (see References for Internet address) that collects data and can be connected to sensors and to a graphics calculator for analysis. It is important that mathematics educators understand how to communicate mathematics visually (In-Service Education and Training (INSET) days could be developed to suit these needs). Cunningham (1991) states that an instructor using visualisation must:

- determine exactly the critical mathematical details to be presented in an image and show these either by highlighting them or by removing conflicting information.

- determine the order in which material is to be demonstrated by the images and present this material in a logical and connected sequence.

- offer students options in ways that expand their mathematical knowledge without confusing or overwhelming them.

- look for opportunities to present dynamic or developing mathematical processes and give students appropriate opportunities to explore or control them.

- consider carefully how students will learn visually, how to evaluate such learning, and how to integrate this learning with other parts of their mathematics studies.

A methodology is required that can support and nurture these issues, hence the need for constructivism. The constructivist philosophy acts as a vehicle for putting them into practice. The employment of visualisation in mathematics education from a constructivist perspective is discussed in greater detail in Chapter 3.

2.4.2. Creating Mental Images for Students

Mental images are very difficult to analyse since there is no direct access to them. The mental image cannot be measured without being disturbed. Attempts to measure a person's mental images are based on interaction with this person, and to magnify these measurement problems, mental images tend to be inherently vague (Dreyfus, 1995). You cannot express a vague image without making it precise; therefore the very nature of the mental image is destroyed in the measurement process. The best we can hope to do is assess the effectiveness of a person's mental images by measuring conceptual understanding after having subjected the student to some form of visual imagery, and see what effect these images have had on the learning process. The case-study in Chapter 5 confirms that visualisation exercises have indeed led to enhanced conceptual understanding.

The diagram in Fig. 2.14 below is known as a 'proof without words' (Eisenberg and Dreyfus, 1991). This diagram is not strictly considered to be a mathematical 'proof', but it is an excellent visual device for illustrating that the limit of the series $1/2 + 1/2^2 + 1/2^3 + ... + 1/2^n + ...$ is 1. This diagram provides the student with a useful, meaningful visualisation of the concept of the limit of this, and other convergent, series.

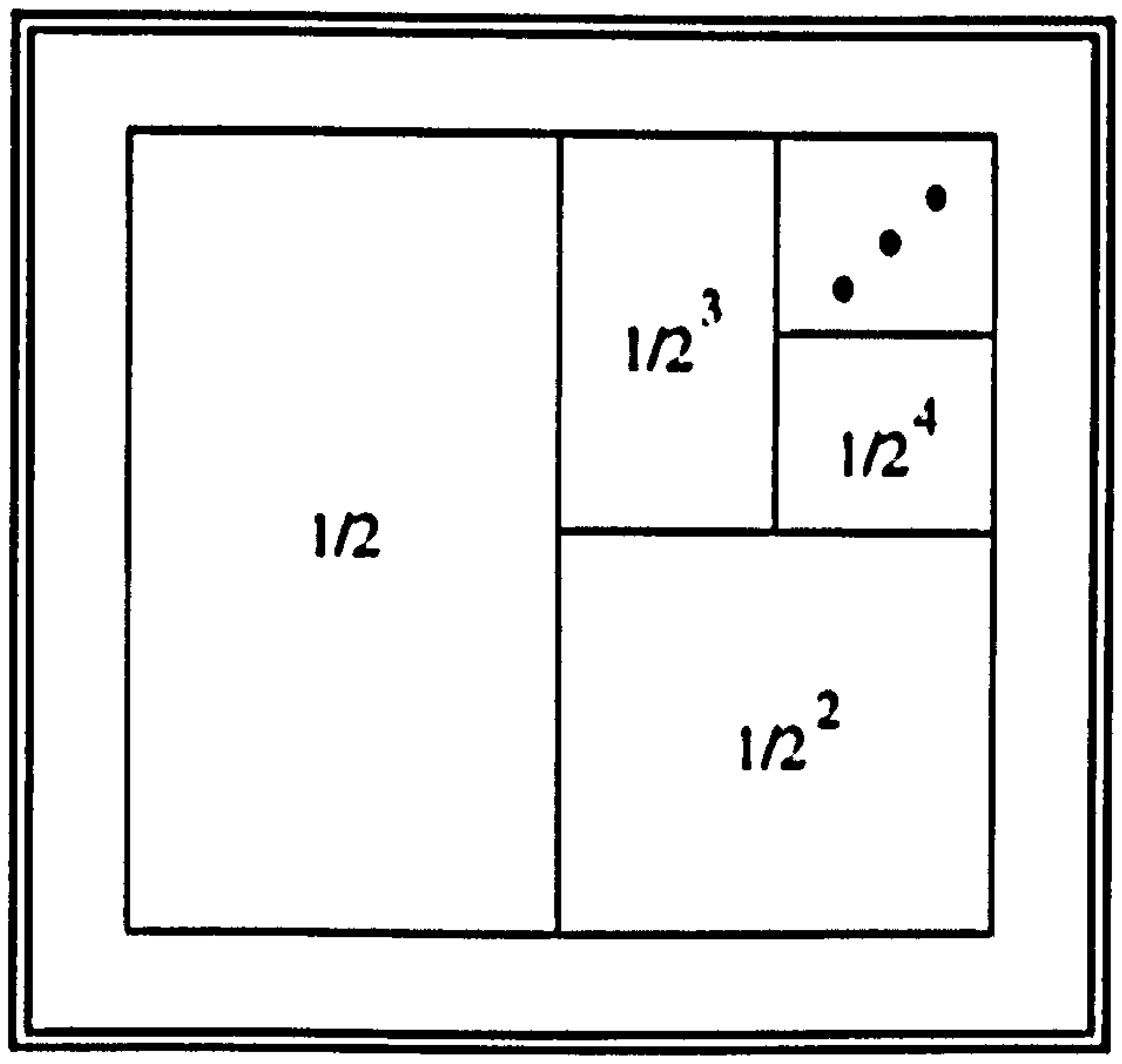


Fig. 2.14.

Visualisation can serve as a catalyst for understanding (Ben-Chaim et al., 1989), for example the generalisation for the sum of the first n odd numbers can be done visually and students can easily see why $1 + 3 + 5 + ... + (2n - 1) = n^2$ by considering Fig. 2.15 below:

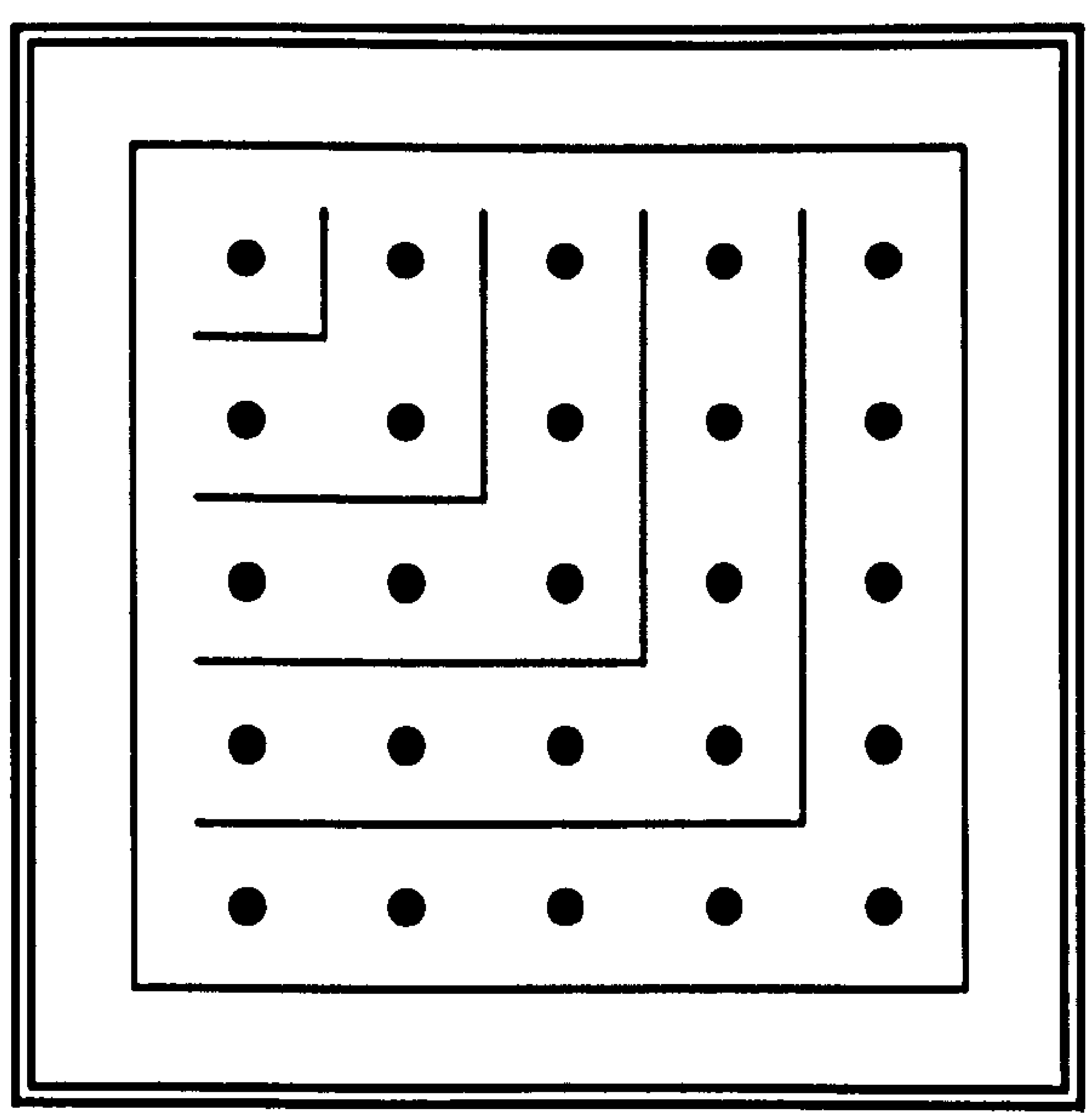


Fig. 2.15.

Most students would be able to understand this 'proof', whereas an algebraic proof, for example using knowledge of arithmetic series, would be beyond many. It would be interesting, however, to see whether or not students, given the picture, could conclude from it that $1 + 3 + 5 + ... + (2n - 1) = n^2$, i.e. use the visual device as a means of conjecturing symbolic results, rather than supporting earlier symbolic work.

Although in many cases a visual proof is seen as being not as rigorous as an algebraic proof, it can be shown that it is possible to give perfectly valid proofs using visual representations (Barwise and Etchemendy, 1991). An example given by Barwise and Etchemendy is that of the Pythagorean Theorem. One familiar 'proof' of this theorem involves a construction that first draws a square on the hypotenuse, and then replicates the original triangle three times as shown in Fig. 2.16 below.

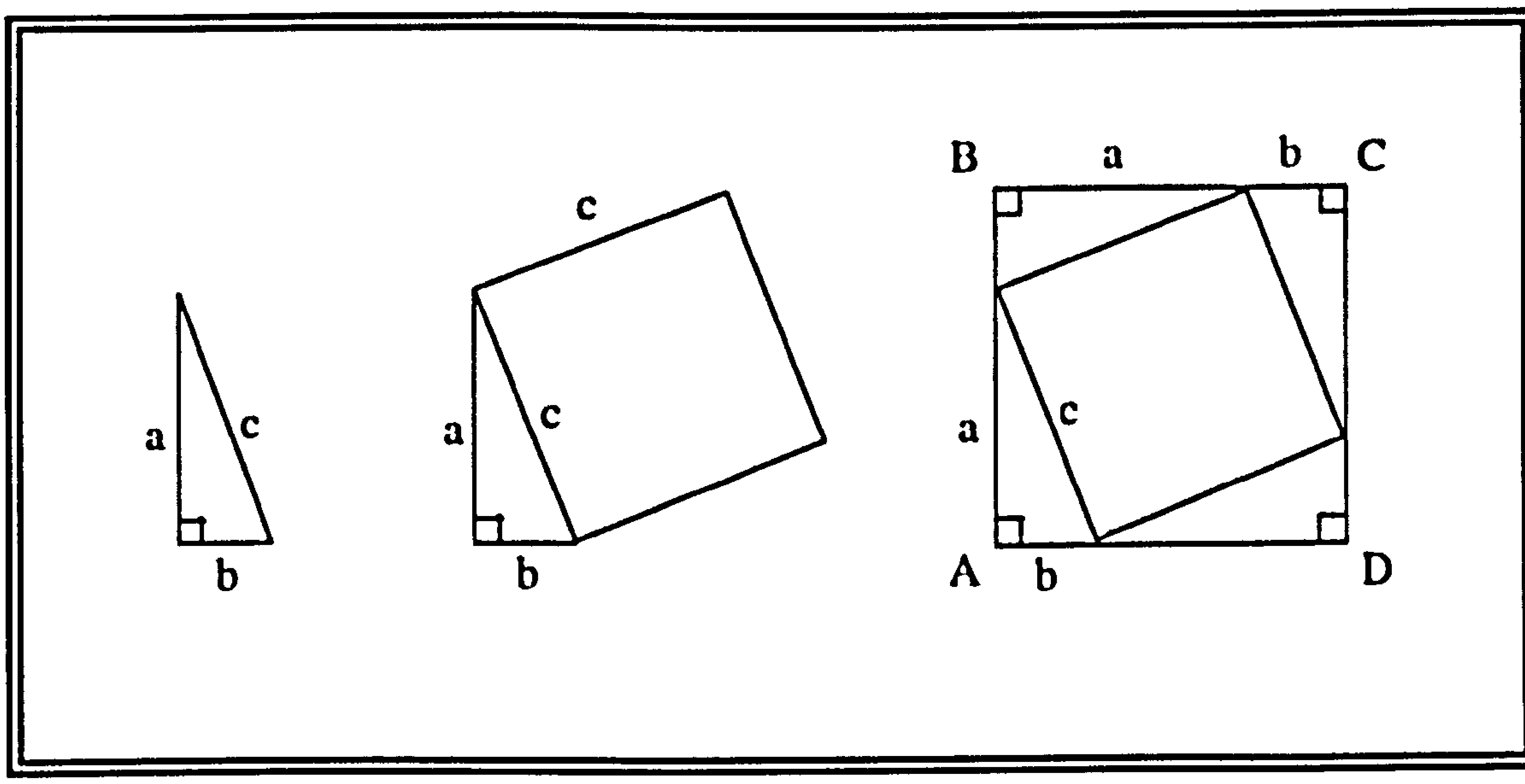


Fig. 2.16.

Using the fact that the sum of the angles of a triangle is 180^0 (i.e. a straight line), it can be easily seen (by most students) that ABCD is itself a square, one whose area can be computed in two different ways. On the one hand, its area is $(a+b)^2$. On the other hand, it can be seen by inspection that its area is also $c^2 + 4(ab/2)$. This gives the equation $(a+b)^2 = c^2 + 2ab$, which leads to the desired equation $a^2 + b^2 = c^2$.

It seems clear that this is a legitimate proof of the Pythagorean Theorem. Note, however, that the diagrams play a crucial role in the proof. Barwise and Etchemendy are not saying that one could not give an analogous (and longer) proof without them, but rather that the proof as given makes crucial use of them. To see this, it only needs to be noted that without them, the proof given above appears to have no rationale. This proof is an interesting combination of both geometric manipulation of a diagram and algebraic manipulation of symbols. Barwise and Etchemendy stress that both symbolic and pictorial representations have their place. They do not advocate that logical proofs should be anything other than rigorous. Rather, they advocate a re-evaluation of the doctrine that diagrams and other forms of visual representation are unwelcome guests in rigorous proofs.

Dynamic diagram build-up with the aid of a computer can lead to proofs that would otherwise be inaccessible and/or difficult to comprehend. An example of this is Langton's Ant (Stewart and Cohen, 1997). Stewart and Cohen describe how the ant lives on a grid of squares, which can be either black or white, and it obeys simple rules that determine the colour of each square. The rules are:

1. One pace forward.
2. If square moved onto is white, turn right.
3. If square moved onto is black, turn left.
4. Change colour of square you have just come from.
5. Go to step 2.

For the first five hundred or so steps, the ant keeps returning to the central square, leaving behind it a series of symmetric patterns. For the next ten thousand or so steps, the picture becomes very chaotic. Then, suddenly, almost as if the ant has finally made up its mind what to do, it builds a 'highway'. It repeatedly follows a sequence of precisely 104 steps that moves it two cells southwest, and continues this indefinitely, forming a diagonal band. The three distinct stages (Simplicity, Chaos, and Emergent Order) in the dynamics of Langton's Ant are illustrated in Fig. 2.17 below. These stages occur in spite of the fact that the ant follows the same rule throughout.

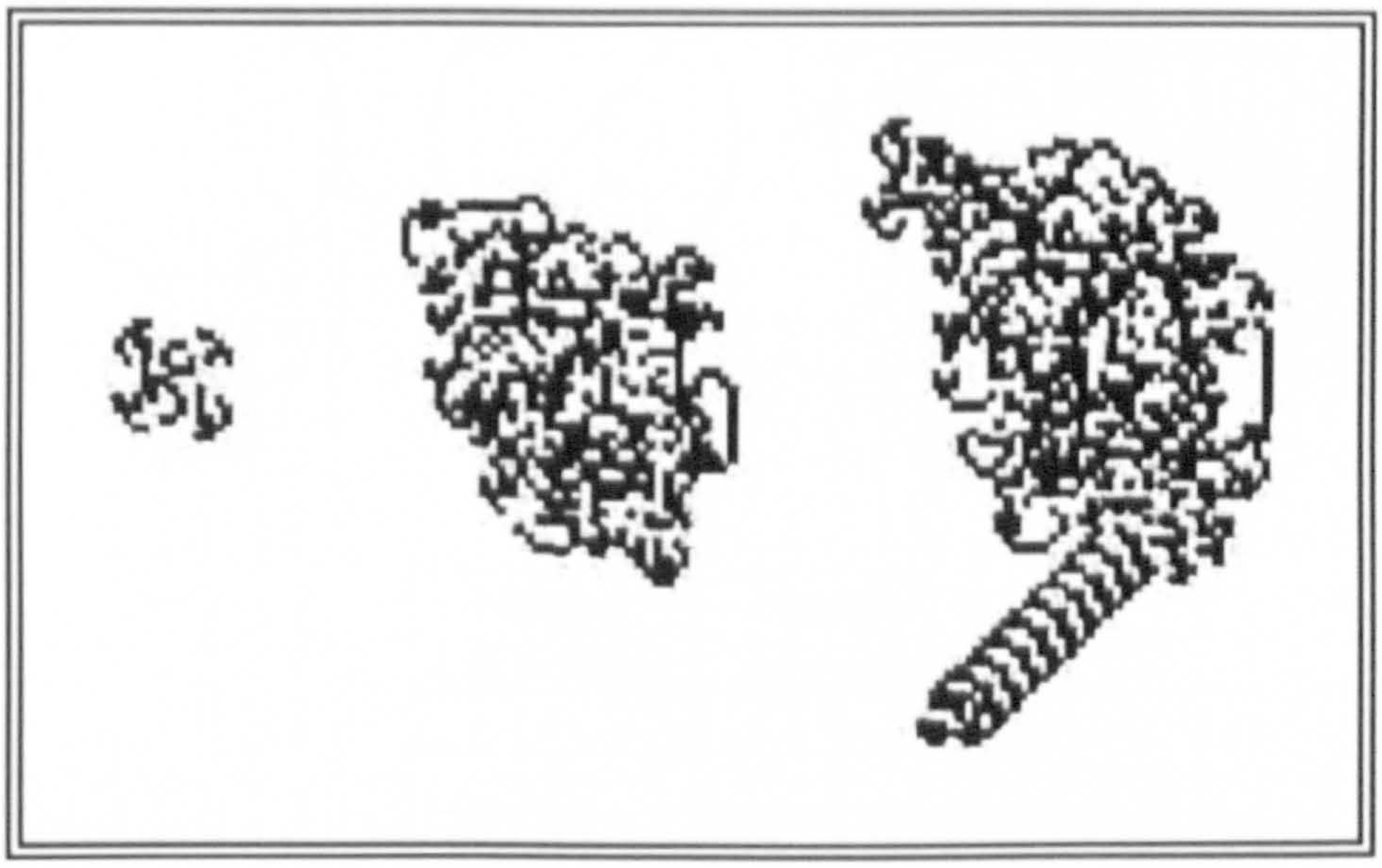


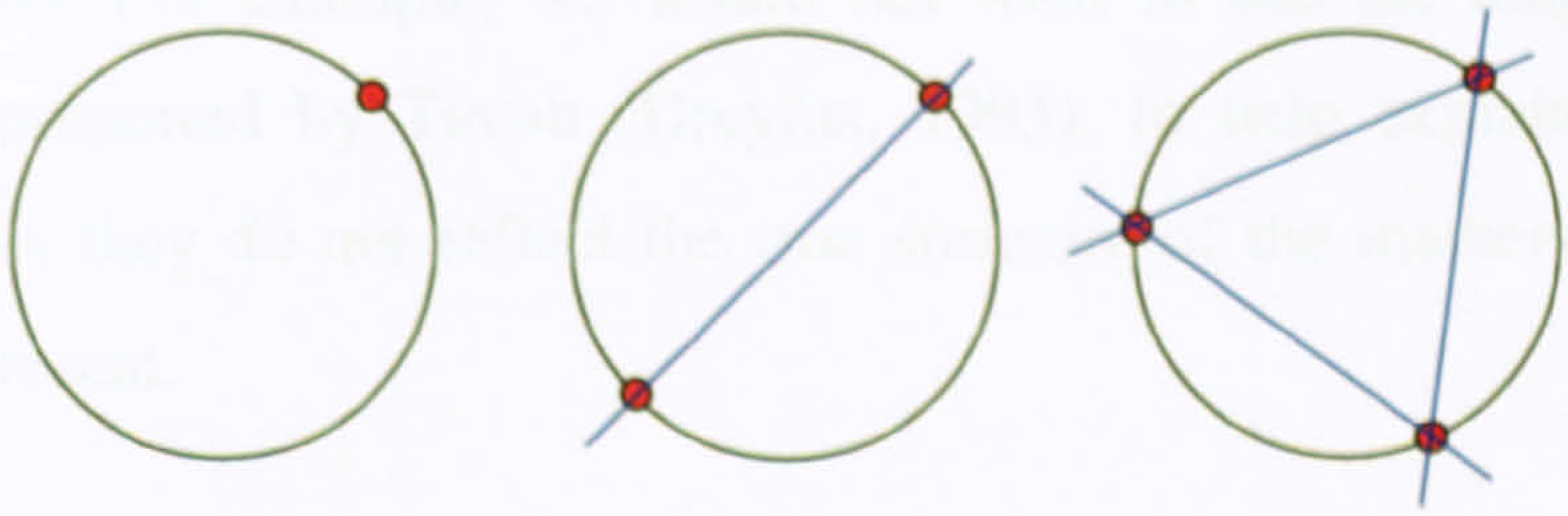
Fig. 2.17.

This large-scale feature emerges from the low-level rules. It makes an apparently irregular pattern although following fixed rules. It cannot be random, however, because the rule is known. It is not that there isn't a pattern to begin with – we just can't see it. Nobody has ever been able to prove why the ant always builds a 'highway', even if the initial conditions are changed by scattering black squares around the grid before the ant starts. This is an example of how the computer can facilitate the dynamic build-up of a process. It has provided the opportunity to spot a pattern, which would have been almost impossible analytically.

Conversely, there are instances where students can initially 'see' patterns that are *not* sustained in more complex cases. An example of this is the following 'pizza cutting' exercise:

What is the maximum number of pieces into which a circular pizza can be divided by applying straight cuts between *n* points on the circumference?

Initial pen-and-paper trials give the following results:



etc.

1 point 1 piece — 2 points 2 pieces — 3 points 4 pieces

Number of points	Number of pieces
1	1
2	2
3	4
4	8
5	16

At this point, most students would 'see' a pattern emerging, i.e. if number of points $= n$, then number of pieces $= 2^{n-1}$. However, continuing the table would produce the following unexpected results:

6	31 (not 32)
7	57 (not 64)

This is the opposite situation to that of Langton's Ant, as initial results in the 'pizza cutting' exercise point to a regular pattern, but ultimately result in irregularity.

Visualisation should be given a much greater priority by mathematics teachers (Ben-Chaim et al., 1989), and it is this use of visualisation to enhance the learning process on which this thesis concentrates. We must help the student to construct useful mental images. There is no point using diagrams, or dynamic imagery, just for the sake of it. It is important that the student is creating the right sort of images that can help in the understanding of mathematical concepts. It is essential that the student is provided with beneficial images that portray the concepts that we wish to get across. For example, we would not want to use the diagrams in Fig. 2.18 below, presented by Tirosh (Dreyfus, 1995), to help explain the concept of a 'third', as they do not reflect the true structure of the mathematical concept that they represent.

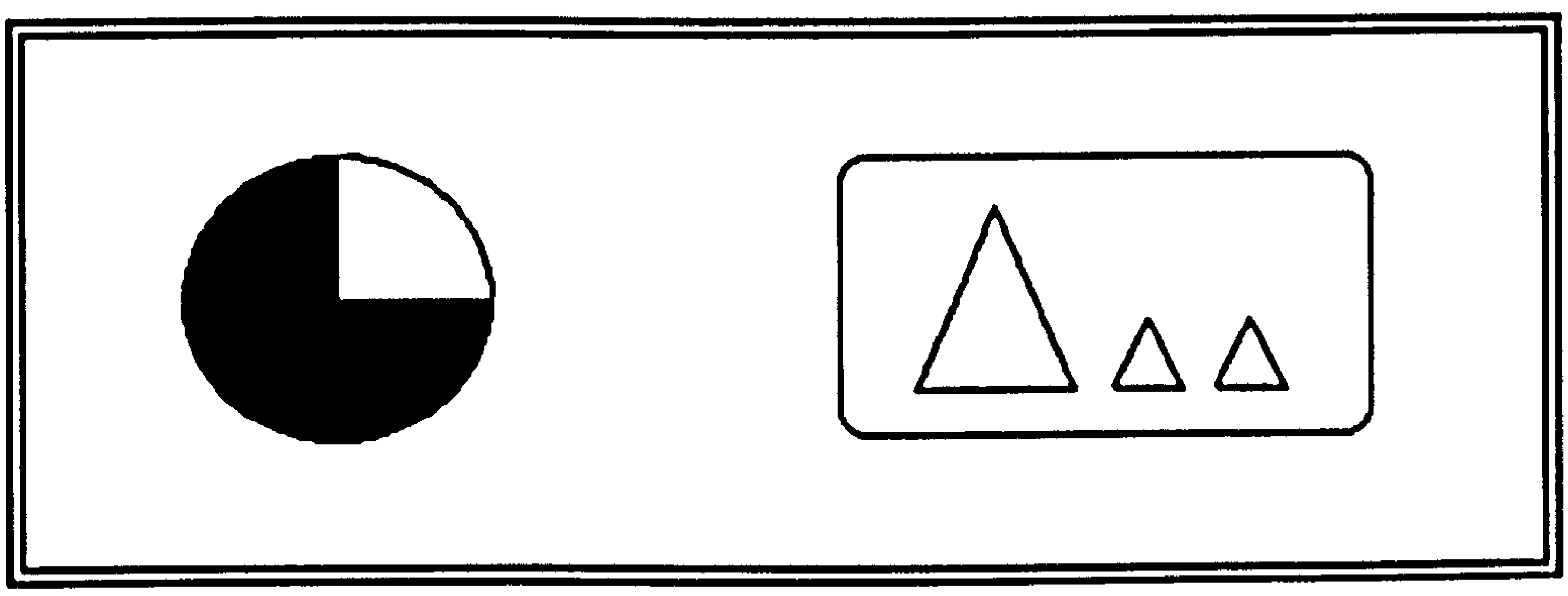


Fig. 2.18.

Firstly, the diagram on the left (above) is meant to show the unshaded area to be a third of the shaded area. The student, however, is much more likely to view the unshaded area as a quarter of the whole. Secondly, the diagram on the right shows the large triangle to be a third of the total number of triangles. The student, however, may well feel that the difference in the areas of the triangles is significant; the area of the large triangle is certainly not a third of the total area.

It is also difficult to know whether or not a particular diagram will be understood. For example, Jahnke (Dreyfus, 1995) came across unexpected difficulties when the diagram in Fig. 2.19 below was used to help teach a learning disabled child subtraction. It was used to help illustrate '6 - 4 = 2'.

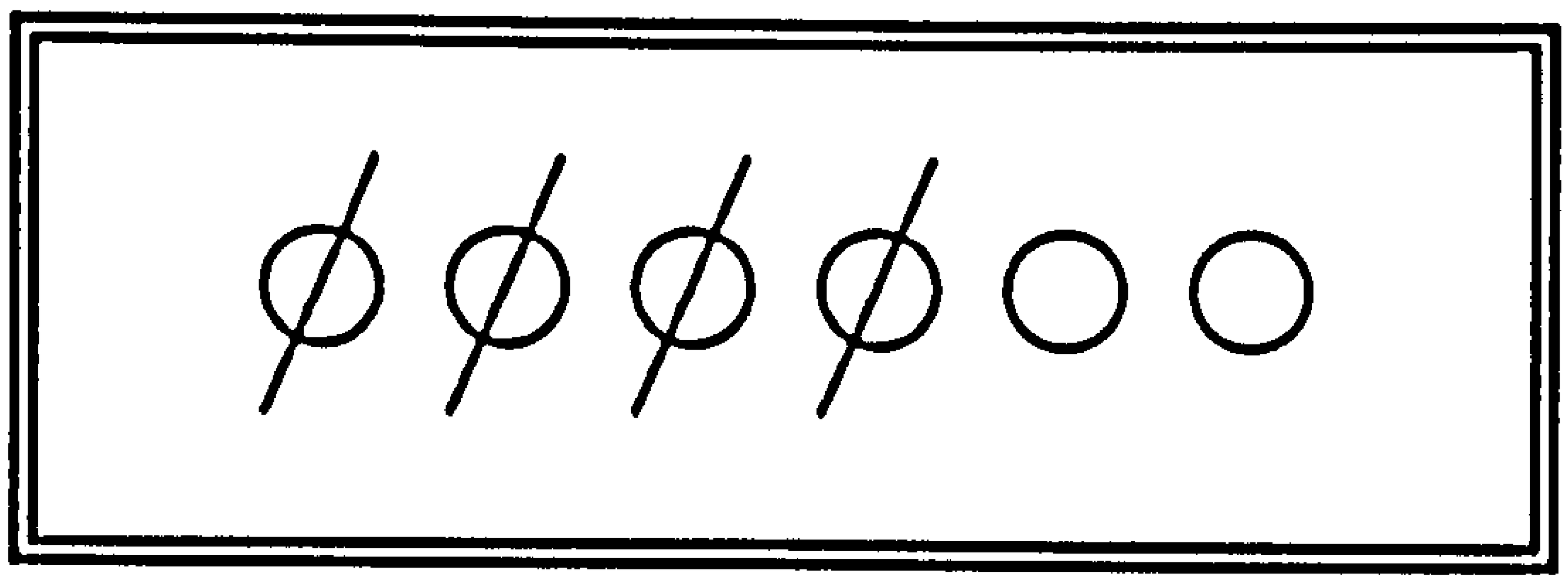


Fig. 2.19.

However, years later, during a discussion with his therapist, the child explained how he still did not understand why the teacher halved the cakes! This example surely indicates the importance of recognising the target audience during the creation of diagrams that are intended to assist the learning process (and care needs to be taken by the teacher). Again, the problem here is that the student has not received (or the picture has not delivered) the message that the teacher wanted to send.

The next example shows how the unintentional misuse of graphing technology can be a hindrance to the development of understanding (Moschkovich et al., 1993). Students used a graphing program to graph $y = 2x + 1$, $y = 3x + 1$, and $y = 4x + 1$, and then stated what they observed. Instead of observing that the lines were increasingly steeper and all passed through (0,1), they commented on the fact that some of the lines were more jagged than others! Moschkovich et al. explain that each graph is composed of a series of pixels on the screen. This results in the lines not being perfectly smooth, and as a result the greater the coefficient, the greater the 'jaggedness' (dependant on scale settings). This observation is an interesting example of how students can use a piece of technology in such a way as to cause them to see things that were not intended, e.g. jaggedness, and not see things that were intended, e.g. steepness and point of intersection. Hence the technology needs to be used with understanding.

The problems associated with the creation of 'appropriate' mental images can now be appreciated, and must be taken into account during the design of educational diagrams and imagery. It is essential that the diagrams will lead to unambiguous mental images.

It is important to understand the precise role of diagrams in learning mathematical concepts at different levels of ability and age, and we need to know in what kind of learning situations visual imagery can be helpful. Powerful patterns of mathematical reasoning can be based on diagrams and visual imagery (Dreyfus, 1995; Abbott, 2001), and this pictorial approach helps students learn and understand mathematics. The careful linking of both symbolic and pictorial representations (as with the software described in Chapter 5) can lead to students appreciating both forms, and feeling comfortable switching between them.

2.4.3. Theories of Teaching and Learning in Mathematics

This section describes various theories of teaching and learning, which are relevant to the discussion about visualisation and its impact, from renowned authors throughout the literature. It provides the basis for the theoretical rationale for the software design in Chapter 5, and informs the theoretical framework in Chapter 6.

2.4.3.1. White-Box/Black-Box (WBBB) Principle [Buchberger]

The White-Box/Black-Box (WBBB) Principle (Buchberger, 1989) is a teacher-oriented didactic model for the use of a computer algebra system (CAS) in the mathematics classroom. Drijvers (1995) summarises the philosophy of the WBBB Principle as follows. While learning a new mathematical concept or technique, it is important for the student to do the relevant operations 'by hand'. A CAS should not be used for those manipulations because the student has not yet mastered them. The learning process is described by the White Box phase, and there is a danger in using a CAS at this level as the student may lose control and understanding. In the second phase, once the new subject has been mastered, computer algebra can be used to carry out the work that is now trivial. This can be useful while exploring the visual aspects of the concepts on a hierarchically higher level, as the CAS is then used as a Black Box that takes care of the lower level tasks. The student can now concentrate fully on more sophisticated aspects of the topic, such as the relationship between symbolic and pictorial forms, without being distracted by details that are already known (mastered in the White Box phase). This is known as the Black Box use of the CAS. The CAS here plays more or less the role of a servant (Drijvers, 1995).

A variation of the WBBB Principle can also be considered, in which the sequence of phases is inverted. The Black-Box/White-Box (BBWB) approach employs a CAS as a generator of both pictorial and symbolic examples and as an explorative tool that confronts the student with new situations. This exploration of pictures and symbols (the Black Box phase) can lead to interesting discoveries and forms the basis of what is known as the explanation phase. In the latter phase, the results of the explorations will be sorted out, and thus will lead to the development of new concepts (the White Box phase). Instead of being the servant, the CAS is now like a master (Drijvers, 1995).

Visualisation plays an important role in this teaching and learning principle by establishing relationships between symbolic and pictorial forms during the Black Box exploration phase. The exploration of visual forms followed by an explanation of how they relate to the symbolic formalism will lead to a deeper conceptual understanding.

2.4.3.2. Communication Model [Laurillard]

This section considers the 'communication model' of teaching and learning (Laurillard, 1990), which shows how control can be given to the learner. Laurillard describes a familiar model, the 'didactic model', whereby descriptions of the world have been constructed for the student by the teacher. In this scenario, the teacher's role is to transmit this constructed knowledge to the student. The teacher has total control over both the material to be taught and the manner in which it is taught. By contrast, within the communication model, knowledge is not a given body of facts and theories, and is not regarded as something static and unchanging. Knowledge is not therefore something that can be given from one person to another. The student constructs their own descriptions of the world and the teacher's aim is to facilitate the student's development of their own perspective on the subject (as per the discussions on constructivism in Chapter 3). Students can therefore take more responsibility for what they learn and how they learn it.

In the communication model, visualisation helps students to explore mathematics as they construct concepts from their own individual perspectives. Dialogue between student and teacher is facilitated by visual representations.

2.4.3.3. Construction of Conceptual Understanding [Hitt]

Understanding a concept implies coherent articulation of the different representations which come into play during problem solving (Hitt, 1998b). Hitt's research identifies the following levels in the student's construction of a concept, in particular the concept of function:

Level 1 Imprecise ideas about a concept (incoherent mixture of different representations of the concept).

Level 2 Identification of different representations of a concept.

	Identification of systems of representation.
Level 3	Translation with preservation of meaning from one system of representation to another.
Level 4	Coherent articulation between two systems of representation.
Level 5	Coherent articulation of different systems of representation in the solution of a problem.

Visualisation features strongly here in all levels, and becomes more prominent as a student progresses through the levels of concept construction.

2.4.3.4. Mental Processes [Dreyfus]

This section looks at work by Dreyfus on thinking and processes, and stresses the point that understanding must be the important goal (Dreyfus, 1991). Understanding is based upon a long sequence of learning activities, involving a great variety of mental processes. The component mental processes required for understanding are therefore extremely difficult to analyse. To define and exemplify is not enough; students must construct properties of the concept for themselves (see discussions in Chapter 3), and reflect about mathematical experiences. Mathematical and psychological processes can rarely be separated. For example, on considering the building of a graph, the mathematical process corresponds to following the rules of plotting (or sketching via key features such as roots, turning points, asymptotes, etc.), and the psychological process corresponds to the generation of the mental image of the graph in a holistic sense.

Students' mental representations of a particular concept may be vastly different from each other. Educators therefore need to 'force' the mental image making process via the appropriate use of visual imagery. Visualisation has played a key role for eminent mathematicians - they think in terms of pictures, not words and language (Dreyfus, 1991). Mental representations are created on the basis of concrete representations, for example visual representations. A rich mental representation contains linked aspects of the concept. Students therefore need to develop the ability to switch between representations when appropriate.

Generalising involves moving from particulars and conjecturing to a general case. *Synthesising* involves combining or merging parts to form a whole, a single picture. Dreyfus describes how the ability to relate different topics, for example, is sorely lacking in students. *Abstracting* is taking the concrete and moving it to an abstract notion (this is very much linked to generalisation). The process of abstraction makes the heaviest cognitive demands on students. It is a different mental process to generalising and synthesising. Rather than thinking about objects themselves, it is concerned with concentrating on properties and relationships. If examples are too complex, then abstraction can be difficult, as the crux of the process is to ignore all the detail.

Dreyfus states that the learning process involves an ordering of stages such as:

1. a single representation.
2. more than one representation in parallel.
3. links between parallel representations.
4. integrating representations, and flexible switching.

These four stages are involved hierarchically as learning develops. Once this process has been completed, an abstract notion of a given concept has been formed, and an individual then has control over the use of alternative representations. Stages 2, 3 and 4 (and possibly 1) involve visualisation, with conceptual understanding coming from an appreciation of the relationship between the visual and the symbolic, and more importantly the ability to transfer information effectively between the representations.

Discovery is an effective way of learning. The psychological aspects are that it has personal involvement, intensity of attention, and feeling of achievement and success. The checking of 'solutions' can offer alternative procedures leading to increased security. Again, as with discovery, this means transferring more of the responsibility for learning from teacher to student. Student activities need to be designed to include these activities.

2.4.3.5. Concept Definition and Concept Image [Vinner]

Pedagogy is the science of teaching, and must be structured in such a way as to result in students increasing their levels of understanding in some way. Mathematics is deductive in that it starts with primary notions, and everything else is built on these foundations. Textbooks in particular follow this notion of starting with the basics and progressing to more complex ideas. This may, however, be pedagogically wrong as it does not take into account psychological processes of concept acquisition and reasoning which may well be different from student to student (Vinner, 1991), or different student learning styles (Berry, 2002; Smith and Berry, 2002).

Vinner defines a concept image as something non-verbal associated in students' minds with the concept name. It could be a visual representation (which is usually the case), which can be verbalised if required, but this verbalisation is not the first thing that is evoked in the students' memories. Concept images are individual and very personal by nature, and may well be a different image in a different situation. To acquire a concept is to form a concept image for it, and not just to know a definition. This concept image helps to construct examples that can be used in problem solving. Definitions are useful, however, as they can help form a concept image, and once formed the definition becomes dispensable, i.e. the 'scaffolding' can be taken away. A concept image, on the other hand, might set traps if incorrectly formed or ambiguous (Graham and Rowlands, 2000; Giraldo et al., 2002). The definition can prevent this from happening.

In order to present Vinner's ideas by means of diagrams, the existence of two cells in a person's cognitive structure needs to be assumed. One cell is for the definition of the concept and the other is for the concept image. The concept image cell is considered to be empty as long as no meaning is associated with the concept definition. This could happen, for example, where the definition is memorised in a meaningless way. Fig. 2.20 shows the interplay between concept image and concept definition. This is a two-way process because the information in each cell supports and reinforces the information in the other. The concept image cell is empty at first, but after several examples and explanations it is gradually filled.

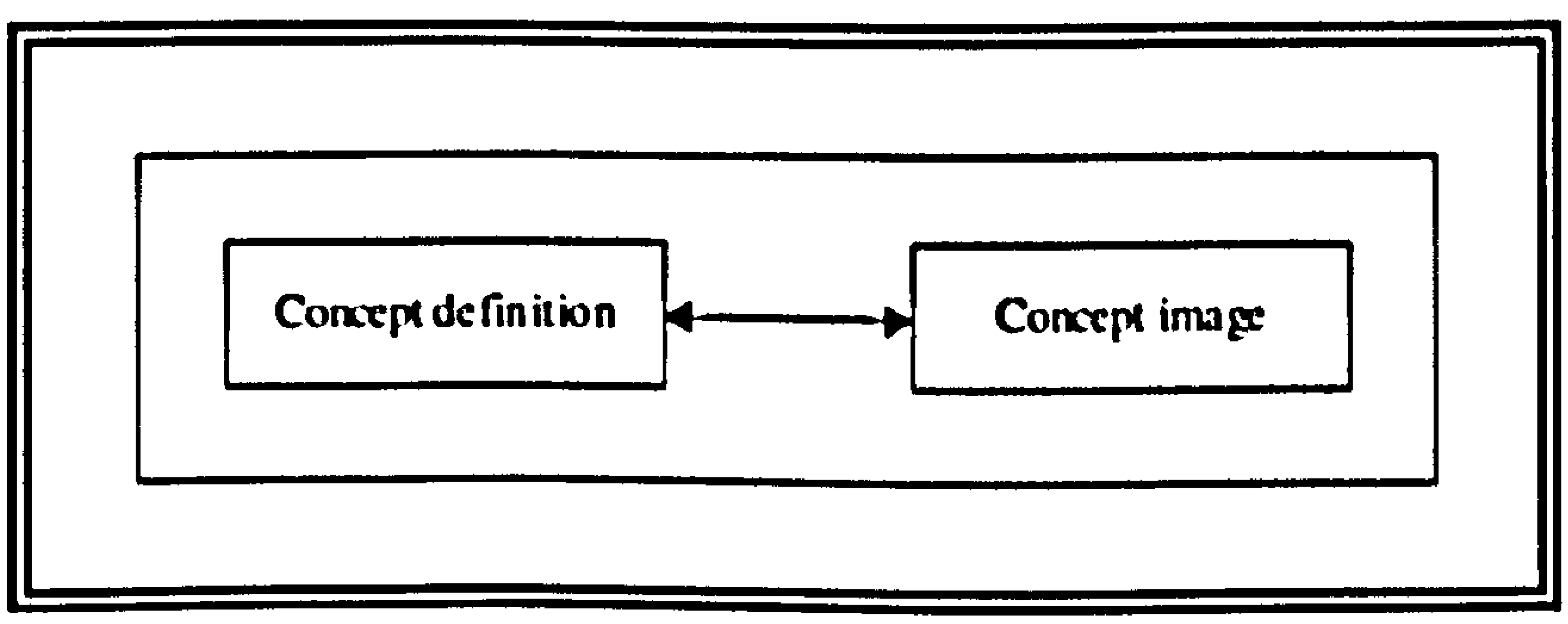


Fig. 2.20.

The concept image can be either static or dynamic, depending on the nature of the concept definition and the type of understanding that the educator wishes to convey. For example, when considering the displacement-time graph of a bungee-jumper (Gretton and Challis, 1996, 1999), it would be desirable for students to appreciate the dynamics of the process as well as merely the resultant curve.

Fig. 2.20 therefore refers to the long-term process of concept formation as a result of the interplay between the cells. Many teachers, however, expect a one way process for the cognitive growth of a concept, as shown in Fig. 2.21, namely that they expect a concept image will be formed by means of the concept definition alone.

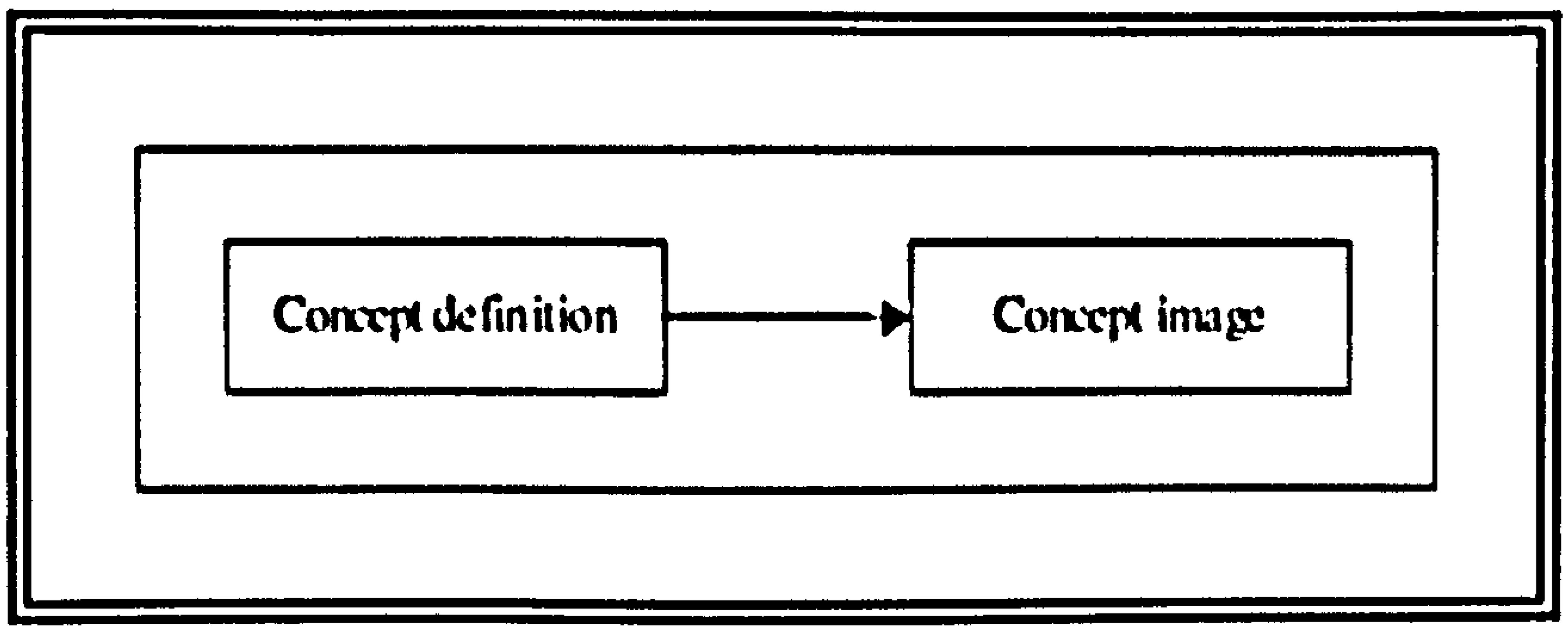


Fig. 2.21.

In addition to the process of concept formation, there are also the processes of task performance. When a cognitive task is posed, the concept image and concept definition cells are activated. The processes involved with the performance of a task are expressed by one of the following three diagrams. Fig. 2.22 illustrates the interplay between definition and image:

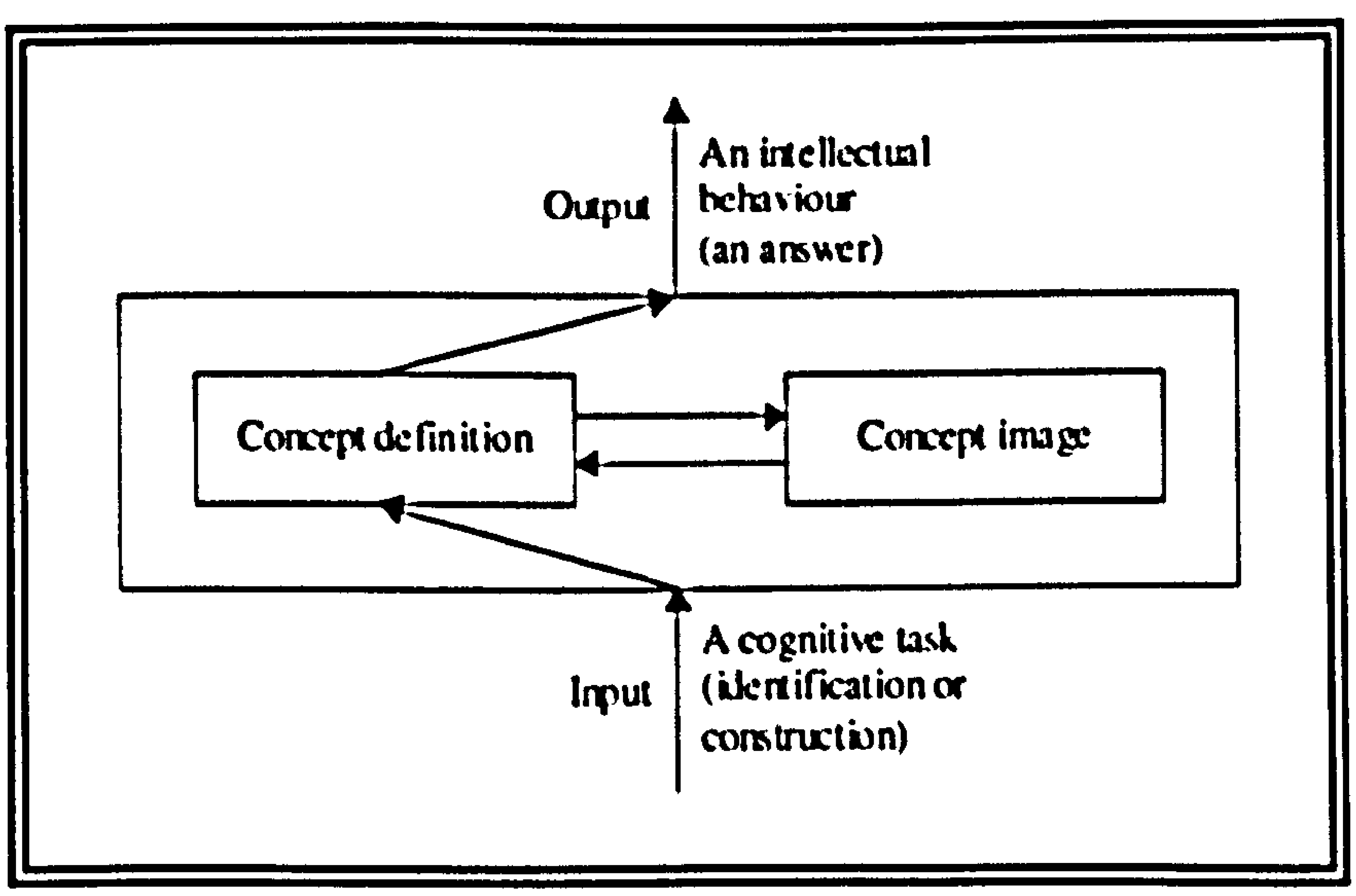


Fig. 2.22.

Fig. 2.23 illustrates a purely formal deduction, where no reference is made to the concept image. Either a concept image has not been formed (i.e. there is no conceptual understanding of the situation) or it has been deemed unnecessary to consult the concept image during the performance of the task.

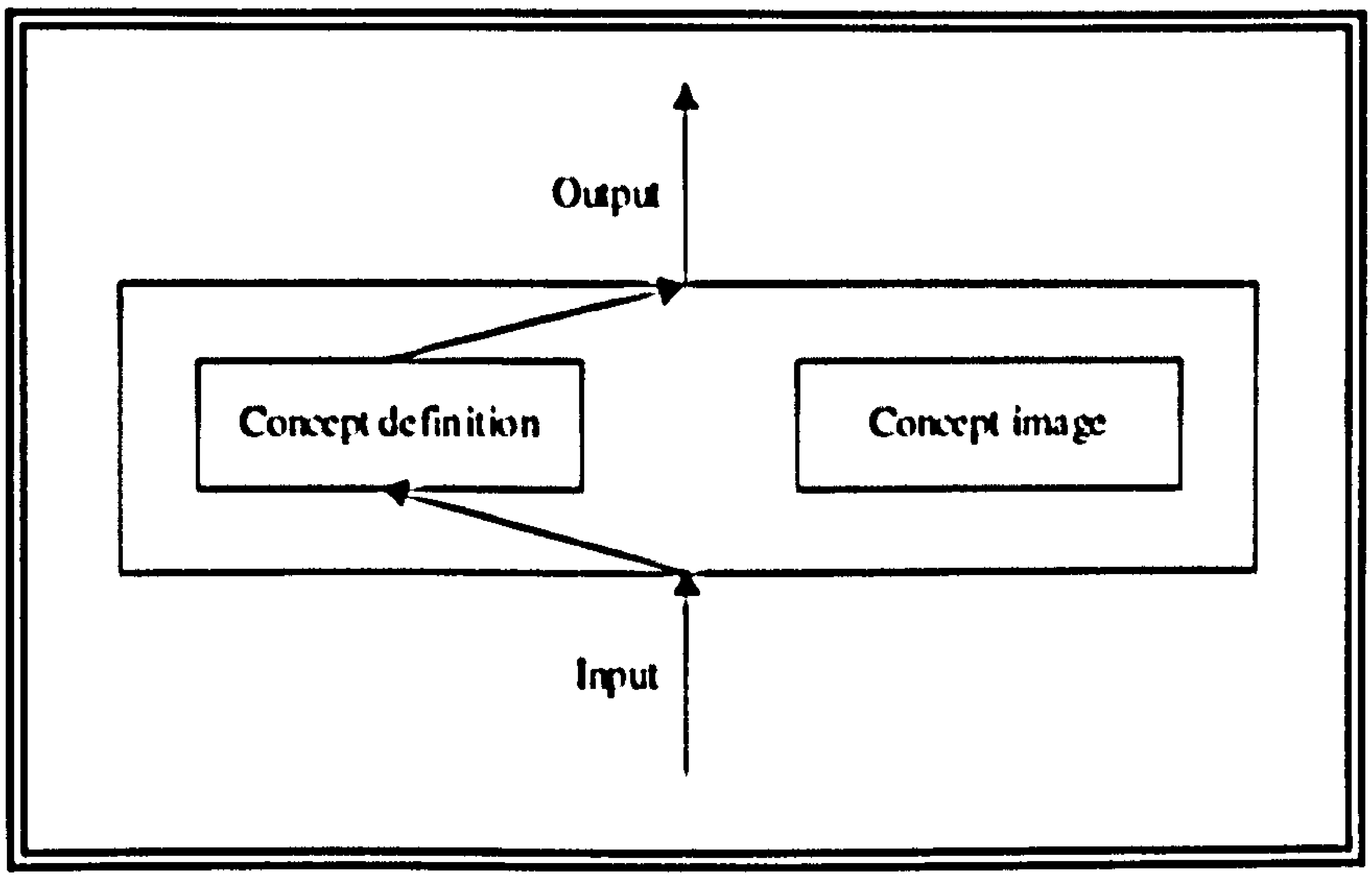


Fig. 2.23.

Fig. 2.24 illustrates a deduction following intuitive thought, where the concept image is consulted in the first place to provide a conceptual understanding of the task, and to provide meaning for the concept definition.

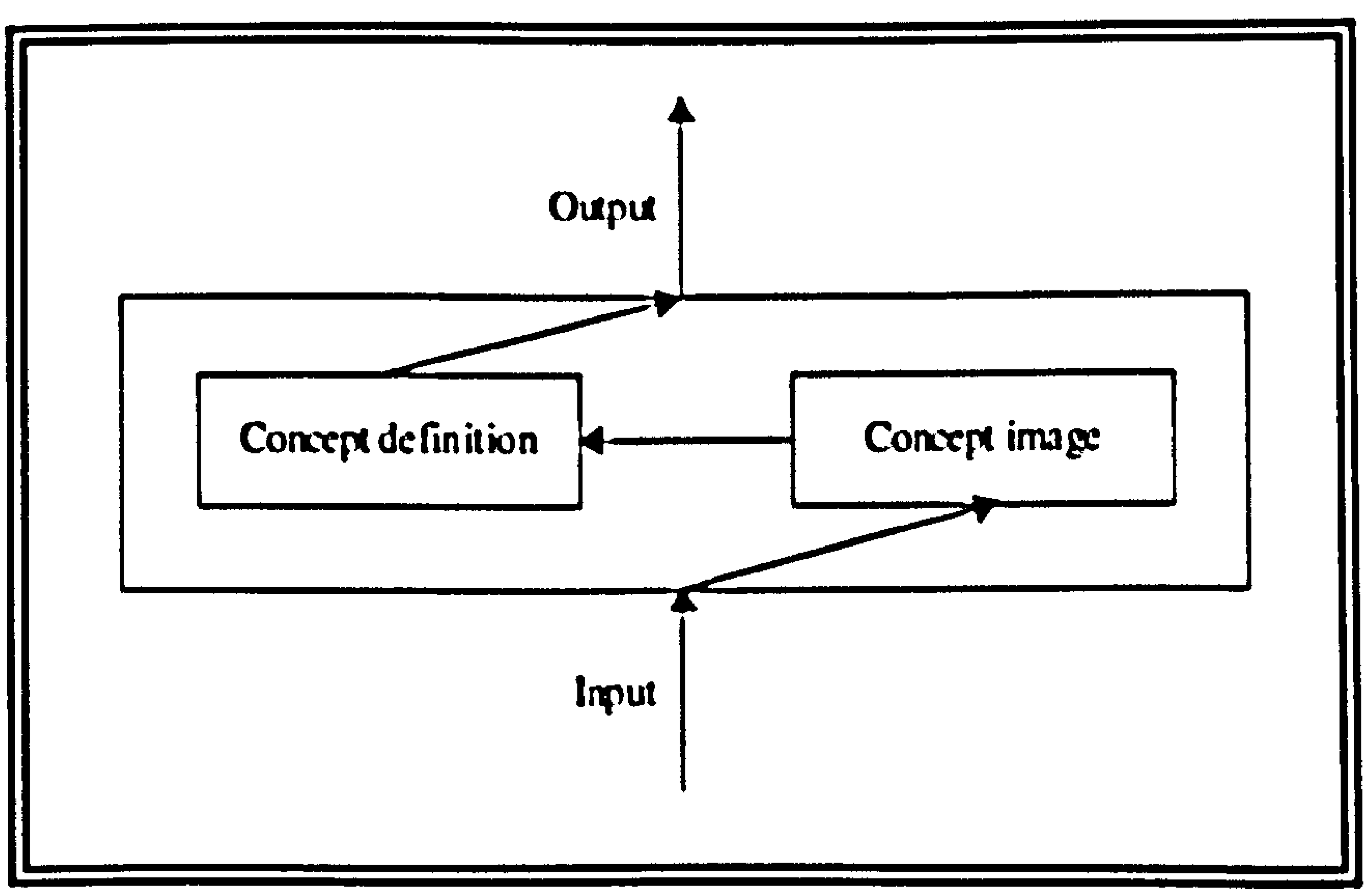


Fig. 2.24.

The common feature of all the processes illustrated in Figs. 2.22, 2.23 and 2.24, is that no matter how a person reacts when a problem is posed, a solution should not be formulated before consulting the concept definition. This is desirable (as the concept definition can prevent misconceptions as a result of an incorrectly formed or ambiguous concept image), but often does not happen in practice. Fig. 2.25 illustrates a deduction following intuitive thought, as with Fig. 2.24, but this time it is a more appropriate model for the processes that normally occur in practice:

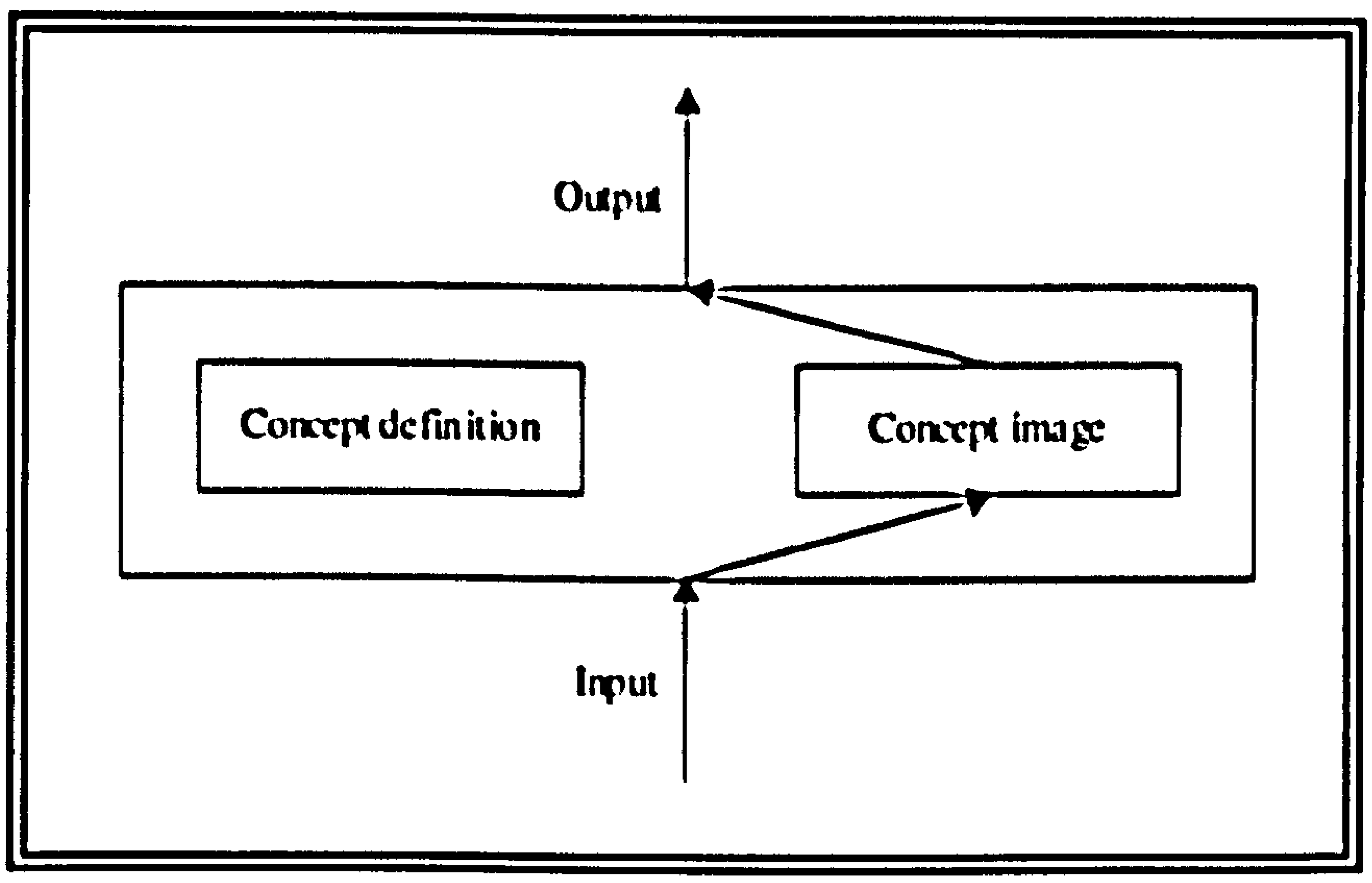


Fig. 2.25.

In this situation, the concept definition cell is not consulted during the process of task performance. The 'everyday' thought habits take over and the respondent is unaware of the need to consult the formal definition. In most cases, reference only to the concept image will be successful. This fact, therefore, does not encourage people to refer to the concept definition cell. Only unfamiliar problems, in which incomplete concept images might be misleading, can encourage people to refer to the concept definition. Vinner has observed that students therefore consider such problems as unfair.

It is very difficult, if not impossible, to learn about a person's concept image, and so the correctness of its construction is vital as the learning stage is the only thing over which educators have any degree of control. Students can build up incorrect concept images, and even when new definitions are taken on board the old images may still be retained, confusing the issue (for examples of misconceptions with the use of visualisation in mathematics, see work reported earlier in this chapter by Moschkovich et al., 1993, and Dreyfus, 1995). Educators need to attempt to change the thought habits of students from 'everyday' to 'technical' in order to aid the correct construction of concept images.

2.4.3.6. Conceptual Entities [Harel and Kaput]

This section considers conceptual entities and their symbols in building concepts (Harel and Kaput, 1991). Mathematical thinking employs mental objects. *Reflective abstraction* is a physical or mental action that is reconstructed and reorganised on a higher plane of thought and so comes to be understood. A *conceptual entity* is a cognitive object for which the mental system has procedures that can take that object as an argument, as in input. It is the awareness of acting on a process as a whole, as a totality (not point by point), that constitutes the conception of that process as an object. For example, when considering the effect of the constant, c, on the graph of $f(x)$ in expressions such as $f(x+c)$, $f(x)+c$, and $cf(x)$, the function $f(x)$ can be regarded as an object, and therefore all three expressions are merely variations of that object, as opposed to three separate, unrelated functions.

The *vertical growth* of conceptual understanding describes the development of conceptual entities that can be further operated on. The *horizontal growth* of conceptual understanding is complemented by vertical growth, and is associated with the interplay between different representations.

Conceptual entities alleviate working memory (as discussed in Section 1.3.2.2 of Chapter 1), or processing load, facilitate comprehension, and assist with the focus of attention. Harel and Kaput believe that computer-based activities can facilitate the construction of conceptual entities, which in turn assist in the vertical growth of mathematical ideas.

The power of conceptual entities is closely related to the role of mathematical symbolism. Mathematical notation plays an important role in the development of conceptual entities (in a similar fashion to the role of the concept definition in helping to form the concept image, as discussed in the previous section). Using mathematical notations, complex ideas can be grouped together and thus represented by physical notations that can be manipulated to generate new ideas.

2.4.3.7. Reflective Abstraction [Dubinsky]

Consideration of how a student constructs knowledge is important for the development of mathematical thought (Dubinsky, 1991). Dubinsky states that reflective abstraction is the construction of mathematical structures by an individual during the stages of cognitive development (similarly described by Harel and Kaput in the previous section). The process of reflective abstraction is facilitated by constructivism (see Chapter 3). It is essentially the construction of mental objects. Reflective abstraction leads to mathematical thinking in which processes are separate from content, and that ultimately processes are converted to objects. Dubinsky defines the following different methods of construction in reflective abstraction that are important for advanced mathematical thinking.

- ***Interiorisation*** is the translation of a succession of material actions into a system of interiorised operations. It is the construction of internal processes as a way of making sense out of perceived phenomena.

- ***Coordination*** is the composition of two or more processes to construct a new one.

- ***Encapsulation*** is perhaps the most important and most difficult for mathematics students. It is the conversion of a process into an object. Actions can therefore become entities that can then be used in higher levels of mathematical thinking.

- ***Generalisation*** is the application by an individual of an existing concept to a wider subject area. The individual has therefore become aware that the concept remains the same but now has a wider applicability, or has encapsulated a process into an object.

- ***Reversal*** is the switching between an original process and a new process. It is where the individual thinks of an internalised process in reverse, not necessarily in the sense of undoing it, but as a means of constructing a new process that consists of reversing the original process.

Working with examples serves to reinforce concepts, but may not help very much with the actual construction of concepts. During meaningful learning, such as using the software described in Chapter 5, understanding the mathematical concepts comes from the construction aspects of reflective abstraction (how this is achieved, by considering the five methods of construction in terms of software usage, is described in Section 5.3.3.2 of Chapter 5). In order to construct a mathematical idea, it is necessary for the student to be mentally active. Educators need to be concerned about the nature of these mental constructions, and need to develop activities that encourage their appropriate development via reflective abstraction. Some insight into the nature of students' conceptual structures (concept images) can be achieved by observing students as they learn (Berry, 2002; Smith and Berry, 2002), ascertaining how students have arrived at a certain solution (correct or incorrect), or how that solution can be justified (an analysis of student solution processes is provided in Chapter 5).

2.4.3.8. Relational and Instrumental Understanding [Skemp]

This section discusses why it is preferable for students to develop relational understanding as opposed to, the more common, instrumental understanding (Skemp, 1976). Relational understanding, knowing not only what method works but why, enables pupils to relate the method to the problem, and possibly to adapt the method to new problems. Instrumental understanding is simply 'rules without reasons', where understanding is taken to be the possession of a rule, and the ability to use it. It necessitates memorising which problems a method works for and which not, and also learning a different method for each new class of problems. Knowing how rules are interrelated enables one to remember them as parts of a connected whole, which is easier. There is more to learn (the connections as well as the separate rules), but the result, once learnt, is more lasting. Skemp believes that there is less re-learning to do, and long-term the time taken may well be less altogether.

Skemp provides examples of instrumental understanding such as 'turn it upside down and multiply' for division by a fraction, and 'take it over to the other side and change the sign'. If the teacher asks a question that does not quite fit the rule, then the students will get it wrong. For example, while a certain student on teaching practice was teaching the topic of area, he became suspicious that the children did not really understand what they were doing. So he asked them, 'What is the area of a field 20cms by 15 metres?' The reply was, '300 square centimetres'. He asked, 'Why not 300 square metres?' They answered, 'Because area is always in square centimetres'.

To prevent errors like the above (assuming only instrumental understanding), the pupils need another rule that both dimensions must be in the same unit. This would not be necessary, however, if the children had relational understanding. Instrumental understanding usually involves a multiplicity of rules rather than fewer principles of more general application.

Although we will see the benefits of teaching for relational understanding in Chapter 5, there are various reasons why this approach may not be adopted in the classroom. An individual teacher might make the choice to teach for instrumental understanding on one or more of the following grounds:

1. That relational understanding would take too long to achieve, and to be able to use a particular technique is all that these pupils are likely to need.
2. That relational understanding of a particular topic is too difficult, but the pupils still need to study the topic for examination reasons.
3. That a skill is needed for use in another subject before it can be understood relationally.
4. That he is a junior teacher in a school where all the other mathematics teaching is instrumental.

2.5. Software Design Issues Incorporating Imagery

This section concentrates on factors that influence the design of educational software that incorporates imagery. Consideration is given to various strategies for increasing motivation in students, factors that have a positive effect on retention level, and a case-study concerned with the learning of functions and graphs specifically.

2.5.1. Increasing Motivation

Three factors that influence motivation to learn are considered here. Firstly, visually stimulating environments can offer an approach to learning that is an alternative to instructivism. Secondly, educational software can be designed in the form of mathematical games that can make learning fun. Finally, various characteristics can be built into the design of educational software in order to help maintain interest.

2.5.1.1. An Alternative Approach to Learning

Environments can be created that help to motivate students to think mathematically by providing mathematics activities whose purposes go beyond just learning mathematics 'for mathematics sake' (Pea, 1987). Examples of such environments include recreational mathematics websites containing computer simulations of the national lottery, puzzles, magic squares, etc. Other activities include the mathematics of computer games, forecasting of the stock market, sporting odds, etc. The application of mathematics to the solution of a real problem, i.e. mathematical modelling, can be used as a teaching vehicle in this way (Houston, 1997; Houston et al., 1997; for further publications with an emphasis on the applications of mathematics and mathematical modelling, see Teaching Mathematics and its Applications: An International Journal of the IMA). The student thus sees mathematics used in a problem solving context, and can appreciate that the mathematics has a point, and has been used for a specific purpose. The fact that this is an alternative approach to learning mathematics (compared to an instructivist approach) can be motivational in itself.

This application-led approach is closely related to problem-based learning, or PBL (Stepien and Gallagher, 1993; Woods, 1994), which consists of carefully designed problems that require students not only to demonstrate acquired knowledge and problem solving proficiency, but also self-directed learning strategies and team participation skills. The process replicates the approach to solving problems in the real world, often with insufficient information, instead of having neat solutions to contrived problems.

Environments can be intrinsically motivating, i.e. people can be motivated to learn in the absence of obvious external rewards or punishments (Malone and Lepper, 1987). Malone and Lepper have studied how to make learning more interesting and enjoyable by considering computer games. They considered the difference between toys and tools. Toys are objects that are used for their own sake with no external goal, whereas tools are objects that are used as a means to achieve some external goal. Unnecessary difficulty with a tool can prove frustrating, whereas toys are often made intentionally difficult to use in order to enhance their challenge. Many people find that difficulty enhances challenge, which in turn enhances pleasure (Malone and Lepper, 1987). The level of difficulty, i.e. degree of challenge, of educational software must therefore be regarded as an important aspect.

It is certainly not the case that all motivating environments are games, or that all educational problems can be solved using games. However, games do provide particularly striking examples of highly motivating activities (Malone, 1980; Malone and Lepper, 1987; Soloway, 1991; Lepper and Henderlong, 2000; note that although the first reference dates back to over twenty years ago, nearly all reported work since is based on the original material by key authors Malone and Lepper). They show how the computer is used to create motivating environments, which are a source of insight for designing motivating educational software. There are strong links between the features that make a package fun and those that make it educational. For example, environments that vary the level of difficulty increase both challenge and the potential for learning, and environments that evoke curiosity and then satisfy it can be both captivating and educational.

"The puritanical attitude that the mind is a muscle to be exercised through mechanical repetition is giving way to a richer view of the creative, exploring mind, which can be nurtured and guided to discover and learn through meaningful problem-solving activities" (Pea, 1987). Multimedia technology with its rich effects is showing great potential for creating fascinating educational environments (Al-Jumeily, 2002; Pappas et al., 2002). Great potential, however, does not

guarantee wise use - consider for example the television! It is most important to find a happy medium between educational content and entertainment.

2.5.1.2. Making Learning Fun

Consideration is given here to a number of activities designed to make learning fun, carried out with students of a variety of age ranges.

A study was carried out in which a group of 6-7 year old children experimented with a variety of mathematics based learning environments (Panoutsopoulos and Potari, 1995). The designers attempted to exploit the creative potential of the computer by using colour, animation, graphics and sound to stimulate situations that may be challenging to children. The aim was that the children would find this approach to learning mathematics more stimulating than the traditional 'chalk and talk' techniques. Panoutsopoulos and Potari describe how this proved to be the case, as the children enjoyed using the software. The package encouraged group discussion about mathematics, and the class teacher commented that the programs were received with enthusiasm by all the children, *regardless of their mathematical ability*.

Mathematical games, which were designed to make learning more fun, have also been developed (Phillips et al., 1995). Whilst playing these mathematical games, the students show interest and excitement, and express a willingness to continue far beyond their usual concentration span. Screen imagery can often be emotionally loaded - it has the power to involve its audience and captivate their imagination. If the mathematics questions can be presented interestingly enough, then the students will explore, and in turn acquire new thinking skills. However, some mathematics educators, described by Phillips et al., deny that computer software can be motivating in itself, and argue that it is the mathematical activities themselves which motivate the students. Phillips et al. point out the fact that they have watched students' reactions, and it would seem beyond doubt that the particularly visual mode of presentation has a strong effect on students' interest and willingness to engage in mathematical activities. It is not merely the use of the computer itself that serves to motivate students, but also the ease with which they can switch between different representations, thus building a deeper conceptual understanding of mathematics. It is conjectured here that this observation applies to all ages and levels of ability.

The design and ease of use of advanced graphics calculators, such as the Texas Instruments range, make them reminiscent of many handheld computer games such as Nintendo GameBoy and Sega GameGear, but it is possible to meet some serious mathematical ideas and to have fun at the same time (Gretton and Challis, 1996, 1999). Gretton and Challis describe classroom experiences where they have combined the TI-82 with the CBL (Computer Based Laboratory). Physical quantities can be measured using the CBL, and then the information can be transferred onto the TI-82 for discussion and analysis. For example, heart-rate can be measured and the graph can be observed on the screen, as in Fig. 2.26, and the temperature of a cup of tea can be measured as it cools down, and again the graph can be observed on screen, as in Fig. 2.27.

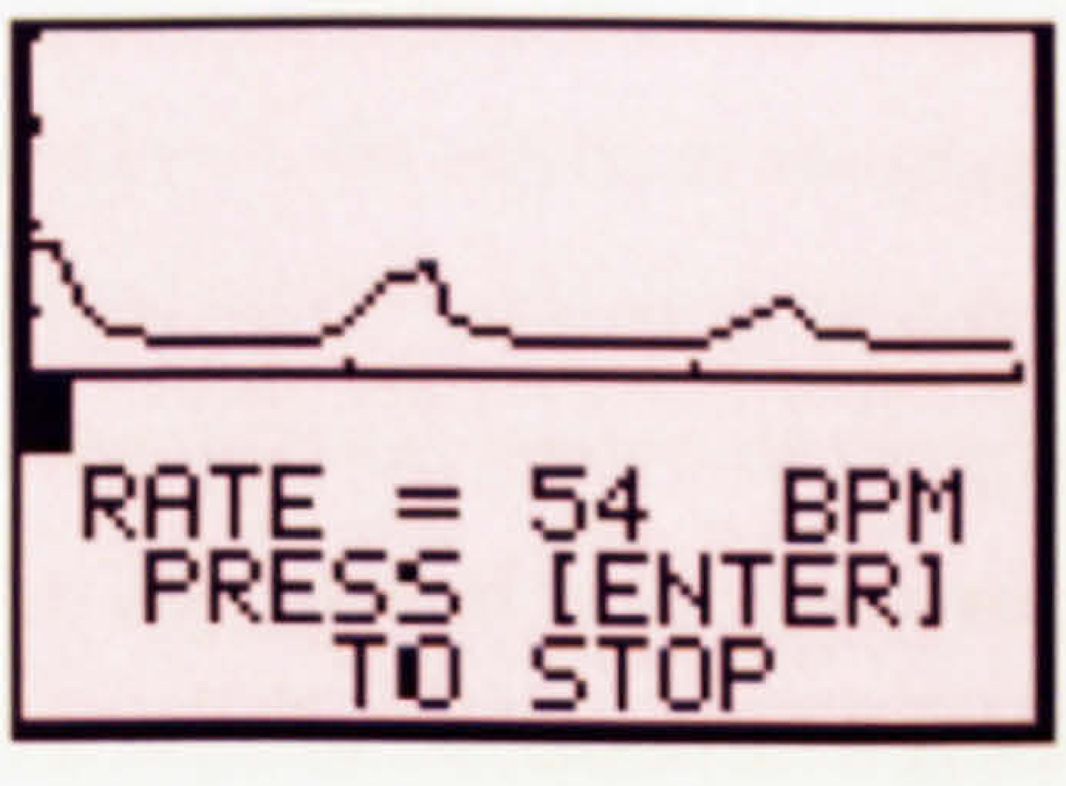


Fig. 2.26.

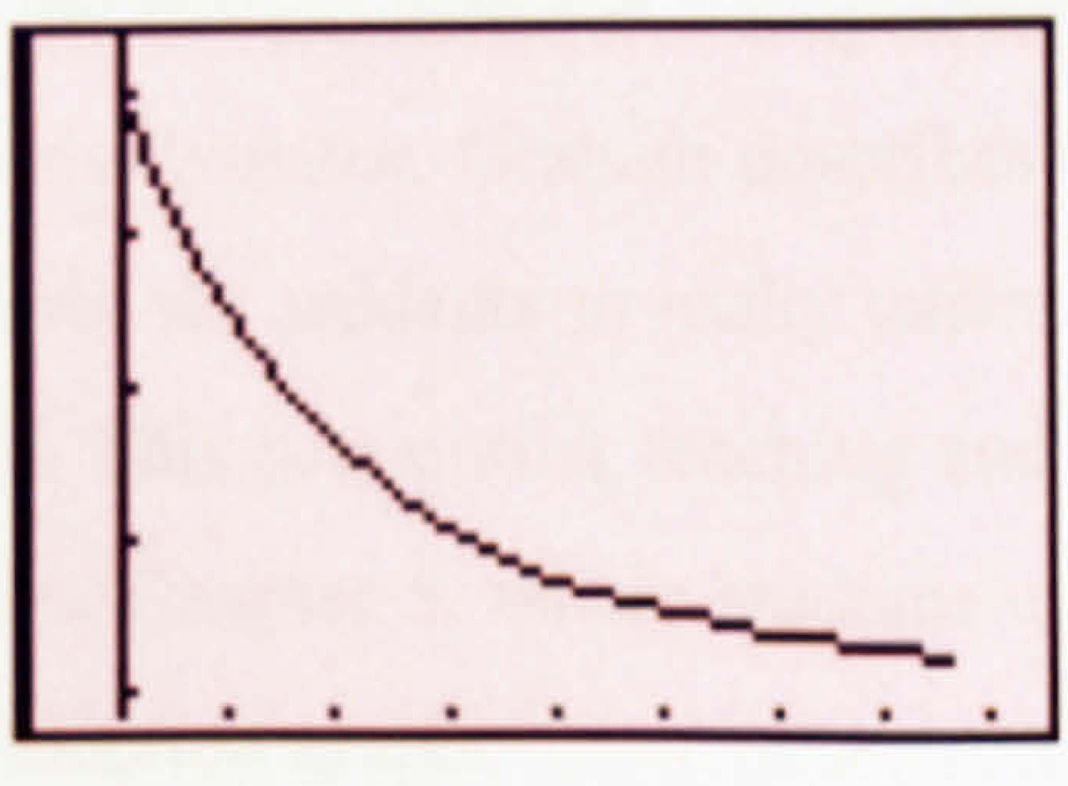


Fig. 2.27.

A coke can and a piece of elastic can even be used to imitate bungee jumping, as in Fig. 2.28. Gretton and Challis describe how the mathematically sophisticated ideas of periodic behaviour and decay arise naturally here.

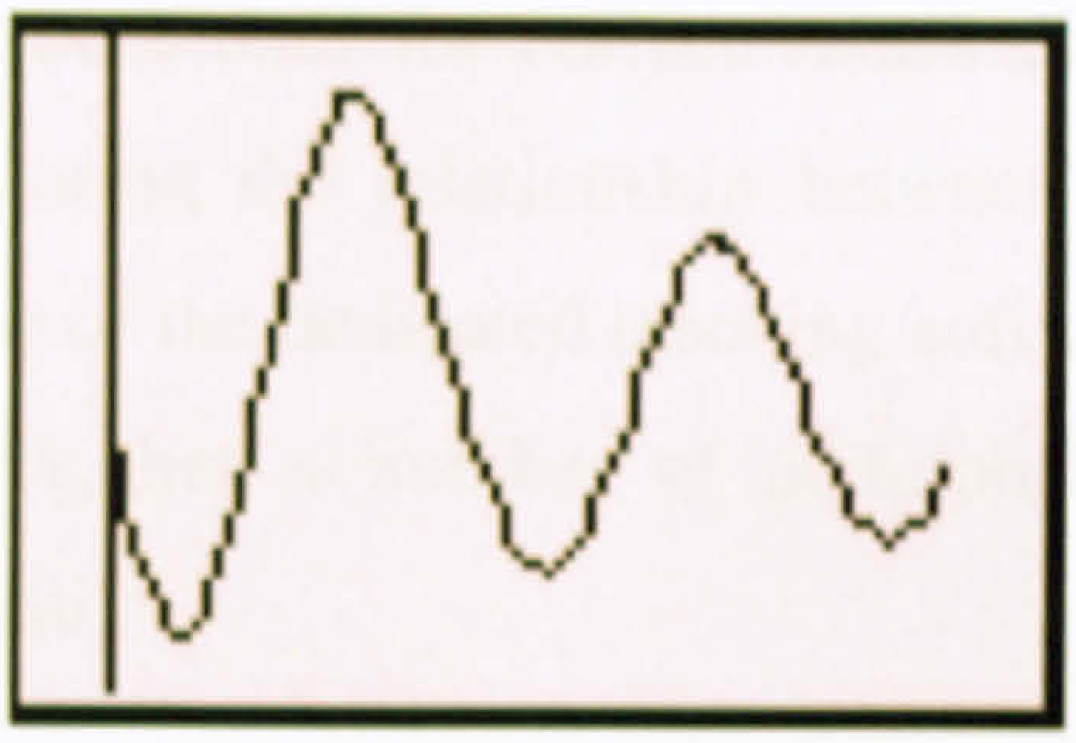


Fig. 2.28.

Via this approach, students can interpret gradients, etc. in terms of real-world applications, **without necessarily having knowledge of the symbolic notation for the graphs**. Hence the higher level skill of interpretation can be developed, followed by the reinforcement of these ideas by **linking the concepts in the pictures to the symbolic formalism**. These examples clearly illustrate how visualisation plays an important role in the conceptualisation of the mathematical ideas, and they exemplify the constructivist philosophy, whereby students are actively involved in the learning process by having ownership of the task.

Innovative teaching and learning with the CBL has been successfully demonstrated with students at the University of Plymouth (Graham, 1996). Students are challenged to reproduce a graph by behaving in the correct manner in front of a motion detector. As they move in front of the motion detector, a displacement-time graph appears on the screen of the graphics calculator. Graham describes how this has proved to be very effective, as it requires the students to really understand the nature of the different aspects of the graph. This is a similar teaching and learning strategy to that adopted in the case-study in Chapter 5, where students attempt to provide the algebraic notation for a given function graph.

In a similar fashion, Sharp describes the use of the TI-83 to help consolidate students' understanding of the straight line graph (Sharp, 1997). The student is presented with a straight line graph, and is prompted to supply values for the slope and intercept. In this way, the student can observe the effect of the input on the graph. This process continues until the correct values are provided for the given graph. This idea of exploring the relationship between symbolic and graphical forms is the central theme of the dedicated teaching software described in Chapter 5. The software, however, has a number of additional advantageous features, which are discussed in detail.

An appreciation for the beauty of mathematics is a possible and desirable outcome of visual thinking and a legitimate educational objective (Zimmermann, 1991). The aesthetics of visual mathematics should be used to full effect to motivate and inspire students' interest in the subject. The Escher sketches in Fig. 2.29 (see References for Internet address) illustrate the 'art' of mathematics. Escher built up his drawings on a mathematical grid that gives rise to translational symmetry along the gridlines.

Fig. 2.29.

The images illustrated in Fig. 2.30 are part of a fascinating mathematical discovery known as the Mandelbrot set (see References for Internet address), which is an example of a rich class of objects known as fractals. Users could click on a part of the fractal on a computer screen and zoom in to investigate it further. Fractals could be explored repeatedly in this manner due to their infinite nature.

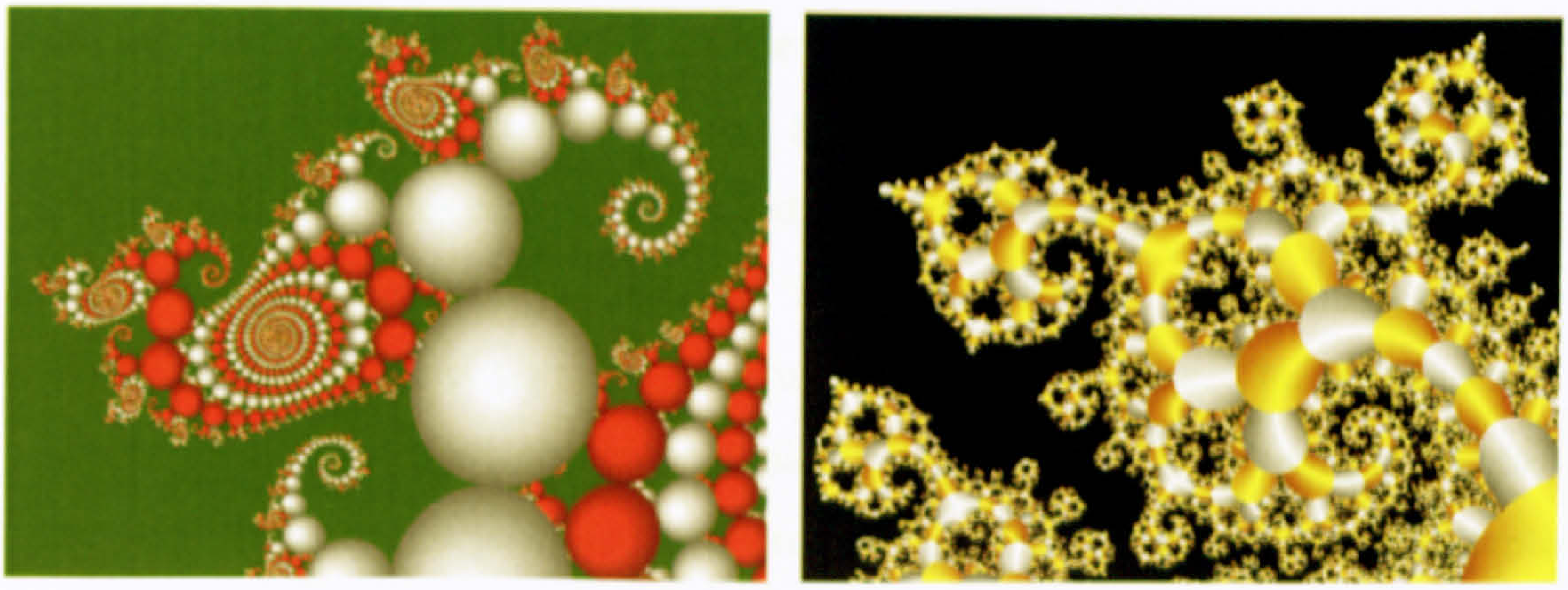


Fig. 2.30.

It is a positive step to encourage students to appreciate the beauty of mathematical graphics. Then, at least in some cases, students can be drawn into a deeper appreciation for the logical or practical dimensions of mathematics (Zimmermann, 1991). It is not necessary to understand the mathematics that produces these images in order to enjoy their beauty, but knowledge of the underlying principles enhances the appreciation.

2.5.1.3. Maintaining Interest

There are four main motivational factors which need to be taken into account whilst developing educational software so that the user's interest will be maintained (Lepper et al., 1993):

- The material needs to be challenging.
- The package must bolster the user's self-confidence.
- It must evoke curiosity.
- It must allow the user to be in control.

Lepper et al. consider each in turn:

The material within an educational package needs to be challenging so that the user will be motivated to use it. If the material is trivially easy or extremely difficult, then it will be of little interest and it will generate boredom rather than serve to motivate. This stresses the need to 'model' a student's ability so that a piece of software can be of the required 'level of difficulty', and moreover can be introduced at the correct point on the student's learning curve. The package also needs to provide goals which can be attained - an activity is challenging if it tests some sort of ability that is valued, and provides tests in which success is desirable (but is not certain).

Interactive packages must always attempt to bolster the user's self-confidence. This can be achieved by praising the user's correct answers, but more importantly by avoiding direct negative feedback. Instead, the user would prefer to see useful hints or other indirect forms of feedback. An example of this is 'assessment software'. The software can be written so as to praise the user's correct answers, or to encourage the user after incorrect answers by offering helpful hints or alternative strategies. In this manner, the user should arrive at the correct answers, at his or her own pace, and these correct answers should then help to motivate the student to progress further.

If the package evokes curiosity, it is likely that the user will be more attentive and will have a more active and deeper involvement. Lepper et al. suggest that curiosity is aroused when people encounter information that implies that their current knowledge is incomplete.

Like curiosity, control is also likely to encourage a more active and deeper involvement in the activity. In addition, the concept of personal control is a source of motivation because many people like to be responsible for their own actions and choices.

The design of the educational package that tackles the learning of functions and graphs via a constructivist, visual approach (described in Chapter 5) has been influenced by the reported work on motivation throughout this section, and a number of the desirable features discussed are evident in the software. It adopts a constructivist approach by using problems in which students are encouraged to explore the conceptual links between graphical and symbolic forms in a visually stimulating environment. Key features of the software are the ease with which students can plot and compare graphs, and the effectiveness of illustrating the relationship between alternative functional forms. Students have fun using the software, and its use promotes discussion with both teacher and peers. As with problem-based learning (PBL), the problems are designed for the development of not only conceptual understanding, but also higher order skills. Finally, the design takes into account the four main motivating factors concerned with the maintenance of interest described by Lepper et al.

2.5.2. Increasing Retention Level

Students can often motivate themselves to perform well in a traditional examination, but often forget what they learnt a short while later. They may well score highly in the examination, but it is not very useful for practical purposes. A learning environment needs to be created which incorporates a different style of learning, so that the students can increase the length of time that they are able to maintain mastery of a particular topic - or at least can recall topics quickly and with little effort.

Well-designed educational packages have an advantage over books. You are free to turn the page of a book, thinking that you understand the contents therein, but the truth of the matter might be that you do not really understand as much as you think. An educational package, on the other hand, may have a small test at the end of a certain topic to check that the user fully understands the contents (e.g.

CALMAT, MATHWISE, etc.). It could quite easily be programmed so that the user cannot proceed until he or she has reached a desired level of understanding. In this manner, unlike with a book, the user cannot easily proceed to topics for which he or she is not properly equipped. Intelligent tutors can supply the user with more material of the same level of difficulty or can direct the user to a higher level of study. Alternative strategies can be provided in situations where the user did not understand at the 'first time of asking'.

Finally, there is the consideration of the effect of active participation on increasing retention level. Evidence suggests that learnt information is retained longer if the student is an active participant in the learning process, rather than just a passive listener (Rickel, 1989). This evidence fully supports the constructivist philosophy. Rickel also explains how information is retained longer if the presentation involves several of the student's senses, and so multimedia must have a significant advantage over books with its audio-visual creative powers. Experts in diverse fields suggest that true understanding of knowledge gained from formal training comes when the student later combines that knowledge with actual experience and application. Educators must therefore endeavour to reduce the passive role of the student and encourage constructivism and interactivity.

The features for increasing retention level reported in this section are incorporated in the software in Chapter 5. Users cannot proceed to the next problem until they have reached the required level of understanding, i.e. successfully completed the problem to hand. The users have ownership of the problems, and the solutions depend entirely on their actions (with support from the teacher). Therefore usage of the software, necessitating interactivity and active participation, clearly exemplifies the constructivist ideal.

2.5.3. Case-Study of Functions and Graphs

Well-designed graphical software can help students overcome mathematical weaknesses. There are numerous software packages available that encourage the exploration of the graphical representation of functions, and innovative 'visual' programming languages allow the production of tailored mathematics software to a near-professional standard (Edwards, 1995).

A report describes various computer programs that have been developed with the main aim of assisting students in the learning of mathematics (Nyondo, 1993). One of the packages is a piece of graph-plotting software called Algebra Graf(x). It enables the teacher to demonstrate numerous functions and their graphs in a way that cannot be achieved by merely using a chalkboard or a piece of paper. The students were able to plot many related graphs and observe the visual effects of changing certain parameters in the algebraic notation. With AUTOGRAPH, pictorial representations can be 'dragged', and the resultant symbolic changes observed. This is a straightforward task using a computer, and yet it can give a deep insight into the relationship between functions and their graphs. Usage of Algebra Graf(x) allows for 'symbols to pictures' transfer, and usage of AUTOGRAPH allows for transfer in both directions. However, the software described in Chapter 5 additionally forces the user to transfer from 'pictures to symbols' initially, thus promoting a constructivist approach, and encourages flexible switching between representations via the evaluation and justification of conjectures.

Function-plotting tools have facilitated the movement away from a focus on calculating values and plotting points, toward a more global emphasis on the behaviour of entire functions, and even families of functions (Dugdale, 1993; Yerushalmy and Schwartz, 1993). In the past, graphs were rarely taught with an eye toward viewing their global features. They were used most often as another way of representing a relationship that was initially depicted in algebraic form. A study is described which illustrates this shifting of emphasis (Kieran, 1993). The students in the study felt that their experience with graphs went far beyond that usually gained by the traditional concentration on the creation of graphs of simple equations. The real thrust of the use of graphical software is that the students themselves can see the benefits. They felt that the computer aided in their conceptual understanding by refocusing their attention in three ways:

1. The computer relieved them of some of the manipulative aspects of calculus.

2. It gave them confidence in the results on which they based their reasoning.

3. It helped them focus attention on more global aspects of problem solving.

Computer technology has allowed educators to fashion new software tools that enable a user to manipulate the graphical representation itself (for example, 'stretch' or 'squeeze' graphs) and to view the impacts on the numerical and symbolic representations that result from this manipulation (Yerushalmy and Schwartz, 1993). An essential feature of this kind of environment is that different representations are linked.

A graph represents a specific function, whereas a symbolic representation often lends itself more readily to generalising to families of functions (Yerushalmy and Schwartz, 1993). Mathematical phenomena that are observed in visual settings can then be profitably explored within symbolic representations. Yerushalmy and Schwartz have found, through their research, that important ingredients of graph plotting software (all of which have been incorporated into the software described in Chapter 5) are:

- *Plotting* - the exposure of the learner to many graphs is assumed to catalyse the ability of conceiving the graph as an entity by itself, without the necessity of seeing the graph as a collection of plotted numerical data.

- *Scaling* - the shape of a graph is determined to a large extent by the choice of scale in the co-ordinate system. This exposes learners to various pictures of the same function, but with different scales. However, this could raise problems as to whether or not the student has the 'whole picture' (see Fig. 2.11 earlier, which illustrates the danger of the 'connected asymptotes').

- *Modifying expressions* - this allows users to explore the graphical role of each of the numerical parameters that appears in the symbolic representation of the function. The user can generate a family of functions by incrementing or decrementing a parameter. This allows the user to investigate the properties of the family of functions that has been created by this parametric variation.

With the aid of computer-based graphing techniques, students can obtain accurate graphs quickly, and it can be seen how visualisation can be used to solve equations (Demana et al., 1993). Without technology, it is not efficient to solve equations graphically because of the time required for most students to draw the graph by hand. With technology, however, students can easily obtain the graph, and conjecture the x-intercepts. Comparing these with the algebraic and/or numerical solutions to $f(x) = 0$ helps establish the connection between solutions of $f(x) = 0$ and x-intercepts of the graph of $y = f(x)$. This is a further example of the effective linking of algebraic and pictorial representations of functions.

A number of the desirable features of graphing software reported in this section have influenced the design of the software in Chapter 5. The 'visual' authoring language TOOLBOOK (see References for Internet address) was used to develop the software so that it could be tailored to support a constructivist approach. Many related graphs can be plotted with ease in order for students to appreciate how pictorial and symbolic forms reinforce one another. Use of the software encourages discussion, resulting in students being able to verbalise the concepts more confidently. The software provides students with opportunities to view graphs as conceptual entities, emphasising global features, and helps students to generalise from specific cases to families of functions. Finally, the important ingredients of graph plotting software described by Yerushalmy and Schwartz, concerned with plotting, scaling, and modifying expressions, have all been incorporated in the design.

2.6. Summary

The purpose of this summary is to illustrate the extent to which the previous work reported in this chapter addresses the aims outlined in Chapter 1, and to discuss the contents of the remainder of the thesis in light of any shortfalls and limitations. Having given a brief overview of the thesis at the end of Chapter 1, a more detailed description of the remaining chapters can now be provided.

Chapter 3

The merits of constructivism (as opposed to instructivism) in the teaching and learning of mathematics have been reported, and it has been seen how a constructivist approach can be particularly useful in the acquisition of conceptual knowledge. Chapter 3 builds on this work by considering the theoretical aspects of the development of cognitive structures (which have been reported in a general sense), and discusses them in terms of how they apply to the teaching and learning of mathematics, concentrating on how constructivism supports the use of visualisation in particular.

Different perspectives of constructivism are discussed, and in order to further existing work these are assessed in terms of the extent to which they support the use of computer-based visualisation and social interaction in mathematics education.

It has been reported how educational technology can be employed to provide environments in which students can explore mathematics. By exploring the visual aspects of mathematics, and then carrying out the symbolic formalisations at a later stage, a more accessible pedagogic path from concreteness to abstraction can be established. In this manner, Chapter 3 describes how the employment of appropriate software can act as a vehicle for supporting the constructivist philosophy.

Chapter 4

Having considered various related skills taxonomies concerned with both the learning and assessment of 'pen-and-paper' mathematics, Chapter 4 adds to the literature by proposing a skills classification, called DEVISe, which is more appropriate for the incorporation of computer-based visualisation in mathematics education.

The earlier work on skills classification is enhanced by showing how visualisation emerges as a common theme. This is an interesting observation which supports the proposal of this thesis, that visualisation is a key ingredient in the conceptual understanding of mathematics, and encourages the mature, insightful discussion and verbalisation of mathematics. The fact that students exhibit difficulties in visualising, however, means that they need to be subjected to appropriate experiences such as those described in Chapter 5.

Chapter 4 adds to the discussion of skills by showing, via alternative scenarios, how a different set of skills is developed depending on the teaching and learning approach adopted. It illustrates how the constructivist use of visualisation is a powerful facility for enhancing higher order skills.

Work has been carried out on the development of students' skills via the appropriate use of technology, but Chapter 4 discusses the extent to which students actually *demonstrate* certain skills when technology is at their disposal in assessment, having experienced a traditional instructivist approach to learning. This preliminary investigation provides useful information on students' skills prior to the case-study in Chapter 5, which analyses the skills students demonstrate after having experienced the constructivist use of computer-based visualisation.

Chapter 5

The case-study is described in detail, which attempts to satisfy the key aims by enhancing visualisation skills as a stepping stone to the acquisition of conceptual understanding and the development of higher order skills.

It has been reported how the appropriate use of visualisation can be used not only to clarify something expressed in symbolic notation, but can act as a vehicle for the acquisition of conceptual understanding. It can give the student an overall 'feel' for the situation, which is often not possible via symbols alone. This philosophy has been adopted in the case-study, which employs a bespoke interactive package that links graphical and symbolic representations. The graphical software allows the student to concentrate on the visual and holistic, and changes the emphasis to more global features of graphs. The bespoke software allows students to observe the effect of changing certain parameters, which can be a powerful way of appreciating the interrelationships between different representations (this is a key feature of the software). The subject domain of functions and graphs was chosen as this is of a particularly visual nature, but the visual skills required to interpret and then conjecture are generic to many other areas in mathematics.

Effective previous work has highlighted various important factors affecting mathematical software design, including the requirement of an explicit goal, visual effectiveness, the importance of interactivity and exploration, enhancement of conceptual understanding, motivation considerations, and the linking of different representations. All of these features have thus been incorporated into the design of the bespoke software.

A detailed rationale for the design of the bespoke software is provided, which is based on the reported merits of constructivism. There has been evidence provided that the employment of computer-based visualisation can enhance conceptual understanding and promote the ability to switch between representations. The bespoke software takes this further by forcing a constructivist approach, which brings with it the reported additional benefits.

Numerous examples have been reported illustrating the positive effect of symbolic manipulation software and handheld technology on student learning. However, when considering the graphical representation of functions (as in the case-study) software such as CAS and graphic calculators merely allow the student to observe what the graph of a given function looks like. The bespoke software, on the other hand, allows the student to interpret a given graph, conjecture its algebraic form, and then plot the conjectured expression, thus encouraging the student to switch between symbolic and pictorial forms in both directions, instead of merely always moving from the symbolic to the pictorial. This assists in the understanding of the relationship between different forms.

There have been a number of pieces of anecdotal evidence to suggest enhanced learning via innovative teaching methods but little evidence of actual measurement or evaluation of these approaches. Work has been done to measure the educational benefits of visualisation and constructivism *in isolation*, but this research goes one step further by providing empirical evidence, by means of a controlled experiment, that quantifies the benefits of an approach integrating *both* visualisation and constructivism. An analysis of data collected from 245 students, on a variety of variables, leads to conclusions as to the effectiveness of the approach in terms of conceptual understanding, the development of skills, and motivation.

Chapter 6

There are many reports that highlight the effective use of the computer in combining symbols and images in the build up of conceptual knowledge, but little evidence of any theoretical underpinning linking subject and skills development together. It has been stated in previous work that "*research must be done on structured testing and evaluation of visual learning and we must build working models of such evaluation, so the agencies which use these evaluation tools will be able to consider visual learning properly*" (Cunningham, 1991), and "*better theories of learning appropriate for practical teaching*" (Tall, 1992) need to be established. The structured testing and evaluation is dealt with in Chapter 5, and Chapter 6 responds to these pleas for research by concentrating on the application of a theoretical framework to model the constructivist use of computer-based visualisation in mathematics education.

This theoretical underpinning of the teaching and learning process is formulated as a direct result of the outcomes of the case-study, and builds on existing theories of teaching and learning. As the case-study concentrates on a localised subject domain, it was considered more useful to apply a generic framework to the process of problem solving using symbols and visualisation.

Chapter 7

Possible future research as a result of the findings of this thesis is suggested, both specific to the case-study and for mathematics education in general.

It has been seen how different types of mathematics questions can now be asked with the aid of technology, containing elements of interactivity, and placing the emphasis on visualisation and exploration, which can help in the development of desirable transferable skills. Chapter 7 considers the impact of this on mathematics education, in particular the A-level curriculum.

Future development of teaching and learning strategies in mathematics education is considered, taking into account developing computer technologies such as Artificial Intelligence and Virtual Reality.

CHAPTER 3

Constructivism: A Theory of Learning

The mind is not a vessel to be filled but a fire to be kindled.

PLUTARCH

3.1. Introduction

This chapter sets the scene for how the constructivist use of computer-based visualisation can be employed in the teaching and learning process. The integration of constructivism and visualisation, coupled with strategic questioning, can encourage the reformulation of conceptual structures and the development of higher order skills, such as the ability to interpret, conjecture, and evaluate. A discussion of instructivism and constructivism in the mathematics classroom is provided, which endeavours to show why the latter is a preferable methodology, in particular when optimising learning via visualisation. This philosophy is then studied further by taking into account both Piagetian and Vygotskian perspectives on constructivism in order to determine which is the more appropriate philosophy for learning via computer-based visualisation. Finally, consideration is given to the best way to employ constructivism in teaching and learning with computer-based visualisation.

Higher mental functions, such as visualisation, originate from the interaction between human beings, or some other external stimuli, but the functions themselves can transcend the context from which they originate (Rowlands et al., 1996). For example, visualisation can be nurtured in a particular setting, but then used elsewhere. A student's understanding of mathematics may not be evolving as you would wish, such that the understanding is still specific to the examples given and the way the subject is taught. A *fully evolved* understanding of mathematics, at any level, ought to be independent of the specific examples used, and the approach taken, by the teacher. For example, the software described in Chapter 5 looks at translations specific to certain functions, but the newly acquired conceptual structures can be applied to any f(x), i.e. not dependant on specific functions. The software uses examples as generic organisers (Tall and Thomas, 1991) in order to help with the abstraction of concepts.

Visually stimulating computer environments can allow students to become immersed in their own knowledge construction. However, it is not a trivial matter to utilise this considerable technological capability most effectively for educational benefit, emphasising the importance of a teaching and learning methodology that can provide a link to other factors influencing the use of visualisation.

3.2. Instructivism versus Constructivism

Instructivist and constructivist approaches to teaching and learning are discussed, focusing on the use of visualisation in mathematics. The impact on teaching practice is considered, taking into account the role of the teacher in each philosophy.

Instructivism as a mode of teaching indicates that a particular set of representations is employed with the aim of mediating mathematical instruction (O'Reilly et al., 1997). Instructivism reflects the traditional hierarchical view of mathematical study, where instructive representations are finely tuned to a particular purpose, making limited use of visual approaches. O'Reilly et al. describe how this is perhaps not appropriate for a whole range of situations. Students who are subjected to this instructivist approach need to be able to discriminate between contexts in order to appreciate when one finely tuned representation is needed as opposed to another. This is clearly a non-trivial process. An example of where this might occur is in the teaching of sets, logic, and boolean algebra as three distinctly separate topics. It would be more pedagogically sound, from a constructivist perspective, to teach the topics simultaneously, highlighting the clear links between the disciplines. In set theory, concepts are represented visually in the form of Venn diagrams. However, these diagrams can also be used to support the other two topics. For example, in logic, the universal set maps over to true, and the empty set maps over to false. Representations can therefore be mixed for the benefit of all three topics, typified by the following three expressions:

$$A \cup \bar{A} = U \quad \text{(i)}$$

$$p \vee \neg p \Leftrightarrow T \quad \text{(ii)}$$

$$p + p' = 1 \quad \text{(iii)}$$

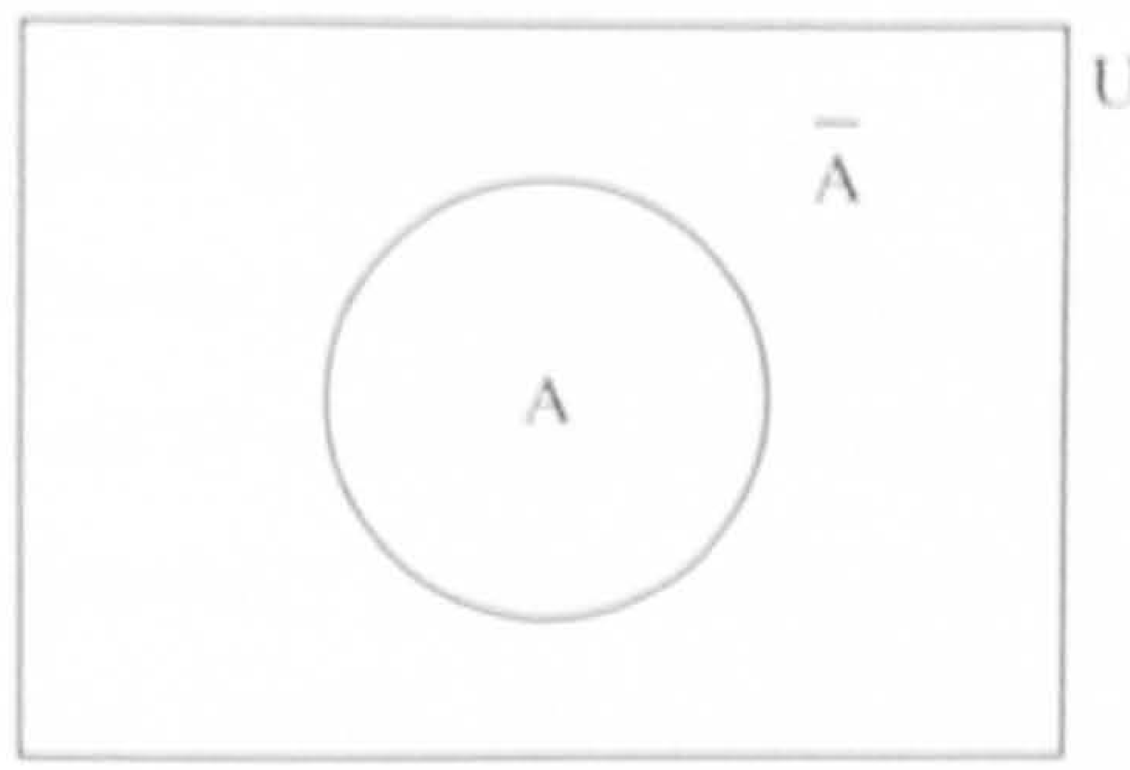


Fig. 3.01.

Fig. 3.01, above, illustrates expression (i) from set theory, but also serves to support expressions (ii) and (iii) from logic and boolean algebra respectively. The three expressions have the same level of abstraction, as they have the same underlying structure. A constructivist approach would, for example, invite students to conjecture logic laws on the basis of the set laws. Given the Venn diagram in Fig. 3.02, below, students could conjecture an appropriate logic law. Here the Venn diagram clearly reinforces, and provides meaning for, the logic law provided.

$$p \vee (p \wedge q) \Leftrightarrow p$$

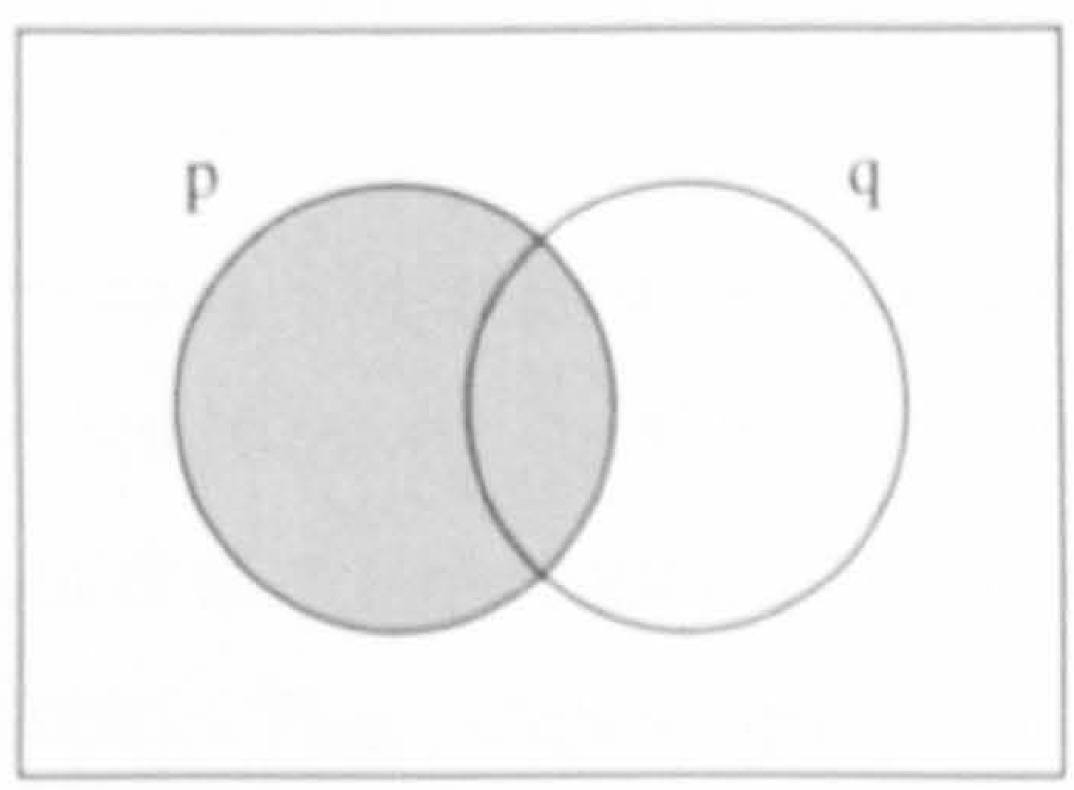


Fig. 3.02.

The instructivist, or behaviourist, approach is to pre-plan a curriculum by breaking down a subject area (usually seen as a finite body of knowledge) into assumed component parts, or required skills (see Chapter 4), and then sequencing these parts into a hierarchy ranging from simple to more complex (Fosnot, 1996a).

Instructivism assumes that listening to explanations from teachers, or engaging in experiences and observations will result in complete learning. Learners are viewed as passive, and educators spend their time developing a sequenced, well-structured curriculum and determining how they will assess, motivate, and evaluate the learner. The learner is expected to progress in a continuous, sequential fashion as long as clear communication and appropriate reinforcement are provided. Via this approach, progress by learners is assessed by measuring observable outcomes, i.e. 'behaviours' on predetermined tasks (normally symbolic). Fosnot describes how much of the current teaching practice follows this instructivist psychology, regardless of the age of the learners or the mathematical topic under study. This approach tends to explain change in procedural ability well, but it offers little in the way of explaining change in conceptual understanding.

Schifter sums up the instructivist way of thinking as follows: "*The teacher shows the students procedures for getting right answers and then monitors them as they reproduce those procedures. To ask a question without having previously shown how to answer it is actually considered 'unfair'. People acquire concepts by receiving information from other people who know more; that if students listen to what their teachers say, they will learn what their teachers know; and that the presence of other students is incidental to learning*" (Schifter, 1996).

As a result of schools (and often universities) adopting an instructivist approach to teaching, it has been reported that students are unable to solve problems successfully in the real world (Honebein et al., 1993; Smith, 1998; Corfield, 2001; Davis, 2001; Nyman and Berry, 2002). They cannot generally apply their knowledge to unfamiliar problem solving situations.

A different type of learning activity is required for the development of problem solving skills, i.e. constructivism. Here the concern is not mastery in a test of procedural skills, but rather the ability to function successfully in unfamiliar problem solving situations, often requiring a balance between symbolic and pictorial competence. For example, an instructivist question concerning functions might be to find turning points, asymptotes, etc., and then, almost as an afterthought, to plot the graph. An example of a constructive question, however, could be to consider some function, $f(x)$, and then determine what happens when a particular symbol within the expression is altered; the students would then be encouraged to explore and investigate. The constructivist philosophy thus invites students to find answers for themselves (Group A skills of the MATH taxonomy being displaced in favour of Group C skills).

The focus with constructivism is to be able to take the knowledge gleaned from local tasks and apply it globally (Honebein et al., 1993). The learning activity has a purpose that goes beyond simply demonstrating mastery of the local tasks; the purpose for a learning activity is driven by the global underlying concepts. This includes the ability to notice when particular skills and information are called for, to be able to recall or find that information, and to be able to apply those skills and that knowledge to solve a real problem (Chapter 4 investigates the types of skills that students choose to apply in problem solving situations). It is therefore not the ability to recall information that educators should be interested in, but instead the ability to apply knowledge and skills in different problem based environments. The constructivist approach, which supports the construction of appropriate mental images, concentrates on a holistic view of learning mathematics, and focuses on deep understanding and strategies, rather than facts and rote memorisation (Honebein et al., 1993; Fosnot, 1996a; Bowles et al., 2002).

Following on from extensive work by Piaget and Vygotsky, constructivism has emerged as a leading psychological theory in teaching and learning. In Higher Education mathematics programmes, it is having a considerable effect on the goals teachers set for learners, the instructional strategies teachers employ in working toward these goals, and the methods of assessment to document genuine learning. These methods include assessment that incorporates the use of technology in which a combination of visual ability and symbolic dexterity can be examined.

Assessment of this nature allows for the testing of not only higher order skills themselves, but also the appropriateness of their application. Implicit in constructivism is the idea that human beings have no access to an objective reality since we are constructing our version of it, while at the same time transforming it and ourselves (Fosnot, 1996a).

Widespread interest in constructivism has recently led to a debate between those who place more emphasis on the individual cognitive structuring process (the radical constructivists) and those who emphasise the sociocultural effects on learning (the social constructivists). Rowlands et al. (1996) consider that the central issue of the constructivist controversy seems to be 'what is knowledge and how is it constructed?' However, Fosnot (1996a) believes that a more important question to be asked is not whether individual cognition or social effects should be given priority in an analysis of learning, but what the interplay is between them, as they are both important factors in a student's cognitive development.

Constructivism focuses exclusively on the processes by which individual students actively construct their own mathematical understanding (Lerman, 1996b). It enables the teacher to view students' perspectives, which is not appropriate with instructivist teaching. Learners are viewed as active meaning-makers, interpreting experiences with their existing cognitive structures. Individuals approach novel situations by interpreting them in the light of their own established structures of understanding. That is, new concepts are constructed when those established understandings do not allow satisfactory accommodation of the novel circumstance. Constructivism provides a means for building on existing cognitive structures hierarchically, whereas instructivism leads to the placing of new understanding adjacent to existing structures. For example, a student may have an initial belief that finding the area between a curve and the x-axis merely involves integrating the expression and inputting the limits. If the student has only encountered contrived examples where the area is completely above the x-axis, then a problem will arise when he or she computes a negative area. The established understanding of the process does not allow satisfactory accommodation of the negative solution. It is only through further exploration, normally by means of visual representations and social interaction, that the student will begin to understand the implications of integrating curves that exist below the x-axis. As a result of this constructivist process, the next time a negative solution is encountered, the student will be able to accommodate it into his or her reformulated cognitive structure.

Another example of building cognitive structures hierarchically is the studying of the process of differentiation only after having explored the properties of curves, such as gradient, continuity, etc. In this constructivist manner, students have a greater ability to handle the types of misconceptions described as theoretical-computational conflicts by Giraldo et al. (2002). These computer-generated images, observed through local magnification, appear to contradict the associated theory. If theoretical-computational conflicts are emphasised, rather than avoided, they can be used to enrich concept images. This constructivist approach leads students *through* the problem, rather than merely providing a 'recipe' for getting *around* it. Giraldo et al. invited students to explore the differentiability of the function $h(x) = \sqrt{x^2 + 1}$ and the corresponding graph for $(x, y) \in [-100, 100]^2$, as in Fig. 3.03.

The conflict is between the appearance of the graph at the origin (which seems to have a 'corner') and the differentiable expression. The appropriate use of technology allows the student to zoom in to the 'corner', producing the graph in Fig. 3.04, in order to build a deeper understanding of the local function behaviour. The constructivist approach encourages the student to switch between symbolic and visual representations.

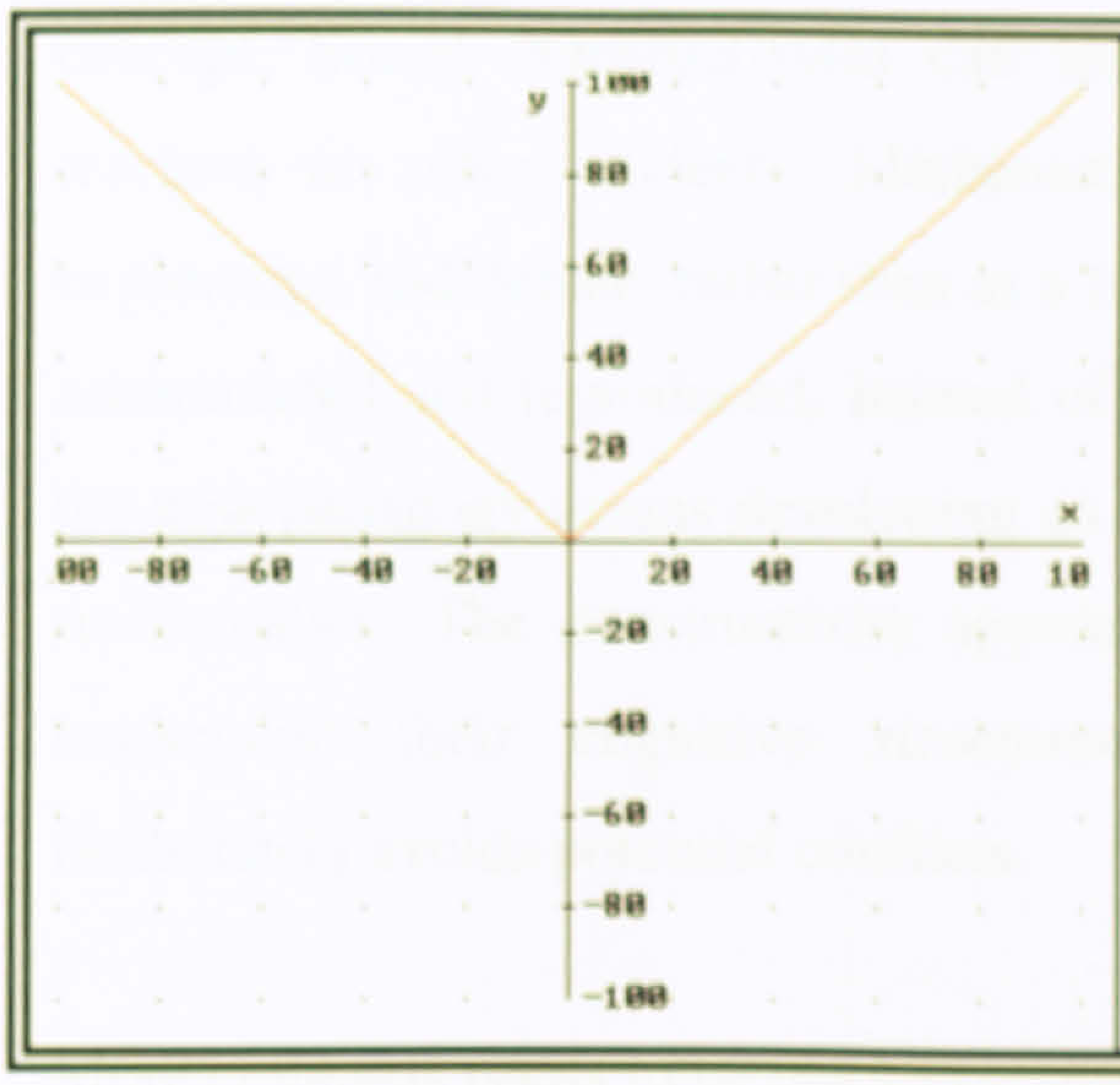


Fig. 3.03.

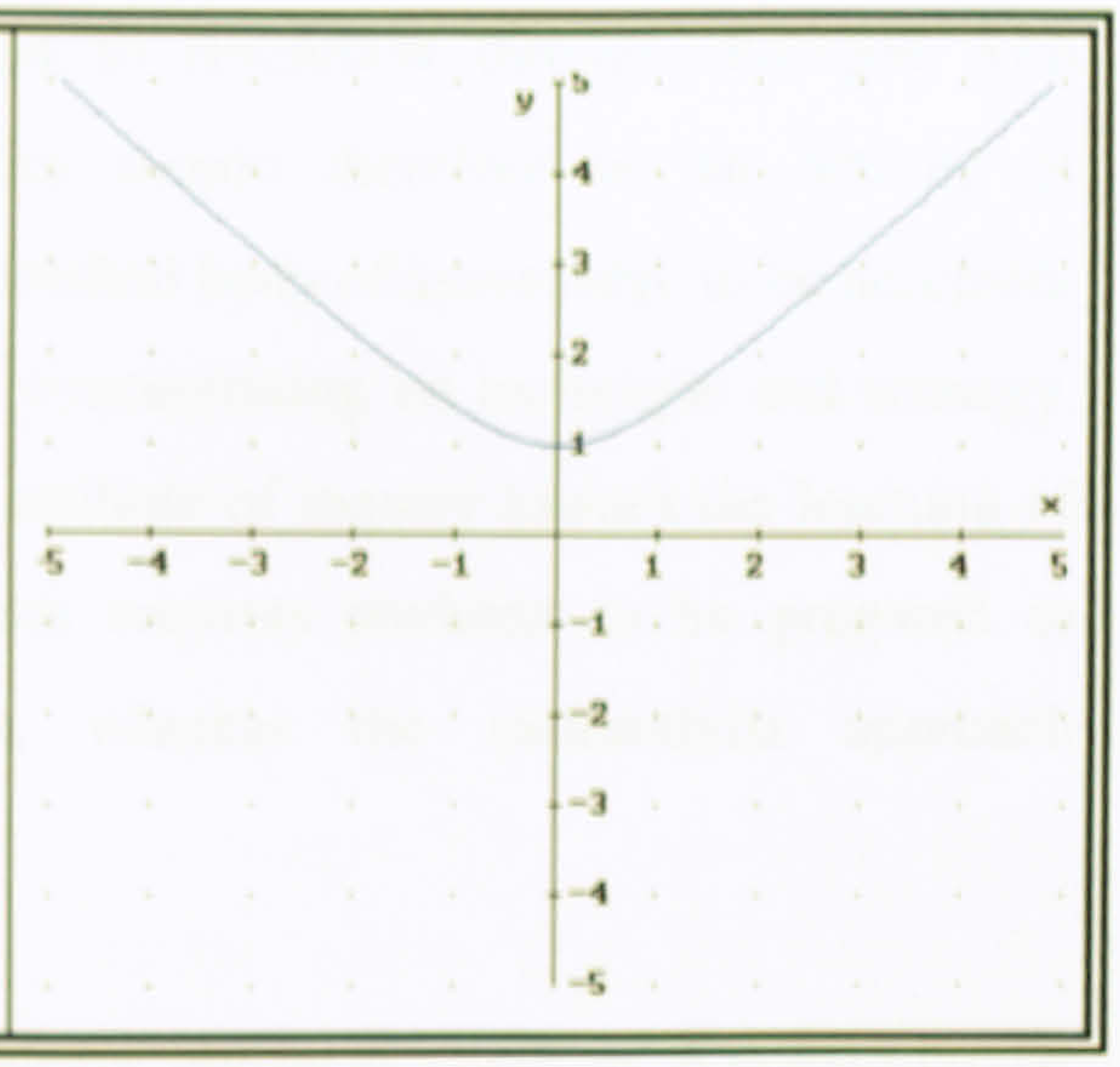


Fig. 3.04.

This type of constructivist activity, as described above, is not simply an individual achievement but is enabled by social interaction (Schifter, 1996). Constructivist learning is a recursive, building process by active learners interacting with the social world (Fosnot, 1996a; Watson, 2001; Pear and Crone-Todd, 2002). The challenge for educators has been to determine what this new paradigm brings to the practice of teaching.

Teaching mathematics from a constructivist perspective involves the provision of activities designed to encourage and facilitate the constructivist process (Holt-Reynolds, 2000). With this perspective, developing an approach to thinking about mathematical issues would be valued more highly than memorising algorithms and using them to obtain correct answers. It also enables students to learn how to construct a mathematical argument and assess its mathematical validity. Teachers need to pose problems such as those from calculus described above, which encourage students to explore mathematical concepts. In turn, students need to feel that they own the problem, which they feel compelled to resolve. This philosophy provides an organising role and a purpose for learning (Schifter, 1996). When students are faced with contradictions to their own conjectures, it is up to them (with appropriate guidance) to find resolution, leading to a deeper grasp of the concept. Social constructivism can lead to resolution through dialogue with teachers or other students. Mathematics should therefore be an activity of exploration and debate, rather than as a finished body of knowledge to be accepted, accumulated and reproduced. Instead of concentrating on technique and strategy, the new pedagogy means developing an attitude of inquiry toward the learning of mathematics. The constructivist approach requires students to be prepared to reassemble their cognitive structures, whereas the instructivist approach deliberately avoids potential conflicts.

More emphasis needs to be placed on students' ability to visualise. A constructivist approach that encourages the use of imagery promotes visual thinking and in turn enhances conceptual understanding. Let us consider how the development of visualisation ability can give students a better conceptual understanding of a mathematical situation. For example, when faced with the problem

$$\int_{-\pi}^{\pi} \sin(x)dx$$

many students prefer (or rather without stopping to think) to perform the integration and input the limits to arrive at the answer (Malabar et al., 1998). Students with visualisation ability, however, will be able to 'see' the graph and deduce straight away that the answer must be zero. They know the answer, but more importantly their mental image provides them with the desirable conceptual understanding of the situation. Is it just the 'bright' students who see the answer quickly, or is it that students are more trustful of an answer obtained by symbolic manipulation? A similar situation arises when, having been taught integration by parts, many students will not hesitate to use that method to show that

$$\int_{-1}^{1} x^3 \cos(x)dx$$

is also zero, without any consideration of graphing the integrand. A constructivist teaching strategy promotes the development of the necessary higher level skills required to tackle such problems more efficiently, whereas the more familiar instructivist approach limits students to the acquisition of lower level, procedural skills. Exploring the implications of integration in terms of positive and negative areas, as described earlier, would provide a useful precursor to the above problems.

Students, regardless of their ability, are not simply empty vessels that require filling with knowledge, but instead, some form of construction of knowledge needs to take place in order for students to learn effectively. Knowledge is not passively received either through the senses or by way of communication; knowledge is actively built up by the individual (von Glasersfeld, 1990).

Constructivism, by its very nature, is a theory about learning, not a description of teaching. The teaching style governs the type of learning that takes place, and therefore needs to be appropriate in order to support constructivism. Fosnot (1996a) provides some general principles of learning derived from constructivism that may be helpful as educational practices are rethought and reformed:

- Learning is not the result of development; learning *is* development. Teachers need to allow learners to raise their own questions, generate their own hypotheses and models as possibilities, and test them for viability.

  [There could, however, be practical difficulties for teachers, as it is more challenging to create and maintain this kind of environment than to teach in an instructivist manner].

- Challenging, open-ended investigations need to be offered, thus allowing learners to explore and generate many possibilities, both affirming and contradictory. Contradictions, in particular, need to be illuminated, explored, and discussed.

[This can be achieved by exploring misconceptions associated with pictorial information, such as the theoretical-computational conflicts discussed earlier (Giraldo et al., 2002)].

- Reflective abstraction is the driving force of learning. As meaning-makers, humans seek to organise and generalise across experiences in a representational form.

  [These representational forms can be symbolic or pictorial, or more powerfully, a combination of the two, which can reinforce understanding].

- Dialogue engenders further thinking. Learners are responsible for defending, proving, justifying, and communicating their ideas to the classroom community.

  [The appropriate use of visualisation can aid this dialogue].

- Learning proceeds toward the development of structures. As learners struggle to make meaning, progressive structural shifts in perspective are constructed. These often require the undoing or reorganising of earlier conceptions.

  [An example of this is the difficulty with calculus at A-level. Students build up conceptions regarding differentiation, but then they have to reorganise their cognitive structures when further study involves the consideration of issues such as differentiability and continuity].

These general principles have indeed been useful for this thesis, and have been assimilated into the teaching philosophy adopted for the case-study in Chapter 5. They have helped to structure the teaching in order to support and encourage constructivist learning. The dialogue between student and teacher (in Section 5.3.3.3 of Chapter 5), which describes the learning process of a particular student whilst using the dedicated software, provides a striking example of how the general principles have been incorporated into the overall teaching style.

Interaction is the process of an individual dealing with previously constructed perceptual and conceptual structures (von Glasersfeld, 1996). Too often, teaching strategies are developed from the assumption that what mathematics educators perceive and infer from their perceptions is there, ready-made, for the students to pick up, if only they had the inclination. This, explains von Glasersfeld, overlooks the basic point that the way we organise our experiences, and the way we relate the various component parts, is an essentially subjective matter. In a practical setting, a teacher cannot possibly know the make-up of individuals' cognitive structures. This is a serious problem in the teaching of mathematics due to its hierarchical nature. Knowledge is an individual (subjective) construction that describes or refers to an individual's experiences, and does not describe or refer to the real world (von Glasersfeld, 1984, 1995, 1996). Hence, when teachers intend to stimulate and enhance a student's learning, they cannot afford to forget that knowledge does not exist outside a person's mind.

Students may perceive their environment in ways that are very different from those intended by educators. A teacher can hope to induce changes in their ways of thinking only if he or she has some idea of the experience, and the conceptual understanding the students possess at any given time. The fundamental principle of constructivism is that learning is very much a constructive activity that the students themselves have to carry out. From this point of view, then, the task of the educator is not to dispense knowledge but to provide students with opportunities and incentives to build it up.

Understanding is therefore embedded in the experience of the individual. 'Experience', as described by Honebein et al. (1993), includes both the physical context in which a person works and the tasks, both cognitive and physical, that a person engages in whilst in that environment. That is, both the physical context for learning and the activities of the learner determine how something is understood, i.e. what is learnt. If the physical context is of a visual nature, and the learner is engaged in activities that necessitate the use of visualisation, then the creation of meaningful concept images can be positively influenced.

Finally in this section, there is the consideration of the 'zone of proximal development' (Scardamalia and Bereiter, 1991; Lerman, 1996a). This is defined as the area in which the student can perform tasks successfully, but only with some assistance. The student therefore works in a constructivist manner, inside an instructional domain. The zone of proximal development is the gap between what students can do on their own and what they can do with input from, for example, a teacher. The learning activity constitutes the zone of proximal development; it is actually the difference in activity between 'with or without' the teacher. As the student and the teacher work together in the zone of proximal development, there is therefore the question of who has ownership of the task (Scardamalia and Bereiter, 1991). Students can benefit from owning the task without being totally on their own. The teacher is there to guide, and to share in evaluating their progress. However, by placing the student in control, the emphasis is on self-directed learning, which supports the constructivist philosophy.

An example of this scenario is the way in which a teacher can guide a student whilst using CABRI-GEOMETRE for the dynamic exploration of Euclidian geometry. Take for instance a situation where the student is exploring the implications of bisecting angles and sides of triangles. In CABRI-GEOMETRE, the student can perform angle bisections and observe that the point of intersection is the centre of an inscribed circle, and similarly can perform perpendicular bisections of the sides and observe that the point of intersection is the centre of a circumcircle. The dynamic software then allows the student to drag vertices of the triangle to illustrate that this is the case for any triangles. The student is therefore in control of the environment, in terms of the functionality of manipulating shapes, etc., and can conjecture proofs, and devise appropriate strategies. The teacher's role is to simply steer the student in a fruitful direction, and to provide guidance if he or she deviates. The aim is to guide students through visual explorations that will enhance their skills set. The teacher can achieve this by asking probing questions such as:

- Do the angle bisectors meet at a point?
- Do the perpendicular bisectors of the sides meet at a point?
- If so, do they always meet inside the triangle?
- Are the properties of these points of any interest?
- Are the altitudes of any newly constructed triangles of any interest?
- Are your findings true for any shape or size of triangle?
- Is there any relationship between any line segments?
- Is there any relationship between the area of the triangle and the area of the circle?
- Is there any relationship between the areas of inscribed and circumscribed circles?
- etc.

Questions of this type encourage students to interpret and evaluate their findings, make comparisons, discuss any implications, and conjecture and justify proofs (desirable Group C skills of the MATH taxonomy).

The zone of proximal development is exemplified by the software usage described in Chapter 5, where assistance in understanding the relationship between graphical and symbolic forms is provided whilst students use the dedicated teaching software. Clearly the aim here is to intentionally increase the size of the zone (the dialogue in Section 5.3.3.3 of Chapter 5 illustrates the role of the teacher).

3.3. Piagetian and Vygotskian Perspectives on Constructivism

This section discusses the radical constructivist perspective of Piaget, and the social constructivist perspective of Vygotsky. Ideas within both theories are considered (Marin et al., 2000) in order to determine which is the more appropriate philosophy for learning via computer-based visualisation.

There are many people in the academic community who are sceptical about constructivism. The resistance to this theme, first met in the eighteenth century by Silvio Ceccato, and in the twentieth recently by Jean Piaget, is not so much due to inconsistencies in their arguments as to suspicion that constructivism tends to undermine too large a part of the traditional view of the world (von Glasersfeld, 1996).

Piaget's constructivist perspective is that the individual is responsible for his thinking and his knowledge. Our knowledge can never be interpreted as a picture or representation of the real world (von Glasersfeld, 1996), but only as a key that unlocks possible paths for us. The Piagetian viewpoint is that all understanding is a matter of interpretive construction on the part of the experiencing subject.

The epistemological problem is how we acquire knowledge of reality, and how reliable and 'true' that knowledge might be. The basic principle of the Piagetian epistemology is that the experiential world constitutes the testing ground for our cognitive structures. In the light of further experience, theories either prove themselves reliable or they do not. Our knowledge is useful and relevant if it stands up to experience. Ideas and theories are structures that are constantly exposed to our experiential world, and either they hold up or they do not. In mathematics, theories can be conjectured, and they are either proved or disproved in light of further specific examples. In a similar fashion, mathematical modelling applies mathematics to real-world situations, not only in light of existing data, but also for further data. The real-world scenario, however, gives more confidence that the hypotheses are correct.

The Piagetian view, therefore, is that constructivism breaks with convention and develops a theory of knowledge in which knowledge does not reflect an 'objective' reality, but exclusively an ordering and organisation of a world constituted by our experience.

If knowledge is to be a description or image of the world as such, we need a criterion that might enable us to judge when our descriptions or images are 'right' or 'true'. The unanswerable question as to whether, or to what extent, any picture our senses convey might correspond to the 'objective' reality is still today the crux of the entire theory of knowledge. The question is unanswerable because, no matter what we do, we can check our perceptions only by means of other perceptions (von Glasersfeld, 1996).

In traditional theories of knowledge, the activity of 'knowing' is taken as a matter of course, an activity that requires no justification and which functions as an initial constituent. With constructivism, however, knowledge cannot be the result of a passive receiving, and must originate as the product of an individual's activity. Learners gradually build up their cognitive structures.

A person evaluates his experiences, and because he evaluates them, he tends to repeat certain ones and to avoid others. The products of conscious cognitive activity, therefore, always have a purpose and are assessed according to how well they serve that purpose.

Knowledge is something that a person builds up in an attempt to order the flow of experience by establishing repeatable experiences and relatively reliable relations between them (von Glasersfeld, 1996). The possibilities of constructing such an order are determined by the preceding steps in the construction. Constructivism must not be interpreted as a picture or description of any absolute reality, but as a possible model of knowing and the acquisition of knowledge in people that are capable of constructing for themselves, on the basis of their own experience, a more or less reliable world.

Piaget firmly believed that knowledge, no matter how it be defined, is in the minds of persons, and that the thinking subject has no alternative but to construct what he or she knows on the basis of his or her own experience. The problem with this, from a teaching perspective, is that students will construct knowledge at different rates from one another. Visualisation can perhaps play an important role here, by speeding up the cognitive structuring process for students who are weaker algebraically.

Knowledge is constructed by means of the physical and mental actions of the subject; the individual is primary, and is the central element in meaning-making (Lerman, 1996b; Rowlands et al., 1996). The construction of knowledge is totally individual because any social/external interaction is interpreted individually anyway (Lerman, 1994). With this perspective, the subject cannot go beyond the limits of individual experience. This condition, however, by no means eliminates the influence and the shaping effects of social interaction. The Piagetian notion of constructivism is that through social interaction individuals attempt to shape their images and theories.

Piaget was the pioneer of the constructivist approach to cognition in the twentieth century. It was his aim to produce as coherent a model as possible, of human cognition and its development (von Glasersfeld, 1995). Piaget believed that the human was a developing organism, not only in a physical, biological sense, but also in a cognitive sense. The main body of his work centred on illuminating the progressive cognitive structuring of individuals; knowledge develops from successive constructions (Fosnot, 1996a). Cognitive structures, when disturbed, generate new possibilities - possibilities of new actions or explanations. Fosnot describes how these possibilities are explored, and correspondences and/or patterns are constructed because of the human's self-organising tendency. Subsequent reflection on these correspondences brings about a structural change - an accommodation that transforms the original cognitive structure, and that explains why the pattern occurs, thus enabling generalisation beyond the specific experience. Piaget terms this process 'reflective abstraction'.

Both **assimilation** and **accommodation** are key terms in Piaget's theory (von Glasersfeld, 1995). Assimilation is treating new material as an instance of something known. Any experienced behaviour is always grafted onto previous schemes, and therefore amounts to assimilating new elements to already constructed structures. Cognitive assimilation comes about when a cognising individual fits an experience into a conceptual structure it already possesses. If an individual is unable to assimilate, there will be a **perturbation**. If the unexpected outcome was disappointing, one or more of the newly noticed characteristics may effect a change in the recognition pattern and thus in the conditions that will trigger the activity in the future. Alternatively, if the unexpected outcome was pleasant or interesting, a new recognition pattern may be formed to include the new characteristic. In both cases there would be an act of learning, which is known as an 'accommodation' (this is a more detailed description of the final bullet-point of general principles of learning provided by Fosnot (1996a) in the previous section, which considers the undoing or reorganising of earlier conceptions in the development of cognitive structures). An example of this process is where a student is faced with the problem

$$\int_{-1}^{1}\frac{1}{x^2}dx.$$

The student tackles the problem in the 'tried and tested' manner, by integrating the expression and inputting the limits, and finds that the area is –2. A perturbation could occur here in one of two ways. Firstly, the student might be aware that the graph exists entirely above the x-axis, and so cannot assimilate the negative area into his or her existing cognitive structure. Secondly, if the student plots the graph (unaware of its nature) in order to confirm the result, as in Fig. 3.05 below, a perturbation will occur, as the student cannot assimilate the existence of the graph above the x-axis with the negative area.

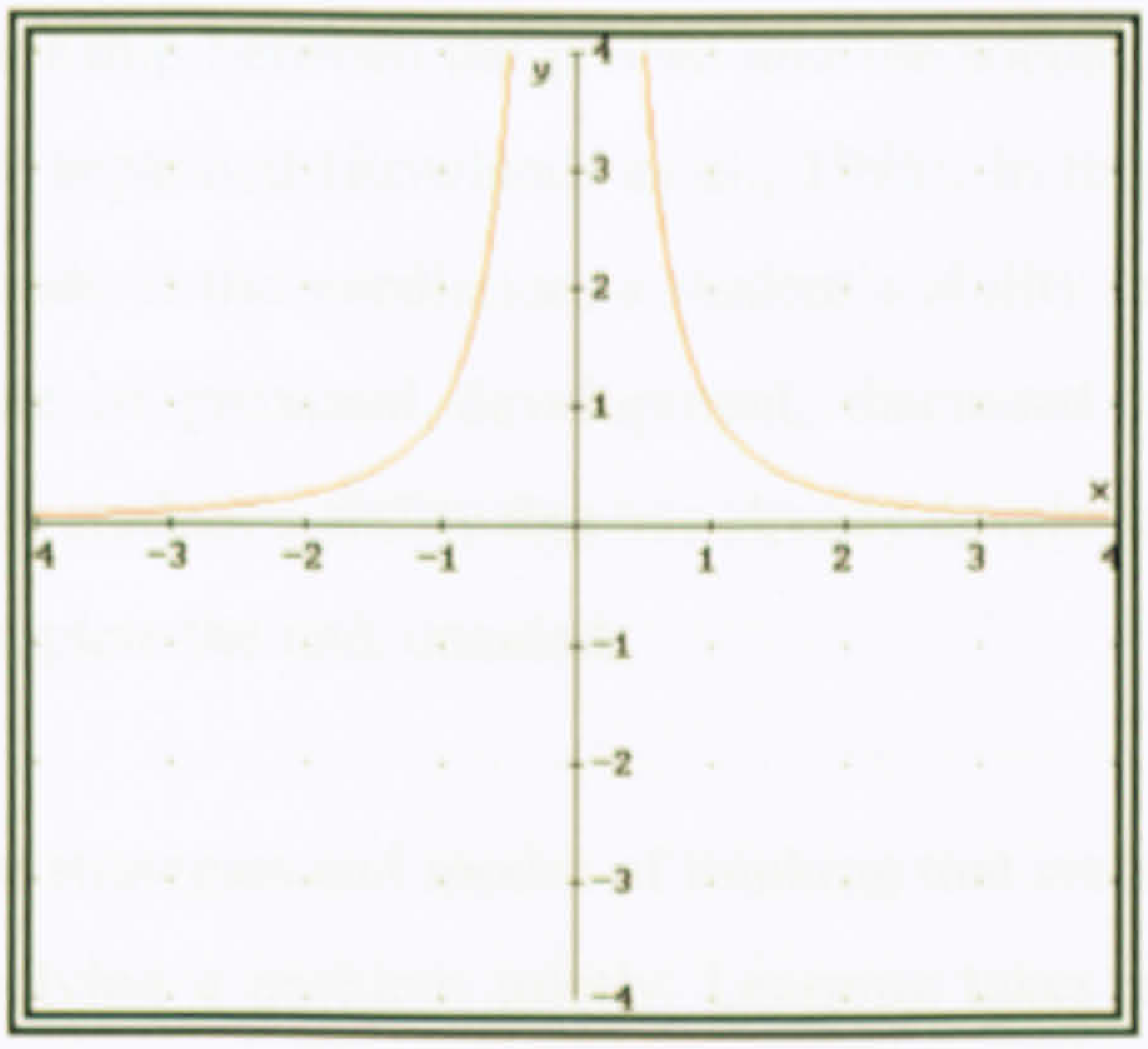


Fig. 3.05.

The visual representation helps the student to resolve the perturbation by illustrating that the area is infinite due to the existence of a vertical asymptote at $x = 0$. This new information results in a restructuring of the student's existing cognitive structure so that an accommodation can take place.

The learning theory that emerges from Piaget's work can be summarised by saying that cognitive change and learning take place when a scheme, instead of producing the expected result (assimilation), leads to perturbation, and perturbation, in turn, to an accommodation (von Glasersfeld, 1995).

The Vygotskian perspective on constructivism is now considered. Vygotsky attempted to develop a fully cultural psychology, placing communication and social life at the centre of meaning-making (Lerman, 1996b). He placed great importance on external influences during the building up of cognitive structures; knowledge is an individual construction in response to experiences in social contexts (Ernest, 1994). The individual constructs knowledge facilitated by a teacher or more able peers. If the mental functions of the student as a developmental process are to be explained, then the completion of a task that the student cannot do unaided must be facilitated (Rowlands et al., 1996). This can be described as a relationship between the private and the social, which, according to Vygotsky, cannot be separated (Rowlands et al., 1996). In this way, as a result of how a student responds to this mediation, a student's ability can be determined as it matures (the zone of proximal development, discussed earlier), rather than simply measuring the student's ability that has already developed, in the case of the student who can complete the task unaided.

Students can acquire strategies and modes of thinking that were first observed with the teacher while solving a problem jointly. Learning takes place in the zone of proximal development as a result of interaction with more knowledgeable peers or teachers (Lerman, 1996a). From this point of view, teaching and learning cannot be discussed separately - which is the viewpoint of this thesis.

The teacher can take on the role of facilitator in the construction of knowledge (rather than a giver of knowledge) by providing props and hints to develop students' cognitive framework. The teacher aids the learner in accomplishing the activity, not by doing the task for the learner or giving the learner the correct answers, but by providing guidance that require learners to formulate their own solution to the problem (Honebein et al., 1993). Probing questions are used as a catalyst to get students to reach the desired goal, without taking away the ownership of the task. In this manner, students can eventually arrive at a required level of understanding for themselves, which is not only advantageous in terms of the learning process, but also increases satisfaction and boosts confidence.

Strategic questioning, known as the Socratic method, can be used to facilitate the construction of a target concept, working within the students' zone of proximal development (Rowlands et al., 1997). Students have misconceptions of mathematics that are resilient to change. If a student realises that his or her intuitive ideas are inadequate, then change is possible. Rowlands et al. explain that this method of strategic questioning challenges (and hopefully removes) misconceptions, and facilitates the construction of knowledge. The key is to ask qualitative questions that lead the student to reach the target concept without it actually being given by the teacher. Consistent with the Vygotskian perspective, these questions provide hurdles to be overcome in order to develop cognitive growth, yet which also serve as props or hints to facilitate the process. The teacher must use questions that challenge students to think according to the properties of the target concept. Rowlands et al. discuss how intuitive concepts stand at one end of the zone of proximal development, and the target concept stands at the other - strategic questions stand in between and facilitate the progression from the former to the latter.

The Socratic method of strategic questioning is precisely the approach adopted in the case-study described in Chapter 5. One of the aims of the case-study is to assess changes in knowledge states, and hence the development of higher order skills (Group C skills of the MATH taxonomy), not merely the ability to gather more factual knowledge (Group A skill of the MATH taxonomy). The scenario in Section 5.3.3.3 of Chapter 5 describes a dialogue between student and teacher whilst the student attempts one of the questions during the case-study. It illustrates an attempt to assess the student's ability to accommodate a different style of learning. The teacher asks probing questions in an attempt to assess the student's ability to respond appropriately, rather than regurgitate information. The relationship between symbolic and pictorial forms plays an important role in unifying the responses.

There are potential problems, however, associated with this approach. Lemerise (1993) reported on a constructivist approach to learning using LOGO. Important differences in the level of involvement were noticed. The students with strong characters benefited more - they were full of ideas, and also reacted very positively to the interventions of the teacher. In contrast, the less able rapidly reached a ceiling in their interest, initiative, and ideas. A resistance to interventions by the teacher suggestive of new approaches or techniques was also observed.

The Vygotskian perspective supports the teaching of decontextualised concepts, enabling cognitive growth and development, leading to the ability to reason (Rowlands et al., 1996). Visualisation skills, together with other higher order skills, can therefore be developed in a particular setting, but can transcend the context from which they originate. Educators should not concentrate too much on specific contexts in the way that they teach. Instead, a more conceptual, global approach to teaching and learning mathematics needs to be encouraged. There are serious implications for contextualising too much. Rowlands et al. (1996) believe that attempts to make the mathematics curriculum more relevant to everyday life has simply diluted both content and the development of skills. Putting everything in context is more time-consuming, and therefore the range of topics has to be diluted as a result. Additionally, if all topics are presented in a specific context, then the students will not develop the ability to generalise globally. For example, real-world applications can help students to manage learning, i.e. they provide an 'anchor', but if everything is contextualised, this will dilute the development of desirable skills such as the ability to apply, reason, etc. If the students cannot divorce the concepts from their contexts, then the props become a 'barrier' to generalisation (this problem is analogous to the restrictive properties of diagrams reported by Presmeg (1986) in Section 2.3.2 of Chapter 2). As a consequence of the content of the curriculum and the instructivist way in which it is taught, the standard of mathematics in the UK is poor compared with other countries, despite continual improvements in GCSE and A-level results (London Mathematical Society (LMS), Institute of Mathematics and its Applications (IMA), and Royal

Statistical Society (RSS), 1995; Rowlands et al., 1996; Learning and Teaching Support Network (MSOR), Institute of Mathematics and its Applications (IMA), London Mathematical Society (LMS), and Engineering Council, 2000).

Vygotsky, like Piaget, assigned great importance to the development of cognitive skills (Group C skills of the MATH taxonomy) along with the more routine learning of mathematics such as algorithmic approaches and algebraic techniques. It is through the learning of concepts separate from the immediate and the concrete that cognitive structures are built (Vygotsky, 1962). For example, in mathematics for sport science, teaching the modelling of the flight of balls generally, to include impact, spin and aerodynamic concepts, will lead to students being able to apply the concepts to any ball sport (Group B skill of the MATH taxonomy). The understanding of concepts leads to the development of mental structures, which in turn provide a platform for the development of further conceptual understanding.

Vygotsky differentiated between what he called 'spontaneous' and 'scientific' concepts (Fosnot, 1996a). He defined spontaneous concepts as those that a learner develops naturally in the process of construction emerging from the learner's own reflections on everyday experience. He proposed that scientific concepts, on the other hand, originate in the structured activity of classroom instruction, and impose on the learner more formal abstractions and more logically defined concepts than those constructed spontaneously. Having made this distinction, one of Vygotsky's main questions became: What facilitates the learning that moves the learner from spontaneous concepts to scientific concepts?

Vygotsky argued that scientific concepts do not come to the learner in a ready-made form. They undergo substantial development, depending on the existing level of the learner's ability to understand the teacher. The case-study in Chapter 5 illustrates how the use of visualisation significantly enhances this development process. As mentioned earlier, Vygotsky's zone of proximal development describes the position at which a learner's spontaneous concepts encounter teacher reasoning. This zone varies from learner to learner and reflects the ability of the learner to understand the logic of the scientific concept. For this reason, Vygotsky viewed tests that only looked at the learner's individual problem solving capability as inadequate, arguing instead that the progress in concept formation achieved by the learner in co-operation with a teacher was a much more viable way to look at the capabilities of learners (Fosnot, 1996a).

To summarise, Vygotsky was interested in the role of the teacher and the learners' peers as they conversed, questioned, explained, and negotiated meaning. He argued that "*the most effective learning occurs when the teacher draws the learner out to the jointly constructed 'potential' level of performance*" (Fosnot, 1996a).

Interactions among students are recognised as much by Piaget as by Vygotsky as an important source of learning and development (Lemerise, 1993). For Piaget, social interaction is favourable as it can lead to a cognitive conflict, which calls for the reorganisation of cognitive structures. For Vygotsky, social interaction allows a student working with another 'more able' student to generate actions that he or she could not do alone, and thus allows the student to enter into his or her zone of proximal development.

From a Vygotskian viewpoint, social interaction provides more direct means for the sharing of knowledge. Followers of the Piagetian philosophy, however, would ask how such sharing of knowledge can take place when the individual is responsible for meaning-making. Both Piaget and Vygotsky place the individual and society at the centre of their theories. The difference in viewpoint, however, is encapsulated in their identification of the source of meaning - Piaget identifying the cognising individual, and Vygotsky identifying society and discursive practices (Lerman, 1996b).

The social constructivist perspective of Vygotsky is the more appropriate philosophy for the case-study in Chapter 5. The case-study places great importance on social interaction during the building up of cognitive structures. It exemplifies the zone of proximal development, as a student's ability is determined as it matures in response to mediation. Probing questions are employed to assess the ability to respond appropriately. The individual constructs knowledge of decontextualised concepts facilitated by the teacher. Visualisation skills are developed in a particular setting, but can be further employed in different contexts. The case-study identifies social interaction as a key source of meaning, as per the Vygotskian perspective, whereas the Piagetian perspective identifies it merely as a means of shaping an individual's already formed cognitive structures.

3.4. Constructivism in Relation to Educational Technology

Chapters 1 and 2 have illustrated the power of technology in the teaching and learning of mathematics in terms of its ability to incorporate visualisation, and the resultant effect on achievement and motivation. This section now considers the design of computer-based learning environments from a constructivist perspective (Dalgarno, 2001; Malabar and Pountney, 2002). The use of technology allows us to create teaching and learning materials that provide a visually compelling learning environment in which students can construct knowledge for themselves. Educators need to allow for investigation and experimentation in student learning and development (Fosnot, 1996b). They need to approach curricula in a learner-centred fashion with the emphasis on investigation, reflection and discourse. The teacher should act as a facilitator in the process, providing support and guidance, and allowing for further investigation and deeper understanding through questioning and probing. Educators must critically evaluate the teaching and learning process, and design teaching activities to promote learner construction. Students need to be allowed to investigate, and to raise their own questions. The constructivist use of technology offers the opportunity to change the nature of the material to be taught and learnt from routine-based to discovery-based activities.

As discussed in the previous section, knowledge is built up from personal experiences, and making these experiences more dynamic will assist in the development of cognitive structures (Tall, 2000a, 2000c). Computer-based attractive environments with visually compelling displays, together with facilities for interaction, can provide the setting for more dynamic, powerful experiences. These environments should be filled with stimuli which encourage rich constructions by students (Nelson, 2000). Graphic representations, coupled with social interactions, are seen as leading to both the development of an individual's knowledge, and the adaptation of concepts (von Glasersfeld, 1992; Smith, 1998; Pesonen, 2002).

Many students, at all levels, find it difficult to answer questions about concepts that have been placed in contexts separate from their immediate concrete experiences (Honebein et al., 1993). As earlier sections of this chapter point out, the constructivist use of computer-based visualisation offers a more powerful means of providing the student with vivid experiences in order to convert the concrete into the abstract more successfully. This can in turn provide students with the appropriate mental structures that can be called upon to utilise conceptual knowledge in unfamiliar situations. Honebein et al. (1993) state that it is only through the richness of prior experience that the learner will be able to assemble the appropriate concepts and strategies to guide performance in a new situation. They believe that the types of activities that can be carried out in a computer environment (exploration, interaction, visualisation, etc.) provide meaningful experiences for learners that help them transfer skills and knowledge to other problem solving activities and subject domains.

It is important that students construct correct knowledge, which highlights the importance of the quality of any mathematical software used. Students' existing knowledge can be incorrect, and this, together with poor experiences, can simply reinforce their misconceptions, and interfere with the learning of new (or further) concepts (Zeuli, 1986; Giraldo et al., 2002). Well designed software, together with working in the zone of proximal development, can minimise the creation of incorrect information, and maximise the development of conceptual understanding. The balance between student-centred and teacher-led work is vital for optimising learning, because the student must feel in control, but the teacher must guide the discovery process. This vital balance is reflected in the design of the software in Chapter 5.

Imagery plays a significant role in mathematical reasoning (Wheatley and Brown, 1994). When students are engaged in the conceptual and relational understanding of mathematics, rather than procedural tasks, it is quite likely they will be using some form of imagery. Wheatley and Brown consider the construction and 're-presentation' of images. While engaged in mathematical activity, students construct images. When they re-present their image at a later date, they are operating from the image that they originally constructed. The nature and quality of the image will influence the re-presentation, hence the importance of quality mathematical software for image generation. This act of re-presentation is a complex one. Piaget has shown that the image constructed may undergo change over time without any intervention - the original image-making process supported by appropriate software is therefore vital. Wheatley and Brown believe that activities that encourage the construction of meaningful images can greatly enhance mathematics learning (e.g. the activities described in Chapter 5). Furthermore, good visualisers are particularly successful in constructing and re-presenting images; students who naturally use images in their thinking easily make sense of novel mathematical tasks while students who are not good visualisers often do not (Wheatley and Brown, 1994; Habre, 2001). It is therefore desirable to develop learning activities that promote the development of image-making skills for all students.

Powerful multiple representation software can be used to encourage the learner to construct meaning for different representations and how they are related. Multiple representation software can demonstrate these links explicitly (O'Reilly et al., 1997). Within such software, constructive changes in one representation trigger automatic changes in another. For example, a change in algebraic representation of a function will immediately promote a corresponding change in the graph of the function. A learning tool cannot be used in a constructivist manner, however, unless the students are genuinely in control. Thus, the pedagogy associated with this mode of use needs to be open and challenging. The dedicated teaching software described in Chapter 5 offers an important educational advantage over standard graph-plotting software. The student has control over where the relational links are made (symbols to picture, or vice versa), which is very much an example of a constructivist representation.

The software described in Chapter 5 has a Piagetian approach in so far as the student constructs knowledge that emerges from an individual's experiences, but has a Vygotskian approach as the teacher prompts and makes hints as to possible solution strategies, i.e. the importance of social interaction. No matter how the students actually construct their knowledge in terms of constructivist orientation, the key issue is whether or not they have actually learnt anything. The extent of learning is analysed in Chapter 5.

One particular cognitive activity that should be promoted in the design of learning environments is the ability to generate and evaluate alternative perspectives (Honebein et al., 1993). Students need to be made aware of the value of trying to see a problem from different perspectives. Consideration of those perspectives can then be used to help develop and refine one's own understanding. This is particularly beneficial for understanding problems of existence in mathematics, for example using visualisation (as an alternative perspective to symbolism) to illustrate that there are no real roots of $f(x) = x^2 + 4$, or that no solution exists for simultaneous linear equations representing lines that are parallel. Collaborative learning, either with peers or a teacher, is one strategy for helping to develop the skills of generating and evaluating alternative perspectives. Collaboration is more likely to yield different, yet equally viable, approaches to problem solving. It is restrictive for students to view all situations or tasks from a single perspective (e.g. symbolic or pictorial preferences), or always try to use the same strategy to solve a problem.

Both Piagetian and Vygotskian perspectives highly value the creation of rich environments capable of inciting student action and involvement. It is necessary to promote a verbal interaction capable of maintaining the initial stimulation (Lemerise, 1993), as these verbal interactions are very important in the learning and development processes. The approach adopted in Chapter 5 allows for social interactions between students, and between students and tutor - it provides a vehicle for constructing knowledge in the zone of proximal development.

The provision of activities to encourage a constructivist process can be achieved readily nowadays by employing visually compelling mathematical software such as AUTOGRAPH, CABRI-GEOMETRE, or a Computer Algebra System such as DERIVE, with which students can explore mathematics. These packages have various features that facilitate a constructivist approach to learning mathematics. AUTOGRAPH allows the user to 'grab and move' graphs, lines, and points on screen whilst observing changes in parameters, and vice versa. CABRI-GEOMETRE encourages the user to drag points around the screen whilst observing the effects of such changes on geometric shapes. DERIVE, with its multiple representation capabilities, allows the user to switch easily between numeric, symbolic and visual representations of information. These examples of software that can enhance constructivist learning can be used effectively to encourage 'what if' situations for students to explore, and to assist the learner to hypothesise mathematical facts prior to attempts at a proof. Enhanced understanding is likely if students can happily move from visual to symbolic representations, and vice versa (as with the software described in Chapter 5), when expressing their mathematical belief about a situation.

Additional to the commercially available software described above, bespoke software based on a constructivist approach has been developed (Sahin, 2003). An example of this is the development of an interactive problem-solving system for rotational dynamics (Dhillon, 1997), which allows students to be the initiators and controllers of their own learning. Evidence of the package's influence on student learning was obtained merely by means of questionnaires and interviews. The case-study described in this thesis employed questionnaires purely to gauge motivation and subject content, whereas the issue of enhanced conceptual understanding and the development of skills was dealt with by a controlled trial.

3.5. Summary

The constructivist philosophy is one that offers a theoretical basis for the development of imaginative teaching methods. In order to adopt the constructivist way of thinking, some of the key concepts underlying educational practice have to be refashioned (von Glasersfeld, 1995). The notions concerning the processes of communication and learning, the nature of information and knowledge, the interaction with others, and motivation, all change when they are seen from the constructivist perspective. Instructivism offers very little - what matters above all is that students learn to think. Traditional assessments in mathematics often require students to do little more than remember what the teacher or the textbook has said; they test memory and rote learning, not understanding.

A variety of reinforcements, such as praise, rewards, and grades, are typically used in education. All of these are examples of extrinsic motivators. There is no question that the procedure works, in as much as it produces the repetition of the reinforced behaviour. The problem is that extrinsic motivators do not motivate an effort to 'understand' (von Glasersfeld, 1995). Whatever students are given as reward for a good performance becomes the reason for performing. This creates a temporary motivation to repeat the successful efforts, but it does not create the desire to learn more, or to seek solutions to novel problems for themselves. The insight 'why' a result is right gives the student a feeling of ability and competence that is far more empowering than any external reinforcement.

Problem solving is undoubtedly a powerful educational tool (and a necessary graduate skill), and its power greatly increases if the students come to see it as enjoyable and worthwhile. The choice of task is also crucial and requires the teacher to use imagination rather than routine. Teachers can motivate students by displaying honest enthusiasm for the topic and its problems. Teachers who feel that their authority lies in knowing all the answers have little chance of awakening genuine curiosity in their students. This raises the issue of how to 'sell' these ideas to teachers who are reluctant to relinquish 'control' of the class. Constructivism suggests that the art of teaching has little to do with the transfer of knowledge - its fundamental purpose must be to foster the art of learning. Concepts are mental structures that cannot be passed from one mind to another. They have to be built up individually by each learner, yet teachers have the task of guiding the students' constructive process. When students are driven by their own interest to investigate a situation, the conceptual changes they are making during the process of reflection will be far more solid than if they were imposed by a teacher. Teachers must not overlook opportunities to explore rich student ideas.

All too often, teachers seem to be convinced that the abstract concepts they are trying to convey are plainly visible in the material they are displaying. This is very often not the case. From the constructivist perspective, concepts are not inherent in things but have to be individually built up by reflective abstraction. Individual students often make abstractions from the presented material that are quite different from those the teacher intends, to whom the material seems unambiguous.

Hence it seems essential to provide a class with a variety of situations that can all be seen as examples of the conceptual construction the teacher wants to convey (von Glasersfeld, 1995).

Whilst challenging students' conceptions, if the teacher reacts by saying that their ideas are wrong and tells them what is considered 'right', the reason why it is considered better may not be understood. It would seem more efficient to present the students with situations where the theory they have been using does not work. The motive to look for a more successful theory may then arise from their own perspective.

The next chapter considers the types of skills students should possess, and which skills can be enhanced via the constructivist use of visual software. This leads to the experiment detailed in Chapter 5 which employs visualisation in a constructivist computer-based environment with the aim of enhancing conceptual understanding of functions and graphs, and at the same time developing higher order skills.

CHAPTER 4

Mathematical Skills and the Role of Visualisation

Given the volume and complexity of scientific data, visualisation in the physical sciences has become a necessity in the modern scientific world.

ROBERT WOLFF

4.1. Introduction

The focus in this chapter is on the need to develop mathematical skills as well as subject specific knowledge. In particular, consideration is given to how the integrated use of visualisation can help to develop higher order skills. Earlier chapters have highlighted the fact that visualisation can be a very powerful educational tool for the understanding of mathematical concepts, but due to the manner in which mathematics is currently taught, curricula make limited use of visual representations. Educators therefore need to design activities that develop students' visualisation skills. Students *need* these skills, as the presentation of mathematical material is becoming increasingly visual.

As the result of an instructivist approach, students do not necessarily have the required higher order skills in order to be able to apply their knowledge to solve unfamiliar problems. A constructivist approach employing visualisation, however, concentrates on a holistic view of learning mathematics, and focuses on deep understanding and the development of skills, rather than facts and routines. This chapter shows how a constructivist approach needs to be adopted in order for higher order skills to be developed.

This chapter proposes a skills classification appropriate for the incorporation of technology in mathematics education. With the aid of this classification, an initial survey of mathematical skills is carried out which gives consideration to the types of skills students possess, and which skills can be enhanced via the constructivist use of visual software. Consideration is given to how the various skills taxonomies discussed in Section 2.3.1 of Chapter 2 are related, and how visualisation emerges as a common theme, underpinning these necessary skills. It is believed here that visualisation is an important factor for guiding concept development, and should be used explicitly as a tool for developing other higher order skills. A comparison of instructivist and constructivist approaches is provided, detailing the extent of visualisation used in each case, and the resultant skills developed. Finally, the extent to which students demonstrate certain skills (as opposed to knowledge), when technology is at their disposal in assessment, is discussed. This discussion is aided by a small investigation that takes a series of traditional examination questions and considers the possible use of a Computer Algebra System (CAS) in attempting to tackle them. This investigation provides useful information regarding the lack of visualisation ability and higher order skills in students, hence the larger case-study in the next chapter which aims to develop such skills via the constructivist use of visualisation.

4.2. DEVISe Skills Classification

All the skills taxonomies discussed in Section 2.3.1 of Chapter 2 are useful starting points in the classification of mathematics questions in terms of skills attainment. A more appropriate set of categories needed to be devised here, however, in order to allow for the introduction of computer-based visualisation as a learning and assessing medium. A skills classification was formulated containing the following five distinct categories:

- the ability to Detect mistakes.
- the ability to Explore.
- the ability to Visualise.
- the ability to Interpret.
- the ability to Select the best method for solution.

This classification is different from the other taxonomies as it is more geared towards computer-based, visual skills coupled with a constructivist approach to learning, whereas the others are concerned with pen-and-paper mathematics. It describes the types of skills gained through the investigative use of mathematical software that employs both symbolic and visual representations.

4.3. Linkages Between the Taxonomies

Visualisation is explicitly mentioned as a skill in the DEVISe classification, but as we shall see, all the attempts at skills classification discussed in Section 2.3.1 of Chapter 2 have a number of linkages, and visualisation is in fact a common, albeit implicit, skill feature.

The five taxonomies considered in this thesis have effectively arisen independently from one another and yet it is of interest to compare them in detail in order to appreciate how they are linked, and how visualisation is prominent. For example, reading from the left, Fig. 4.01 below links the DEVISe categories to the six NCVQ key skills, and links the six NCVQ key skills to the objectives of the Mathskills Discipline Network, and hence to the DEVISe classification. Similarly, the DEVISe classification is linked to the descriptions of the three types of skills questions used by Galbraith and Haines (2000b), and these would seem to indicate that they map closely onto the MATH taxonomy, and hence link directly to the DEVISe classification. The visualisation links and influences are highlighted in red.

4.4. Visualisation as a Developing Skill

Although the five initiatives originated from different perspectives, the above section highlights the fact that they overlap significantly in identifying core skills.

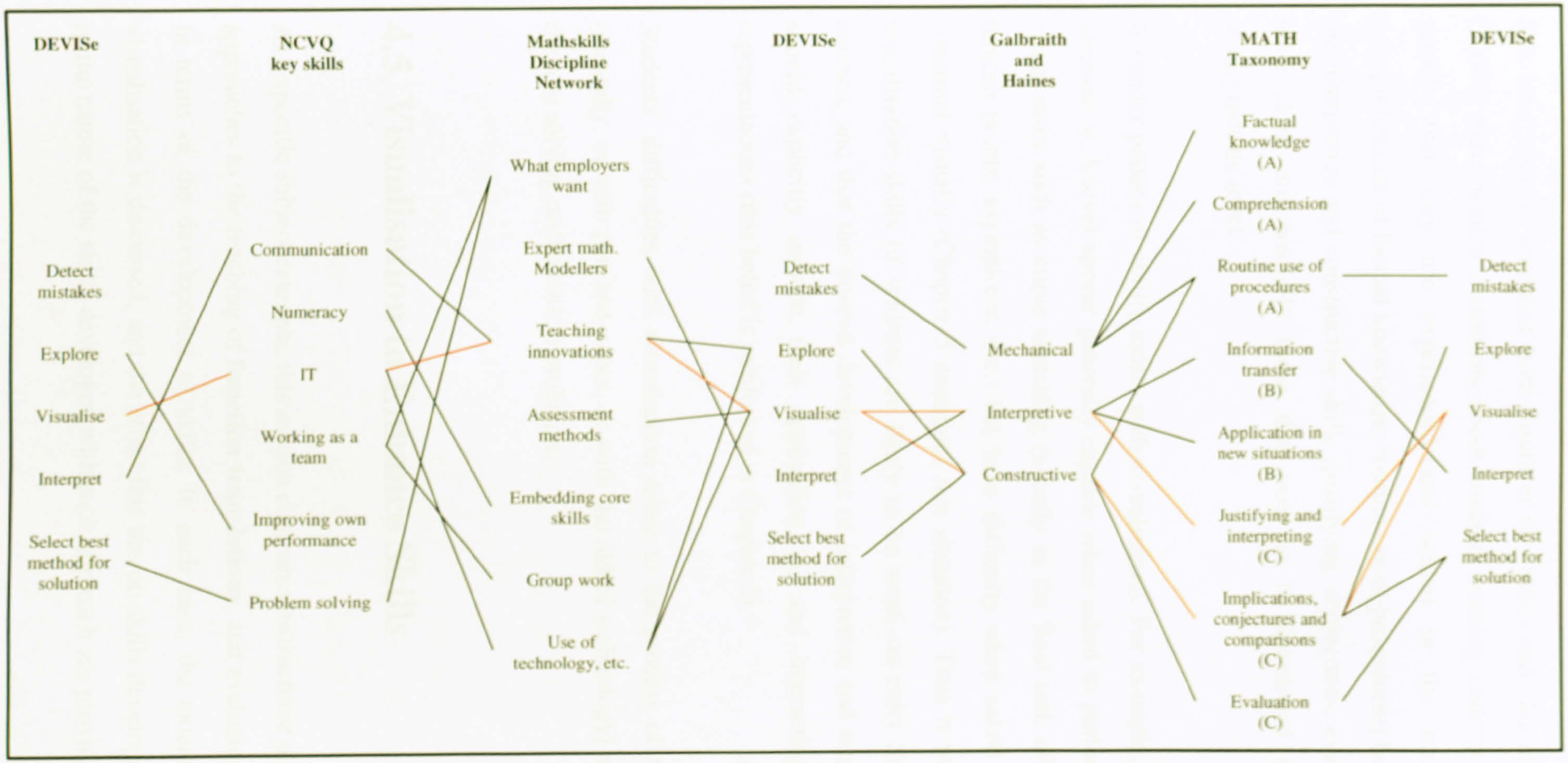

DEVISe
Detect mistakes
Explore
Visualise
Interpret
Select best method for solution
NCVQ key skills
Communication
Numeracy
IT
Working as a team
Improving own performance
Problem solving
Mathskills Discipline Network
What employers want
Expert math. Modellers
Teaching innovations
Assessment methods
Embedding core skills
Group work
Use of technology, etc.
DEVISe
Detect mistakes
Explore
Visualise
Interpret
Select best method for solution
Galbraith and Haines
Mechanical
Interpretive
Constructive
MATH Taxonomy
Factual knowledge (A)
Comprehension (A)
Routine use of procedures (A)
Information transfer (B)
Application in new situations (B)
Justifying and interpreting (C)
Implications, conjectures and comparisons (C)
Evaluation (C)
DEVISe
Detect mistakes
Explore
Visualise
Interpret
Select best method for solution

Fig. 4.01.

It is interesting to note that both Smith et al. (1996) and Galbraith and Haines (2000b) make similar comments about students entering undergraduate courses, namely that they are experienced and skilled in the mechanical tasks (comprehension of factual knowledge, routine use of procedures) but are limited in the interpretive and constructive skills (justifying, conjectures, evaluation). These latter skills are precisely those that need to be developed significantly at undergraduate level.

A similar pattern of ability exists with visualisation. For example, students being assessed at A-level appear generally capable when asked to perform mechanical, visual tasks such as curve sketching (usually as the final task after determining turning points, asymptotes, etc.) but have difficulty when asked to interpret or construct visually (Chapter 5 examines this situation). Thus it would seem that visualisation skills of students are likely to be weak on entry to undergraduate courses, and that the desired development of interpretive and constructive skills should explicitly include their application to, and interaction with, visual representations (this belief is reinforced in Chapter 5).

Students' difficulties with visualisation relate to the process of forming images (mentally, or with pen-and-paper, or with the aid of technology) as well as using them in solving mathematical problems.

4.5. Visualisation to Enhance Skills

As a specific subject example, this section compares instructivist and constructivist approaches to the teaching of **function translations**, and evaluates each approach in terms of the development of skills. In each case, the extent of the use of visualisation is discussed, and the effect this has on skills development. Examples of the nature of the skills developed with each approach are provided, classified by the descriptors of the MATH taxonomy. This taxonomy is appropriate for this purpose as it describes the nature of mathematical activities in terms of a hierarchy of skills. It has been observed that there is clearly an overlap in some of the skill descriptors depending on the nature of the task, and the understanding required (Leinbach et al., 2002).

Instructivist Approach:

Instructions are given as to how any function can be plotted by finding corresponding y-values for a range of x-values, and sketching the curve, or straight line, through the plotted points (there are potential problems immediately here, with the appropriate selection of x-values that serve to illustrate the key features of the graph). This can be followed with an explanation of 'families' of functions. A comprehensive and prescriptive set of rules can now be delivered in order to show the effect of the constant, c, in the expressions $f(x)+c$, $f(x+c)$, and $cf(x)$, for positive and negative values of c. As a result of this instructivist approach, the following skills are attainable:

An understanding of the shapes of specific graphs can be achieved, and from which families they come (**Factual knowledge, Group A**). For example, given the graph of $f(x)=2x^2-3x+1$, students should recognise that it is a parabola, which is a member of the family of quadratic functions.

An understanding of the effect of changing individual parameters on any $f(x)$ can be achieved (**Comprehension, Group A**). For example, in the expression $f(x)=3(x-2)^2+1$, students should be able to understand the effects of the 3, -2, and 1, on the graph.

Students should be able to carry out the steps involved with graphical construction (**Routine use of procedures, Group A**). For example, students should be able to plot a given function, and should be able to substitute different values of a, b, and c, into the expression $f(x) = a(x+b)^2 + c$, and plot (or by now, sketch) the corresponding graphs.

Students should be able to express a symbolic expression as a graph (**Information transfer, Group B**), but it is not as likely that they would be able to transfer information in the opposite direction. Additionally, students should be able to provide an explanation of the shapes and properties of graphs in non-technical terms, for example verbalise the effects of parameters, and the location and nature of turning points, asymptotes, etc.

Students should be able to pick out a rule that will act as a template for solution, i.e. make the problem fit the prescribed rule (**Application in new situations, Group B**). For example, when provided with an unfamiliar graph, i.e. any $f(x)$ in visual form, students should be able to appreciate any translations or transformations, given the rules. It is likely, however, that students will be limited in the ability to apply their knowledge to the modelling of real-life situations, as this requires the type of higher order skills developed via the following constructivist approach.

Constructivist Approach:

Using appropriate graph plotting software, students are invited to explore the effect of the constant, *c*, in the expression $f(x) = (x+c)^2$, for any $f(x)$ of their choice, and for positive and negative values of *c*, to appreciate horizontal translations. Similar explorations follow for the expression $f(x) = x^2 + c$, to appreciate vertical translations. Expressions of the form $f(x) = (x+a)^2 + b$ are then considered, to take account of both actions simultaneously. Students hypothesise rules based on their investigations, and plot appropriate graphs in order to 'prove' their hypotheses. Hence the examples are used as generic organisers (Tall and Thomas,

1991) to abstract the concept of translation. When given an expression such as $f(x) = x^2 + 2x + 3$ to explore, and provided that they have sufficiently explored earlier expressions, students will notice that it is merely a translation, both horizontally and vertically, of the graph of $f(x) = x^2$ (the expression is *given* to the students to facilitate the constructivist process). They can thus conjecture, as a result of their explorations, that the given quadratic expression is expressible in the more desirable form that they used in earlier explorations, i.e. $f(x) = (x + a)^2 + b$, in this case $f(x) = (x + 1)^2 + 2$. In this manner, students can concentrate on the concepts of translations, rather than the procedures involved. The formal aspects of completing the square, the steps involved with the construction of graphs, and the facility of sketching via key features, can be dealt with once the desirable conceptual structures have been established (knowledge of completing the square would have to be a prerequisite for the instructivist approach). This approach can clearly be adopted for other families of functions, as well as other transformational properties such as scaling.

The constructivist approach provides students with the necessary Group A and B skills (as with the instructivist approach), but develops them further. For example, the constructivist approach helps students to view graphs as conceptual entities, which encompass the global features, rather than point-by-point representations of symbolic expressions. Students should therefore be able to select an appropriate range of x-values that will illustrate the key features of the graph, thus eliminating the potential problems encountered with the instructivist approach to graphical construction.

Visualisation has been employed from the outset as students have switched between graphical and symbolic representations in both directions. As a result, students will feel comfortable providing the algebraic notation for a given graph (unlike with the instructivist approach), thus demonstrating the ability to transfer information in both directions (**Information transfer, Group B**).

Students will not only be able to pick out a rule that will act as a template for solution, but will understand the reason for the rule. They will understand how and why their knowledge can be applied in new applications (**Application in new situations, Group B**).

Students should be able to discuss translations in a generic sense, and provide appropriate justification in the forms of examples that illustrate the relationship between the pictorial and the symbolic (**Justifying, Group C**). They should also be able to discuss what the graph of a given symbolic expression will look like, and additionally, describe the location of a graph in terms of the corresponding symbolic notation, and the key features such as local extrema, asymptotes, etc. (**Interpreting, Group C**). Students can reinforce their interpretations with examples and counter-examples as a form of justification. Additionally, students can interpret information in an alternative, more appropriate, form in order to tackle a certain problem. For example, students can appreciate that the form $f(x) = a(x+b)^2 + c$ is a more appropriate form than $f(x) = \alpha x^2 + \beta x + \gamma$ in order to tackle a problem concerned with translations and transformations, but less appropriate for finding the roots of a quadratic. Students can thus choose an interpretation appropriate for the problem (this example is clearly linked to the **Group B** skills of **Information transfer** and **Application in new situations**). Mathematical models of real situations need to be interpreted, for example when modelling the path of projectiles (under gravity only), students must interpret the positive root of the quadratic equation as the position where the projectile hits the ground.

Having formalised their explorations in an algebraic context, students should be able to change the shape and location of a graph and describe what implications these changes have on particular components of the symbolic notation (**Implications, Group C**). During the exploration stage, students constantly conjecture the form of the graph from the symbolic expression, and vice versa, and ultimately have to conjecture a more appropriate form of the symbolic notation in order to proceed (**Conjectures, Group C**). Students develop the ability to compare translations of different families of functions, for example the effect of the constant, c, in $f(x)=(x+c)^2$, $f(x)=\sin(x+c)$, or $f(x)=e^{x+c}$. In this manner, students build up a generic understanding of translations, which is independent of specific cases (**Comparisons, Group C**).

Students should be able to evaluate their explorations in terms of the best method, or optimum strategy, for the solution of a problem (**Evaluation, Group C**). This evaluation should include the appropriateness of the use of visualisation.

Visualisation can not only provide an additional perspective, for example confirming graphically the existence, or non-existence, of algebraic solutions to equations, but can act as a vehicle for mathematical discovery. The instructivist approach uses visualisation merely as the product of following rules, for example plotting a graph, whereas the constructivist approach integrates visualisation from the outset, using it as a key ingredient in the exploration of concepts. Visualisation acts as a support, rather than an alternative, to symbolism. It would be very difficult to conjecture facts regarding the relationship between functional forms without the visual support.

Both of these approaches are fairly extreme cases; in reality teachers would probably wish to choose an approach somewhere in between. The two cases have been provided to illustrate that although the actual subject knowledge attained would probably be similar with each approach (in terms of preparation for national assessment, given its nature), an approach including more of the constructivist activities is preferable as it can help in the development of higher order skills.

4.6. Preliminary Investigation of Mathematical Skills in a Computer-Based Environment

The previous section has illustrated how the constructivist use of visualisation can enhance skills development. This section now considers the extent to which students demonstrate visualisation ability or higher order skills when appropriate technology that supports the use of visualisation, in this case a Computer Algebra System, is at their disposal in assessment (Malabar and Pountney, 2000). Consideration is given as to whether or not students' skills are being tested properly given that computer technology is available. This investigation provides useful information on students' skills prior to the more extensive case-study in the next chapter.

Consideration is given to the possible use of a CAS with algebraic and graphical display capability in attempting to tackle a series of questions, at the same time taking into account which skills are being both assessed and developed. As the MATH taxonomy deals with the structuring of assessment tasks in terms of the skills required to complete them, this will be used as a framework for a discussion of how traditional examination questions fare in the presence of a CAS when considering skills development.

The outcomes are described, and possible implications for the development of assessment, which incorporates technology to facilitate the assessment of a wider range of skills, are discussed. A group of undergraduate mathematics students, who were familiar with using a CAS in examinations, tackled an assortment of questions from public examination papers where a CAS is not allowed.

4.6.1. Background to the Investigation

"Should a CAS be used in formal written examinations, and if so, how should traditional examination questions change (if at all)?", is a question debated amongst teachers and students of mathematics at all levels as technology advances and handheld 'calculators' with algebraic and graphical functionality become more available and affordable to schools and students. Issues relating to the appropriate use of a CAS in teaching, learning and assessment are not new. For example, the issue of symbolic mathematics systems in mathematics education was addressed at ICME 5 (International Congress for Mathematical Education, Adelaide, 1984). In a paper entitled 'Should Students Learn Integration Rules?', Buchberger (1989) formulated his White-Box/Black-Box principle (see Section 2.4.3.1 of Chapter 2) to describe an approach to govern the use of symbolic computation for both students of mathematics and for non-specialist mathematics users such as engineers and scientists. Pedagogical issues and assessment issues relating to handheld technologies were raised, for example, by Etchells and Monaghan (1994), and more recently by a variety of authors (Kutzler, 1999; Gardiner et al., 2000; Connors and Snook, 2001; Forster and Mueller, 2002). This issue is now being looked at again on the basis of testing mathematical *skills* rather than just subject knowledge and procedures.

Ruthven (1997) produced a report surveying the availability and use of CAS in A-level mathematics and compared this with usage internationally. The report continued with a review of published research into the use of CAS in mathematics education up to that point and concluded with a considered view on how assessment policy with regard to CAS might develop nationally. Amongst other considerations, Ruthven commented that:

> "The central issue is to ensure that examination questions and marking procedures take appropriate account of whatever technology *ALL* students have available. This means not only

> avoiding items that can be answered more or less directly through routine use of the technology, but devising new types of item which test the capacity of students both to use the technology effectively and to demonstrate understanding of relevant concepts."

The 'routineness' of A-level mathematics papers has been reported (Monaghan et al., 1998; Malabar and Pountney, 2000), which highlighted the fact that lower attaining students obtain proportionally more marks on routine parts of questions. Yet a CAS can often reduce such routine, procedural parts of questions to mere button pressing. Clearly, if a CAS is allowed in such examinations, any replacing of 'routine' questions (or changing the marks allocated to such questions) must be well thought through. The presence of a CAS in an examination has the potential to allow the examiner to test higher mathematical skills and more conceptual understanding than previously, at the expense of time spent on routine procedures and mental manipulations. The challenge to examiners, as Ruthven suggested, is to set appropriate questions, possibly of a totally different nature from those traditionally set.

4.6.2. Testing Mathematical Knowledge and Skills

Smith et al. (1996) suggest that "assessment drives what students learn. It controls their approach to learning by directing them to take either a surface approach or a deep approach to learning." They also stress the point that any classification has to be subject to the knowledge of the students' prior learning history, i.e. has to be put into context. For example, a proof question could be a Group C task if previously unseen by students but perhaps a Group A task otherwise. This emphasis on skills classification is similar to earlier work by Nagy et al. (1991).

It seems a reasonable expectation that students progressing from A-level mathematics to undergraduate studies and beyond should progress from a majority of Group A tasks at A-level to a majority of Group C tasks by graduation, irrespective of the particular mathematical topics studied. A cursory investigation of recent A-level papers does suggest a concentration of Group A assessment tasks along the lines of Monaghan's findings and others (see for example Boaler, 1997). A similar study of final year undergraduate degree assessments, across different universities and different combinations of mathematical subject areas, would be of interest to see if the higher order skills are assessed in a significant way. As the Mathskills Discipline Network found, many prospective graduate employers seem to recognise and require such skills rather than specific mathematical subject knowledge (as discussed in Section 2.3.1.2 of Chapter 2).

The use of CAS can obviously influence this skills transition, but does it do so in a positive manner? Buchberger's black boxes are in effect Group A tasks but they have only become so after the white box phase that may incorporate tasks from Groups A, B and C. But where does the white box end and the black box start? It is conjectured here that the use of a CAS can promote more Group C tasks perhaps at A-level or earlier, and not just for 'clever' students.

4.6.3. Skills Investigation

As a preliminary study, a group of 20 undergraduate students were set a number of examination questions for which they would not normally have access to a CAS. The students (a mixture of first year and final year students for comparison purposes) were all undertaking a degree in Mathematics, Statistics and Computing at Liverpool John Moores University. Throughout this course they are accustomed to using a CAS (in this case DERIVE) both in courseworks and timed written examinations as the appropriate use of technology is one of the principal aims of the course.

The four questions are described in detail in the following section and range from traditional A-level questions to Mathematical Olympiad standard. The first two are from subject areas familiar to the students on the course, the latter two were of a nature beyond their experience (but not necessarily beyond their capability). The four questions were chosen to determine whether the nature of a question affects the extent to which students use visualisation and other higher order skills. With the aim of gaining insight into the types of skills that students choose to demonstrate, qualitative data was gathered to begin to answer such questions as:

- Would students show a bias towards algebraic methods of solution even when a graphical/visual approach was possible (with or without a CAS)?

- If the students could solve the problem *without* using the graphical capabilities of a CAS, would they do so even if the solution took longer to achieve?

- Would students use a CAS at all times, even for 'simple' algebra and calculus problems?

- Would students recognise when a CAS was of little apparent use?

- Would a topic apparently new to the students influence their use of a CAS?

- Would students provide 'unexpected' (i.e. valid, but not expected by the examiner) solutions?

The students were also given the MATH taxonomy of Smith et al., and asked to classify each part of each question. They were also asked to comment on the allocation of marks in Question 1 in relation to the task to be completed.

4.6.4. Outcomes

Question 1. This question is taken from the report by Monaghan et al. (1998).

The gradient of a curve is given by $\frac{dy}{dx} = 3x^2 - 8x + 5$.

The curve passes through the point (0,3).

(i) Find the equation of the curve.

[4]

(ii) Find the coordinates of the two stationary points on the curve. State, with a reason, the nature of each stationary point.

[6]

(iii) State the range of values of k for which the curve has three distinct intersections with the line $y = k$.

[2]

(iv) State the range of values of x for which the curve has a negative gradient. Find the x-coordinate of the point within this range where the curve is steepest.

[3]

Student Responses:

All students from both groups attempted the first two parts of this question without using a CAS at all, commenting that it would take longer to key in expressions than to write down the answer. 30% suggested that they would have used a CAS if the expression for the gradient had been more complicated, and a similar figure suggested a CAS use to confirm their answers given time.

For parts (iii) and (iv), the solution strategy varied. 40% could 'see' a mental picture of the curve and solve the parts without using a CAS. Another 40% used the CAS to plot their answer in part (i) and used the graph to read off the answers. The rest tried to answer these parts using algebra and came unstuck, for example one student attempted to solve a cubic for three real and distinct roots in part (iii).

All students classified parts (i), (ii) and (iv) as Group A tasks and part (iii) as a Group B task. All students felt that parts (i) and (ii) had too many marks at the expense of part (iii) and possibly part (iv).

Author Commentary:

For this given gradient expression, one would expect students to be able to (wish to) integrate without a CAS in part (i). Obviously, for more complicated expressions a CAS would be more appropriate without affecting the conceptual context of the question. The ideas of indefinite integration and the fixing of the arbitrary constant must be understood, especially using a CAS where the arbitrary constant of integration is often omitted. These are Group A skills.

For part (ii), students need to know (factual knowledge, Group A) the definition of a stationary point. Routine use of procedures suffice for a pen-and-paper solution. A student with a CAS having graphical display capabilities (or indeed just a graphical calculator) might plot the curve and read off the stationary points (zoom in, zoom out, cross-hairs, etc.) justifying their nature by interpreting (Group C) the shape of the graph (see Fig. 4.02 below).

For part (iii), the visualisation and interpretation is more evidently a higher order skill. Yet only 2 out of 15 marks are awarded. Similar comments apply to part (iv).

[Anecdotally, the same sort of imbalance occurs in A-level Mechanics questions. The mark scheme reveals that an often significant minority of marks is available for applying mechanics principles and concepts to obtain the relevant equations, with the majority of marks available for subsequent manipulation of the equations to produce the required result.]

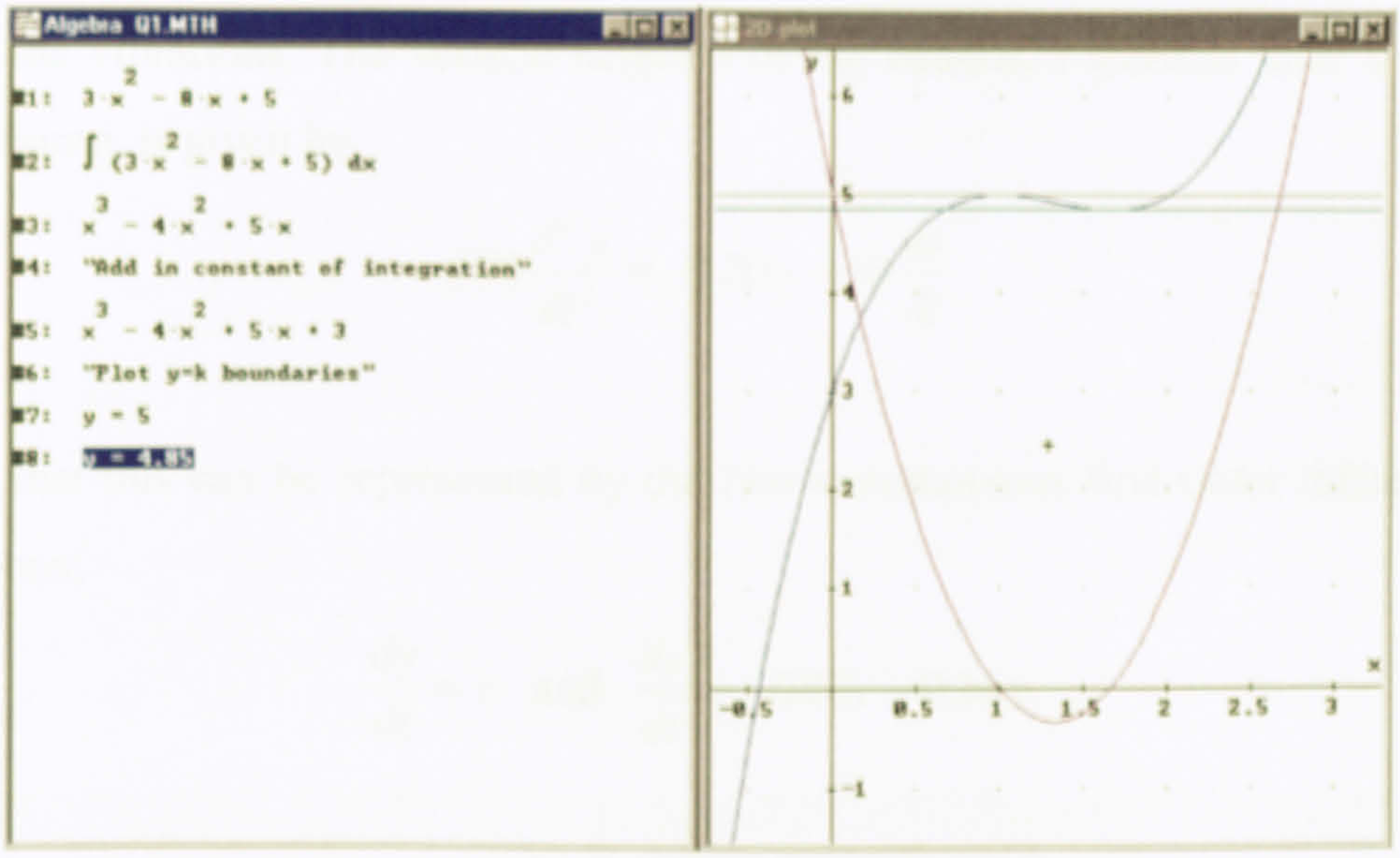


Fig. 4.02.

It would appear that the use of a CAS that allows the interaction between algebraic and graphical representations in a straightforward manner, helped the 'weaker' students here to progress more smoothly through the whole question, and hence gave them an opportunity to demonstrate higher order skills, rather than having to mentally 'see' the solutions to parts (iii) and (iv).

In summary, this type of question would appear to give no direct advantage to a student with a CAS in terms of answering the question completely, but the use of a CAS does add to the possibility of a successful solution. A major debating point here is the allocation of marks for apparent application of routine procedures rather than mathematical insight. Although this issue prevailed long before the advent of a CAS, the presence of a CAS has served to highlight it even more.

Question 2. This question is taken from an A-level Further Mathematics specimen paper.

The suspension system of a beach buggy has a spring with a shock absorber to damp the vibrations. The vertical height y of the chassis, t seconds after a wheel hits a bump, is given by

$$500\frac{d^2y}{dt^2} = -120y - 40\frac{dy}{dt}$$

Show that this can be represented by the two simultaneous first order differential equations:

$$\frac{dy}{dt} = v \quad \text{and} \quad \frac{dv}{dt} = -0.08v - 0.24y$$

If $y = 0$ and $\frac{dy}{dt} = 10$ when $t = 0$, sketch a graph of y against t.

Student Responses:

Most (>90% from both groups of students) thought this the most straightforward question of the four (once they could recall the method to obtain the simultaneous differential equations) and classified it as a Group A task. All students commented that they would have used DERIVE's utility command to first attempt a solution to the given second-order differential equation if the reduction to a pair of first order differential equations hadn't been asked for. Similarly, all used DERIVE to 'sketch' the graph.

Author Commentary:

If the 'show that' part of the question had been omitted, then the whole problem could have been done using the CAS as a black box (see Fig. 4.03 below). The pen-and-paper approach using a routine solution procedure is a Group A task. The sketching of the curve by hand is perhaps bordering on a Group B task. In this case, following Ruthven's quote earlier, the question would seem to be best avoided. However, the 'show that' part, although of a routine nature, is a pre-processing procedure commonly needed with a CAS that numerically solves higher order differential equations by reducing them to a system of first order ones. Hence, this part does test use of technology quite reasonably if the extension to reduction of higher order equations is apparent to the student.

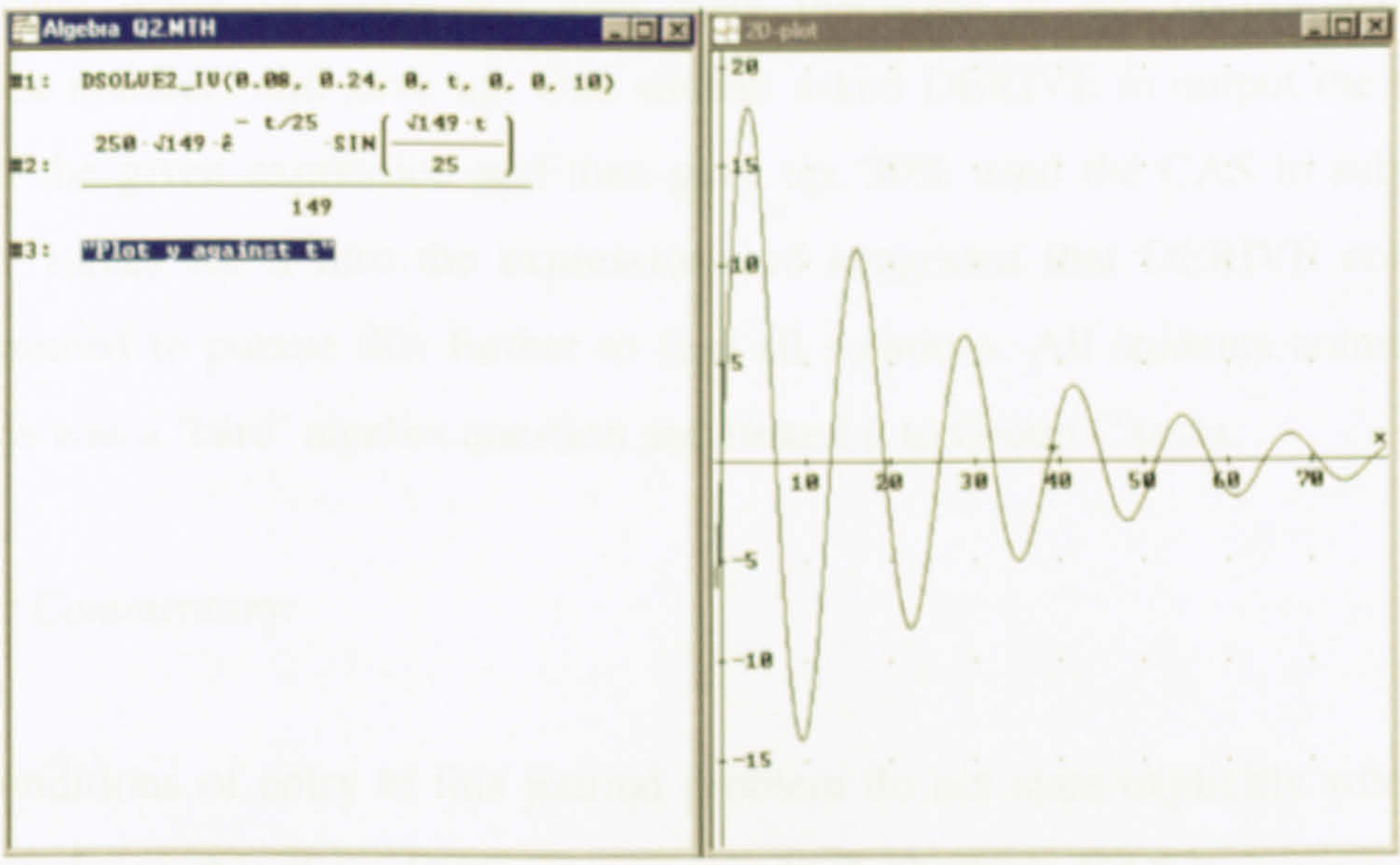


Fig. 4.03.

From a visualisation perspective, the question stops too early. The student is not asked to interpret the graph and show knowledge involving exponential envelopes, limiting issues as time increases, etc.

Question 3. This question is taken from the Student Problems page in the Mathematical Gazette, Vol. 82, No. 495, Nov. 1998, pp. 512-513, and is a regular feature of this journal. Students up to the age of 19 are invited to enter and prizes are awarded 'for the most impressive solutions'. The journal publishes solutions and commentary on the entries received.

Find all the integers n for which $n^4 - 4n^3 + 14n^2 - 20n + 10$ is a perfect square.

Student Responses:

No student got the correct answer but all made some attempt at a start. 25% (with the majority of these being final year students) tried to factorise the given quartic expression using DERIVE but then were dismayed to see factors involving complex numbers and gave up. One student asked DERIVE to output the square root of the given expression and then gave up. 30% used the CAS to substitute integer values for n into the expression and suggested that DERIVE could be programmed to pursue this further to find all solutions. All students commented that this was a 'hard' algebra question and linked it to Group C tasks.

Author Commentary:

The conditions of entry to this journal problem do not state explicitly whether a CAS can be used in the solution or not, or indeed whether a CAS is any use or not. 'An impressive solution' is an interesting phrase to use to select a winner but a cursory glance through other Problem Pages suggest that not many solutions using a CAS are deemed to be impressive!

The pen-and-paper solution quoted starts with this as its first line:

Now $n^4 - 4n^3 + 14n^2 - 20n + 10 = (n-1)^4 + 8(n-1)^2 + 1$

$$= [(n-1)^2 + 4]^2 - 15 = k^2 \text{ (say).}$$

The strategy is to try and complete the square and then use the difference of two squares and consider possible factorisations. The difficulty here is essentially one of knowing where to start (why powers of (n-1) for example in the first line?). Once some sort of problem solving strategy has been conjectured (Group C task), the algebra and integer factorisation considerations are relatively straightforward.

Of course, one solution strategy, suggested by some students, is simply to try many integer values of n and observe those that yield a perfect square. This could be done by pen-and-paper but soon becomes tedious. A CAS could be programmed to do this fairly easily but the question of securing all possible answers arises.

The use of a CAS here does offer a 'trial and error' type of starting strategy. One possible starting point is to write the problem as:

Solve for integer n the equation

$$n^4 - 4n^3 + 14n^2 - 20n + 10 - m^2 = 0 \qquad \text{(m, integer),}$$

and then try to FACTOR the left hand side with respect to n. DERIVE gives output as per Fig. 4.04 below.

```
      4      3       2                 2
#1:  n  - 4·n  + 14·n  - 20·n + 10 - m

             4      3       2                 2
#2:  FACTOR(n  - 4·n  + 14·n  - 20·n + 10 - m , Complex, n)

                        2                                2
#3:  (n - 1 + î·√(√(m  + 15) + 4))·(n - 1 - î·√(√(m  + 15) + 4))

                       2                          2
       ·(n + √(√(m  + 15) - 4) - 1)·(n - √(√(m  + 15) - 4) - 1)
```

Fig. 4.04.

An examination of the last two factors and the discriminants (remember m, n both integers) suggests that m^2+15 must be a perfect square. For any chance of achieving integer n solutions, m^2 can only have values 0, 1, 4, 9, 16, 25, 36, 49, 64, 81, ... and hence the only possibilities for m^2+15 to be a perfect square are $m^2 = 1$ and 49 since the difference between successive squares is then greater than 15. Hence integer values for n are found to be -1, 1 and 3.

It is arguable that the CAS here has helped to provide a more obvious starting point leading to a more logical solution process, rather than 'seeing' how to proceed using brain power alone. The CAS on its own cannot solve the problem entirely and human reasoning (Group C task) is still required. It is open to debate whether the above solution is 'an impressive one' - beauty being in the eye of the beholder!

As with Question 1, this question would seem to give little advantage to students with a CAS, other than perhaps to obtain some credit for demonstrating some initial problem solving strategies with a CAS. This question is perhaps typical of those set to differentiate the good students from the weaker ones and clearly requires Group C skills.

Question 4. This question is taken from the 38th International Mathematical Olympiad 1997, set by a mathematician from Iran, and is reported in the Mathematical Gazette, Vol. 81, No. 492, Nov. 1997, p. 480.

An n x n matrix whose entries come from the set $S = \{1, 2, ..., 2n - 1\}$ is called a *silver* matrix if, for each $i = 1, ..., n$ the ith row and ith column together contain all the elements of S. Show that

(a) there is no silver matrix for $n = 1997$;

(b) silver matrices exist for infinitely many values of n.

Student Responses:

"I don't know how to start this problem but I'm sure DERIVE would be useful here because it is good with matrices", was a disappointing but perhaps understandable response from over 90% of the first year students. The final year students again showed some mathematical experience by writing down cases for n = 2, 3, 4 and then conjecturing (Group C skill) that "perhaps silver matrices only work when n is a prime number?" One student suggested she would use DERIVE to check whether 1997 was a prime number, but not one student could prove their conjecture.

Some students suggested that a question like this was unfair as "you could not tell which part of the syllabus it had come from." Again, as in Question 3, students equated 'difficulty' to Group C skills.

Author Commentary:

This looks at first glance as if a CAS could be used to advantage. It soon becomes apparent that a CAS is of little use here even in helping with a starting point as in the last example. Investigations with small size matrices initially (pen-and-paper) and trying to spot 'the big picture' and then conjecture a 'proof' (Group C task) seems the best way forward. These type of questions seem ideal for an Olympiad of this type where higher order mathematical skills are tested, whether or not a CAS might be allowed in the future.

4.6.5. Conclusions

Tentative conclusions about the use of a CAS from the investigation are given. However, these conclusions would benefit from a wider, more quantitative study across a range of institutions and students.

- Those students experienced in using a CAS via their teaching and learning *did* begin to use a CAS appropriately when allowed to in an assessment. The weaker students appeared to have more confidence to succeed in the presence of a CAS. However, the brighter first year undergraduates still preferred to use 'pen-and-paper' solutions with the belief that they were 'safer' in terms of gaining marks (probably only because of the way they had been nurtured at school).

- There was some evidence to suggest that even final year students still select an algebraic solution to a problem rather than a potentially quicker graphical one. It is not clear whether the use of a CAS with the ability to place algebra and graphics on the same screen changes this preference or not.

- Final year students were more inclined to use the CAS as an investigational tool when a problem-solving strategy was not obvious to them. This may well be because of their greater 'experience', or inherent development of skills during their undergraduate studies.

- Students tended to equate an 'easy' problem with Group A skills and a 'hard' problem with Group C skills, although the authors of the taxonomy did not suggest a change in difficulty as one moved across the Groups. Students also felt that the allocation of marks should be biased towards evidence of Group C skills.

In questions 1 and 2, many students (mainly first year) chose not to use visualisation during the solution process, instead opting for a more familiar algebraic approach. Questions 3 and 4, which contained no obvious visualisation elements, were included because they assessed higher order skills. They illustrate that there are some skills and topics with which visualisation cannot help explicitly. Although the ideas cannot be represented visually, they are still not 'handle-turning' types, but instead require the types of higher order skills (Group

C) that can be developed via visualisation (as discussed in Section 4.5 of this chapter). Students were unsuccessful at these questions, thus demonstrating an inability to use higher order skills to generate solution strategies in problem solving.

Students that have not been subjected to visualisation during the learning process do not choose to use visual techniques to tackle problems, and struggle with problems that require higher order skills that can be developed via visualisation.

This preliminary investigation has provided useful information on students' skills prior to the case-study in Chapter 5, which analyses the skills (both visual and higher order) that students demonstrate after having experienced a learning process that integrates computer-based visualisation in a constructivist environment. Teaching, learning and assessment are currently heavily biased towards algebraic approaches, however the constructivist use of the software in the case-study forces a visual approach that promotes the development of higher order skills.

4.6.6. Discussion

The decision whether or not to allow the use of a CAS or indeed any other tool such as a graphing calculator, formulae sheet, etc. in an examination is obviously linked to the purpose of the assessment as perceived by the examiner. Thus for example, if the purpose is solely to test mental manipulation skills and factual knowledge recall of mathematical methods then a CAS is probably not appropriate and should not be allowed. The balance of skills being assessed in an examination again depends on the purpose of that examination. An Olympiad question such as Question 4 is aimed at the most able mathematics students and is testing problem-solving skills explicitly whereas Question 1 leads the student through the problem to a large extent and would be too straightforward for a competition or an

Olympiad. If assessment is intended to test what has been taught and the teaching and learning involves the use of a CAS, then the assessment should do also.

The use of a mathematical skills taxonomy has been found to be helpful in determining if the intended purpose of formal examination questions is matched by the skills being tested. Once the purpose of a question or questions is clearer to the examiner, reasons for any restriction on the use of a CAS, or any other technological aid, should be more apparent. This is only another way of saying that the learning outcomes specified for a course or module, properly expressed in terms of acquired skills, should be clearly and transparently assessed, and that the aims should specifically address the role of a CAS in terms of skills attainment.

If, for example, the purpose of a question is to assess some higher order skills and a CAS makes the solution of the question routine, then either the CAS has to be excluded or the question changed. This is exemplified in Question 2 where the solution of a differential equation would be routine using a CAS and yet the student is asked to remember and reproduce a specific method of solution. What specifically was the examiner testing? An investigation into the effects of changing the numerical values of the parameters of the differential equation on the buggy's suspension characteristics, using a CAS to produce the solutions quickly, would have tested more Group C skills.

It may generally be felt that the use of a CAS can only trivialise a problem. However as illustrated by the examples above, this is not the case. In Question 1, the use of a CAS, albeit the graphical part, appeared to help the student who had difficulty 'seeing' the graph mentally. Of course, it would be ideal if more students could create their own correct mental images in mathematics, but is this the purpose of this question? If the question had been posed with the graph of the curve given and not an algebraic expression for its gradient then the potential is there for more interpretive (Group C) type questions and the use of a CAS is potentially diminished. If the gradient expression had been (say) $x\sin(x^2)$ then the use of a CAS to perform the integration might be more appropriate.

The use of a CAS, as in Question 3, might allow some students to succeed by offering a starting point that might not have been within their capability otherwise, and hence an opportunity to show higher order skills in the rest of the solution. If problem-solving skills can be improved for a wider range of student abilities by using a CAS to advantage, then a CAS is appropriate.

There are many questions that can be set where the advantage of a CAS is perhaps not immediately apparent, as in Question 4. However, these questions are often seen as 'hard' (as evidenced by the students in this assessment investigation) as they test the higher order skills. The lower order skills must not be excluded entirely. The assessment strategy should include a balance of questions to suit the range of abilities of suitable candidates and the danger that the use of a CAS in examinations leads to 'harder' questions is a real one. The use of a CAS in examinations could make them even more challenging for the weaker students in the sense that the amount of procedural questions (on which weaker students rely – their 'comfort blanket') could be diminished. It is interesting to conjecture that an examination question that starts with Group A skills and ends with Group C skills offers more chance of success to more students than one that starts with Group C skills and has Group A skills within it but later on.

So, some traditional examination questions *can* fare well in the presence of a CAS but equally some traditional questions would benefit from a 'face-lift' if examiners wish to test a wider range of mathematical skills. As technology advances and a CAS and other tools become more readily available, it is apparent that setting assessments will require even more care and planning to ensure appropriate testing of relevant skills and abilities.

4.7. Summary

Consideration has been given to various skills taxonomies and the numerous linkages between them. Graduate employability is increasingly important, exemplified by the transferable skills detailed in the NCVQ key skills and the objectives of the Mathskills Discipline Network. Attempts at the classification of mathematics questions in terms of the skills required in order to tackle them have also been reported, such as the MATH taxonomy and the classification by Galbraith and Haines. The author's DEVISc classification is more geared towards IT-based, visual skills coupled with a constructivist approach to learning.

Visualisation emerged as a common theme of the various taxonomies. It can be an important factor for guiding concept development, and should be used explicitly as a tool for developing other higher order skills. Although visualisation is an important skill in terms of having the potential to enhance a global view and understanding of mathematics, students still choose mechanical as opposed to pictorial methods for solving mathematical problems.

Students are generally capable and experienced at algorithmic approaches, but are very limited in the skills required to tackle more interpretive and constructive tasks. Visualisation offers a major support for these Group C skills, and hence its development is seen to be important. Its inclusion will impact not only on teaching and learning, but also on the assessment of any revised curriculum.

Educators need to create a more diverse set of experiences for students in order to develop a range of skills. Students can pass examinations with merely surface knowledge - teachers must however encourage the acquisition of a deeper knowledge by asking the right types of questions. Changes need to be made to teaching with assessment in mind, as the prime motivation for students is to pass.

The preliminary investigation, which considered the extent to which students demonstrated mathematical skills when a CAS was at their disposal in assessment, produced the following observations. The use of a CAS gave 'weaker' students confidence as they attempted to solve problems, however many students still opted for pen-and-paper solutions even when the use of technology was appropriate and advantageous. Even students who were experienced in using a CAS selected algebraic solution processes, rather than opting for more efficient visual ones. Students generally felt that 'difficult' questions are those that assess Group C skills. It is conjectured here that they are not necessarily more difficult, it is that the students have not adequately nurtured the appropriate skills in order to tackle them successfully.

The effective use of technology on skills and the importance of visualisation as a tool for enhancing conceptual understanding are concentrated on in the next chapter in a case-study which employs a constructivist approach to learning. Chapter 3, which considered the benefits of constructivism, and this chapter, which has looked at the importance of visualisation and its prevalence within related skills taxonomies, have provided the necessary background to the following chapter, which describes how these two factors are successfully combined to assess the development of visualisation skills leading to enhanced conceptual understanding of functions and graphs. A constructivist approach to teaching and learning can lead to better visualisation skills.

Chapter 5 illustrates how the encouragement to develop visual images and to promote a general visualisation awareness is very attainable. The structured use of interactive materials can help to encourage the creation of mental images and thus the visualisation process itself. The interactive computer environment described encourages and develops students' visualisation skills, and the relationship between both symbolic and visual abilities.

CHAPTER 5

Students' Ability to Visualise: A Case-Study

The art of asking the right questions in mathematics is more important than the act of solving them.

GEORGE CANTOR

5.1. Introduction

"*Since visualisation ability is an important factor, more research needs to be carried out on methods to improve an individual's visualisation ability*" (Sein et al., 1993). This is precisely what the approach described in this chapter sets out to do, using a practical case-study.

The standard view of visualisation is that it transforms the symbolic representation into the pictorial form, and can enrich the discovery process. For an educational scenario the latter is certainly true, however an alternative way of exploiting visualisation for educational purposes is to start with the visualisation of the symbols and encourage the student to arrive at the symbols that have brought about that visualised entity. The software used in this chapter effectively visualises mathematical patterns for the user to explore, and therefore employs visualisation creatively as a tool for understanding. "*This is the essence of mathematical visualisation*" (Zimmermann and Cunningham, 1991).

Earlier chapters have reviewed and examined the effectiveness of previous work using visualisation, coupled with a coherent learning and teaching methodology to enhance mathematical skills. Consideration is given in this chapter to employing constructivism in teaching and learning with computer-based visualisation in order to develop not only visualisation skills but also the higher order mathematical skills outlined in Chapter 4.

In order to establish any practical evidence of enhanced mathematical skills of students, a controlled experiment was carried out to assess the effectiveness of the constructivist employment of computer-based visualisation. The experiment looks at using appropriate technology that assists in the integration of a constructivist approach to learning and computer-based visualisation in order to enhance students' conceptual understanding of functions and graphs.

The experiment is described in this chapter under the following headings:

Aims:

The experiment aims to provide students with a better understanding of the links between algebraic and graphical representations, and at the same time attempts to enhance higher order skills, such as those described in the MATH taxonomy in Section 2.3.1.3 of Chapter 2. Key and secondary aims stated in Section 1.4 of Chapter 1 are addressed.

Methods:

A rationale is provided for the choice of topic, functions and graphs, and the bespoke software is described, highlighting the positive features and the factors affecting the design. A theoretical rationale is also provided, which takes into account the various theories of teaching and learning discussed in Section 2.4.3 of Chapter 2. An actual dialogue between student and teacher, captured during the case-study, illustrates key features of the learning process, such as the constructivist use of visualisation and the role of strategic questioning. The logistics of the experiment are provided, including the experimental design and a study of the test questions in terms of the range of skills that they assess.

Results:

A detailed statistical analysis of the data is used to evaluate the effectiveness of this constructivist visual approach, and consideration is given to what the students actually thought of the process via feedback from questionnaires.

Conclusions:

Concluding remarks about the whole process focus on students' experiences in terms of the enhancement of mathematical skills.

5.2. Aims

The case-study achieves the following key and secondary aims:

5.2.1. Key Aims

- Evaluation of enhanced student learning of mathematical concepts, and assessment of the extent of any skills development, via the constructivist use of computer-based visualisation.

- Comparison of the performance of students after learning via either an instructivist or constructivist (incorporating computer-based visualisation) approach, in terms of both procedural and visual skills, as well as other higher order skills.

- The software in the case-study aims to enhance student visualisation skills, which in turn help in the development of conceptual understanding, together with other desirable higher order skills. It aims to develop a more holistic view of mathematics, and to provide students with better strategies for problem solving. It aims to develop an understanding that is independent of specific examples used, so that the conceptual knowledge acquired can be applied to any function, i.e. the knowledge gleaned from local tasks can be applied globally.

5.2.2. Secondary Aims

- Analysis of specific comparisons in the case-study:

    - Schools (Schools A, B and C).

- Sex (Male and Female).
- Subject (Mechanics and Statistics).
- GCSE Grade (A*/A and B/C).

(a rationale for the specific comparisons is provided in Section 5.3.4.1 of this chapter).

- Assessment of the motivational effects, in terms of usefulness and enjoyment, of a constructivist computer-based visual approach to learning.

5.3. Methods

A rationale for both the choice of topic and the design of the bespoke software is provided, together with a dialogue between student and teacher illustrating the teaching and learning styles adopted. The logistics of the experiment are described, including the experimental design and a study of the test questions.

5.3.1. Rationale for the Chosen Topic

An investigation of the mathematical difficulties experienced by undergraduates has revealed that many of these difficulties relate to the understanding of graphs (Gill, 1998). The most significant area of skills and knowledge in terms of predicting success in university mathematics examinations concerned graphs, including the interpretation of shape, the verbal interpretation of graphs, and the symbolic representation of graphs.

Traditionally, the typical instructional path through the representations of functions has been from algebraic expressions to graphs (Williams, 1993), and any linking that occurred among the representations tended to follow this path. This has given rise to students' beliefs that graphs were something extra, something appended to functions but not a fundamental representation of the function. Advances in technology have meant that either representation can be thought of as a starting point for translation among representations. The ability to begin with a graphical, as opposed to a symbolic, representation may encourage a view of function as an object, rather than as a process.

Many students appear to have little knowledge regarding the relationship between the algebraic notations of a function and the geometric implications of the function's graph (Myers, 1997). According to Joseph Lagrange, "*As long as algebra and geometry travelled separate paths, their advance was slow and their applications limited. But when these two sciences joined company, they drew from each other fresh vitality and thenceforth marched on at a rapid pace toward perfection*" (Moritz, 1914). Joining the two disciplines can add considerable power to each. The failure to relate algebraic and pictorial representations results in students lacking the ability to view a graph as a visualisation of a function. A graph is a powerful tool for relaying information about a function. The software described in this chapter has been designed to help students appreciate the relationships that exist between the algebraic notation and its corresponding graphical form.

The concept of function is one of the most powerful and useful notions in mathematics. Nevertheless, the learning and teaching of functions are sometimes neglected in comparison to other areas of mathematical instruction (Romberg et al., 1993). For many years, students have been taught both how to construct different representations of functions and how to interpret them. However, it is often the algebraic representations and the subsequent methods of manipulating those representations that have been emphasised in most traditional mathematics courses.

The impact of technology on the way mathematical functions can be represented and manipulated is forcing educators to reconsider the way functions are taught, with the algebraic emphasis being challenged. Technology makes it possible to deal with functions in new ways and to explore new ideas in curriculum and classroom practice. In the past, graphs were often difficult and cumbersome for teachers and students to create or manipulate. Nowadays, with the available technology, the expectation is that emphasising graphical representations will make functions easier to learn for most students. This chapter attempts to confirm this expectation.

An informal survey of some local Merseyside mathematics teachers has shown that questions involving functions and their graphs are the most commonly quoted example at A-level of 'visual difficulties'. Almost all traditional teaching practices and textbooks treat functions symbolically rather than graphically in the first instance. Graphical material is also poorly connected to the symbolic treatment of functions, with graphs being treated as a secondary topic, and not as a primary means of accessing functional information. Many mathematics questions present a function in algebraic form, and the student is then required to plot its graph (with or without a graphics calculator) or describe in some way the key features of its graph. The question would be more 'visual', however, if the student were given a graph, and then asked what type of function, i.e. what algebraic form, the graph represents. It is this learning methodology on which the interactive package, constructed using the generic authoring package TOOLBOOK (described in Section 5.3.3 of this chapter), is based. Whilst using the software, the user attempts to provide the symbolic notation corresponding to the given graph and, using this symbolic form, attempts to reproduce the given graph.

Previous work, discussed in Chapter 2, has described how graphics and symbolic calculators, such as the TI-92 and Voyage 200 (see References for Internet addresses), promote the relationship between different functional forms. This is also true of the bespoke graphical software, but additionally the latter has the advantage of giving students the opportunity to provide the appropriate symbolic form for a given graph, which, it is conjectured, is an even more powerful facility for the development of higher order skills. Algebraic and graphical forms of functions each reinforce understanding of the other. This transfer between representations can help the transition of reasoning from the concrete to the abstract. The purpose of the software, by providing multiple-linked representations, is to encourage the production of a concept image (Vinner, 1991) of function that is more flexible, more powerful, and more compelling than by other means. It is hoped that in this manner students can build the bridge between the concrete and the abstract more effectively.

5.3.3. The Interactive Software

To develop students' conceptual understanding of the relationship between visual and symbolic representations of functions, a piece of bespoke mathematical software was written entitled 'Graphs of Functions: A Constructivist Approach'. Users of the software must attempt to provide the algebraic notation for a given function graph. Note that this dedicated software had to be specially written, as no available software could be found that would handle the transition from the visual to the symbolic. For example, this is not possible directly on a graphics calculator. A graphics calculator can only produce a graph from a given function expression - it cannot produce the function expression of a given graph. This is not as helpful in the acquisition of conceptual understanding. In the 'CTI Mathematics and Statistics Guide to Software for Teaching' (1995), there are many types of graph-plotting software available, but all follow the pathway from symbolic notation to graph, which further supports the need for specialist software. The software described here is unique in that it reverses this structure and concentrates on finding the symbolic notation from the graphical representation. This is an alternative format, which lends itself to a more constructivist approach to learning, as per the discussions provided in Chapters 3 and 4.

The software has four sections. In the first section, the **Introduction**, the user is provided with three further sections (this final page of the Introduction section is shown in Fig. 5.01). An **Instructions** section on how to use the software is provided, then the user moves on to the main section for the exploration of **Graphs of Functions**. This is followed by a **Summary** section.

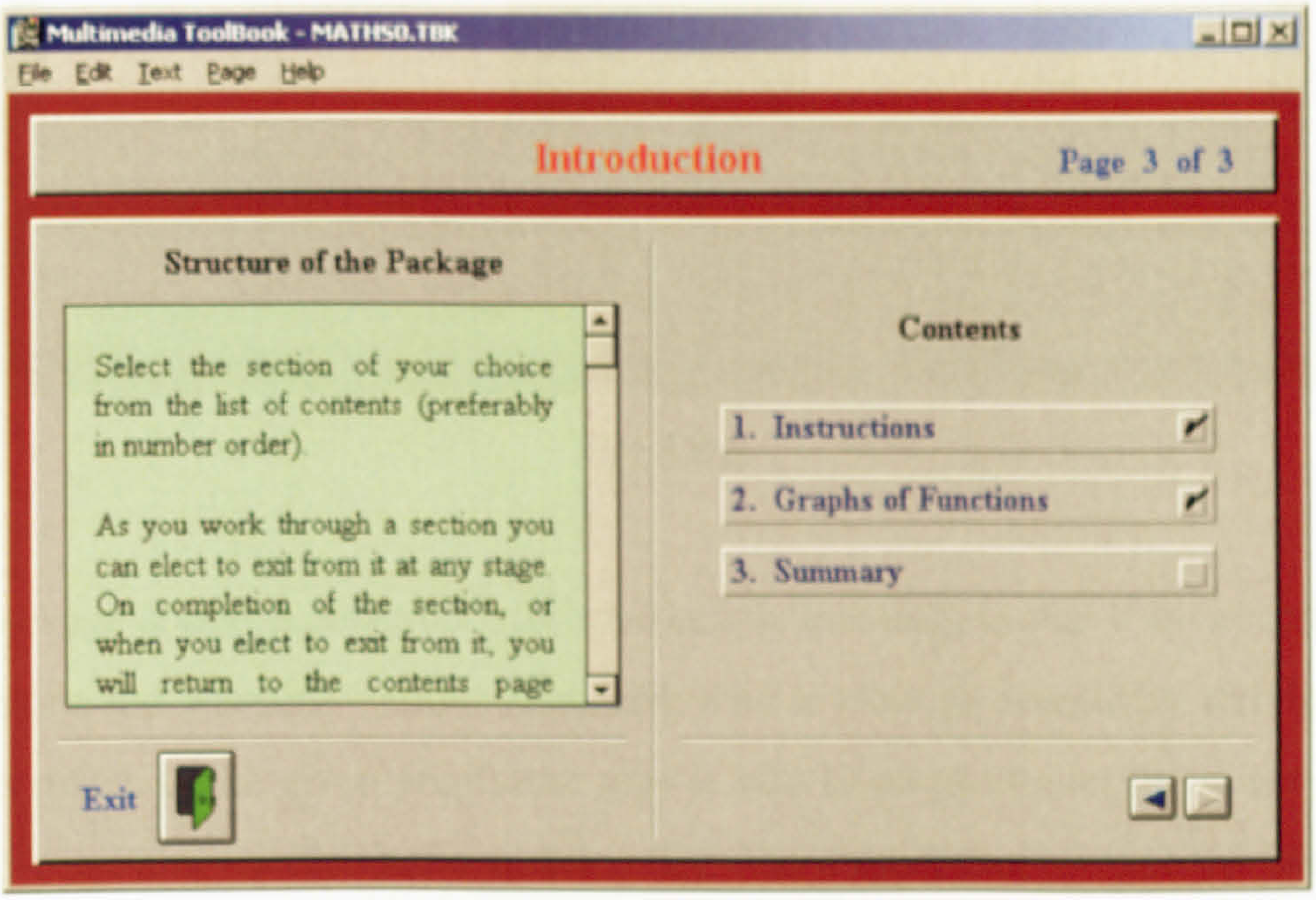


Fig. 5.01.

The user is shown a graph (a translation and/or transformation of an elementary function, or a combination of elementary functions), and then using a keypad (see Fig. 5.02) attempts to determine the correct symbolic expression for the function. Once the user has conjectured a function expression, it can be plotted in a window adjacent to the given graph, and a comparison can be made. See Fig. 5.02 for an example.

This process can be repeated as many times as required. There is also a 'Hint' button in case students are struggling to find a starting point for their explorations. Each graph that is produced tests the user's conjecture. The user can then use this information to test out a further conjecture. This loop is repeated until the correct 'solution' is achieved.

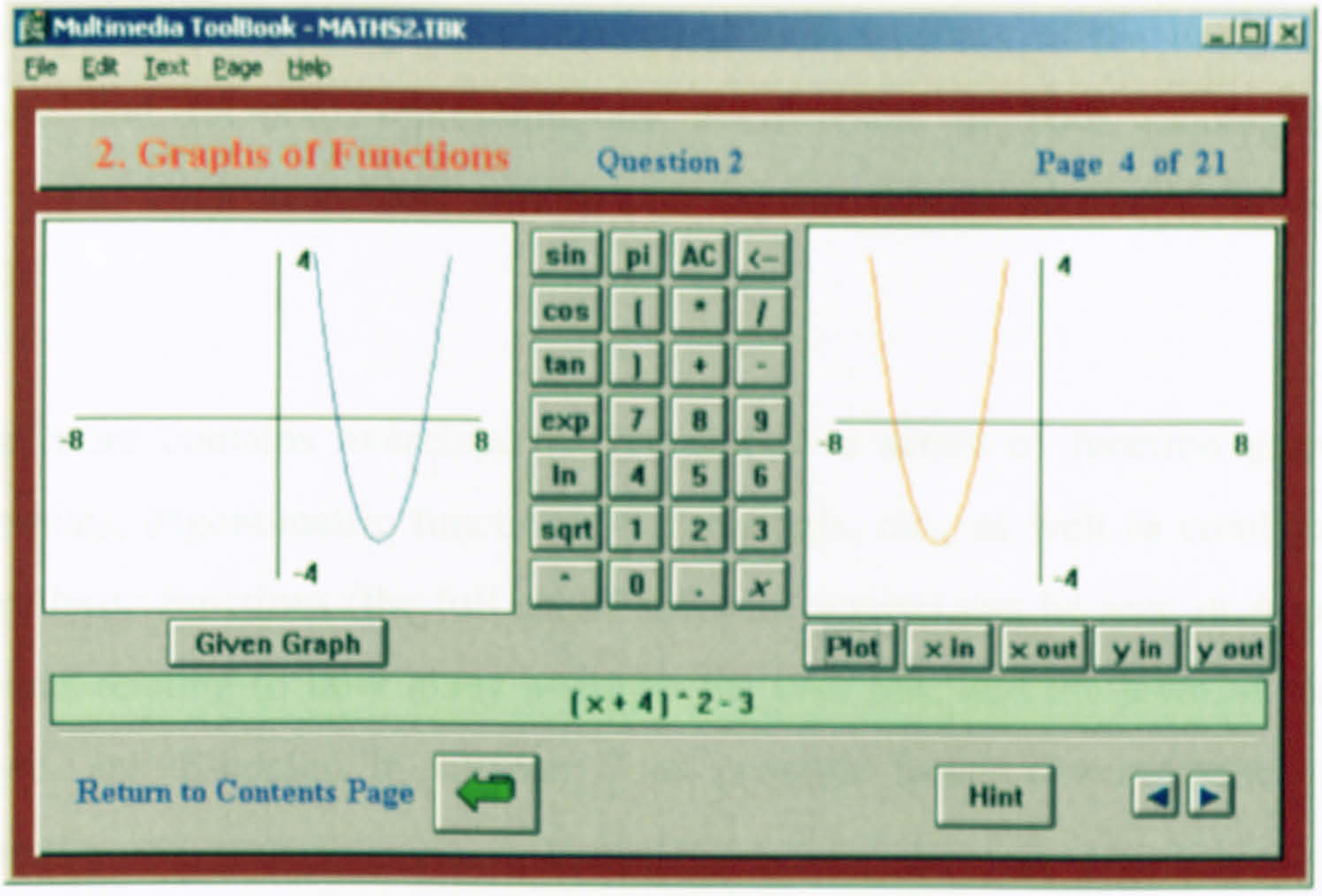


Fig. 5.02.

The package adopts a constructivist approach to learning, in that it invites the user to explore and examine various functions with a view to eventually arriving at a perfect match to the given graph (the user is also allowed to alter the scales of the axes by zooming in or out for either axis). The expectation is that the user will begin to appreciate the effect on the graph of altering various parts of the function (a 'walk-through' example can be seen in Section 5.3.3.3).

Integrating representations in the bespoke software provides an opportunity for direct manipulation of both symbols and graphs. The software supports a pedagogy of exploratory learning. The ease with which graphs can be plotted permits users to shift their focus away from the detail of physically constructing the graph and allows them to benefit from the thoughtful manipulation of the graph as an entity. It encourages them to construct meaning for different representations and how they are related, as well as helping to build the idea of a function not just as a process, but as an object.

The software provides an environment that encourages students to conjecture, make comparisons, observe patterns, reflect on findings, and generalise. The students therefore build up their conceptual understanding of the links between algebraic and pictorial representations as a result of both successful *and unsuccessful* conjectures and evaluations (again, this is illustrated in Section 5.3.3.3).

The software contains exercises that consist of a series of function graphs of polynomials, trigonometric functions, exponentials, etc., as well as combinations of these basic functions (the full set of software screens can be seen in Appendix A). Issues relating to how many attempts the user has, and the level of mastery achieved, are discussed in Chapter 7 as possible future developments of the software.

The diagram in Fig. 5.03, informed by the computer learning process (Sein et al., 1993), summarises the learning process that students experience whilst using the dedicated software. The diagram illustrates how the learning environment, together with the teaching and learning approach adopted, can develop learner characteristics. The form of learning influences the type of mental images that students create, and these in turn determine student performance and attitudes. These learning outcomes can then inform the design of future teaching and learning scenarios.

5.3.3.1. Factors Affecting the Design of the Software

Various design issues for effective educational graphing software (Goldenberg, 1991) have been taken into account during the design of the interactive software. The educational value of a piece of software is in the generalisations students can abstract from particular instances, which is certainly the case with the software discussed here. It must be easy to modify functions, i.e. the interface must make it convenient for students to explore by modifying a single parameter in one form in order to study the effect on another.

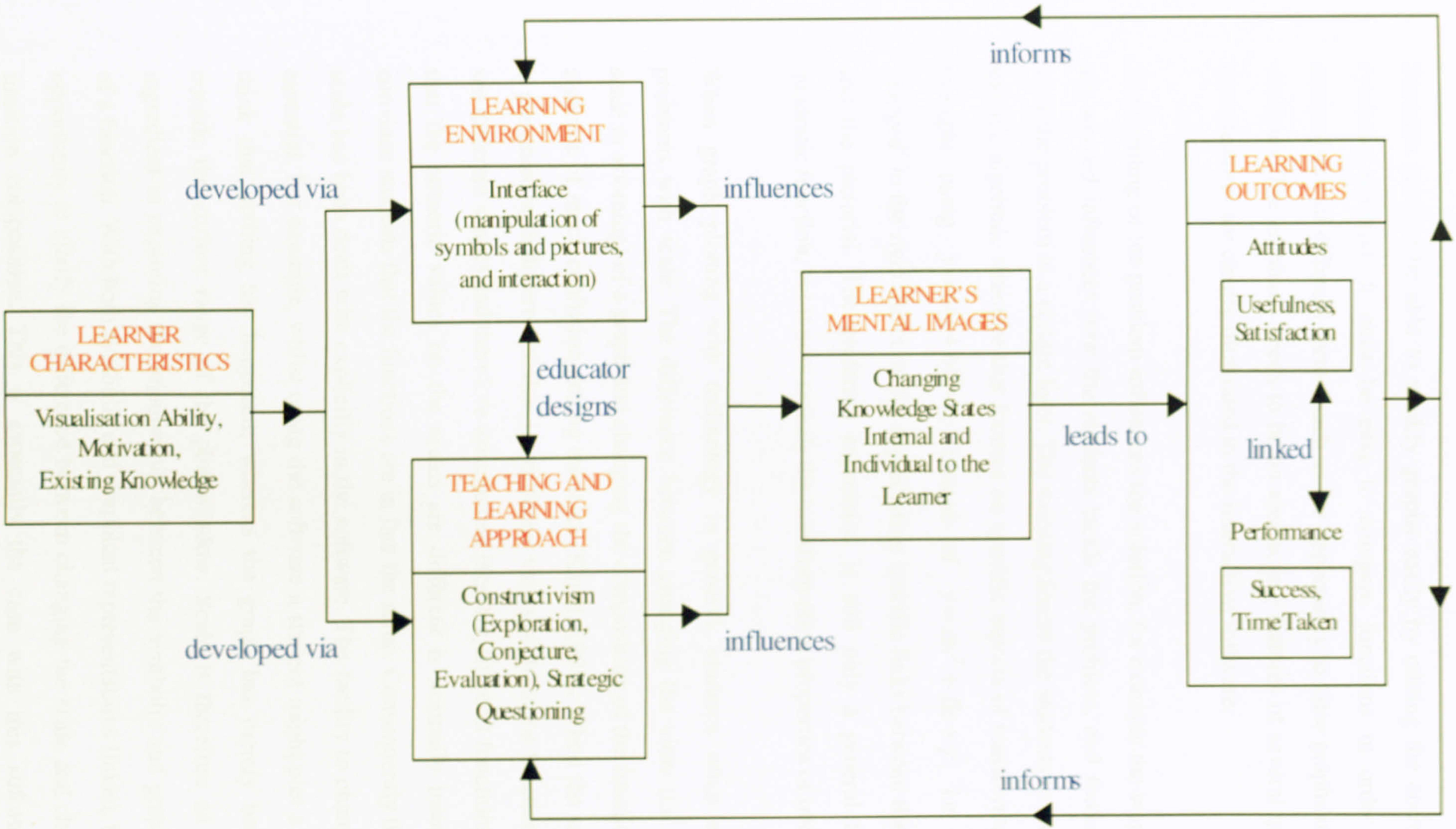

LEARNER CHARACTERISTICS
Visualisation Ability, Motivation, Existing Knowledge
developed via
developed via
LEARNING ENVIRONMENT
Interface (manipulation of symbols and pictures, and interaction)
educator designs
TEACHING AND LEARNING APPROACH
Constructivism (Exploration, Conjecture, Evaluation), Strategic Questioning
influences
influences
LEARNER'S MENTAL IMAGES
Changing Knowledge States Internal and Individual to the Learner
leads to
LEARNING OUTCOMES
Attitudes
Usefulness, Satisfaction
linked
Performance
Success, Time Taken
informs
informs

Fig. 5.03.

Students need to be able to modify graphs easily by editing the corresponding symbolic notation. It must be easy to compare functions in order to make comparisons of different forms, such as the opportunity to view graphs in multiple windows. The emphasis needs to be on abstracting features of several graphs. All these features are clearly illustrated in the interactive software.

The wording of the problem influences the solution, for example the way the hints are worded influences how the students tackle the problem, and forces them to view the problem in a certain light. The wording forces the students to work with a specific algebraic structure that focuses on specific aspects of transformations, for example using $y = a(x+b)^2 + c$ instead of $y = \alpha x^2 + \beta x + \gamma$, i.e. they are 'nudged' in the right direction for appreciating specific links between the symbolic and the pictorial. The preferred expression is still only a general form of a quadratic function, but it now reveals the transformation properties of interest.

When graph plotting with technology in general, students often experience problems with scale. The difference between changing the view (i.e. changing scale or zooming) of a graph and changing the composition of the function itself is a source of much confusion among students (Kaput, 1993). When the same graph is presented with different scales, i.e. different views of the same function, students see different shapes and therefore assume that they are different functions. The fact that the numeric values on the scales are different is normally insufficient to convince students that the functions are in fact the same. Consequently the issue of scale has been dealt with explicitly in the software. The facility to change scale is essential. For example, whilst using the software a student might plot a graph and think that nothing has happened, whereas the graph has merely been plotted outside the current range of the plot window. Scale is therefore an important ingredient in exploring the relationship between the symbolic and graphical form of a function. With both symbolic and graphical representations linked, there is the opportunity to clarify the difference between changing the scale and changing the function composition. This is especially the case with this software, which examines function comparisons in a linked representation environment. Students can control the scale however they feel appropriate, and use it to reshape graphs accordingly (the decision not to provide information regarding intercepts, co-ordinates, etc., as an intentional design feature, is discussed further on page 276 of Chapter 7).

In addition to Goldenberg's design issues, the four main motivational factors concerned with the maintenance of interest (Lepper et al., 1993), as discussed in Section 2.5.1.3 of Chapter 2, have also been incorporated in the design.

5.3.3.2. Theoretical Rationale for the Design of the Software

This section describes how the design of the software incorporates the formal theories of teaching and learning discussed in Section 2.4.3 of Chapter 2. It illustrates how conceptual understanding can be built up via constructivist switching between representations. Traditional teaching approaches, unlike the approach supported here, are very linear, as illustrated in Fig. 5.04 below.

Traditional linear approach

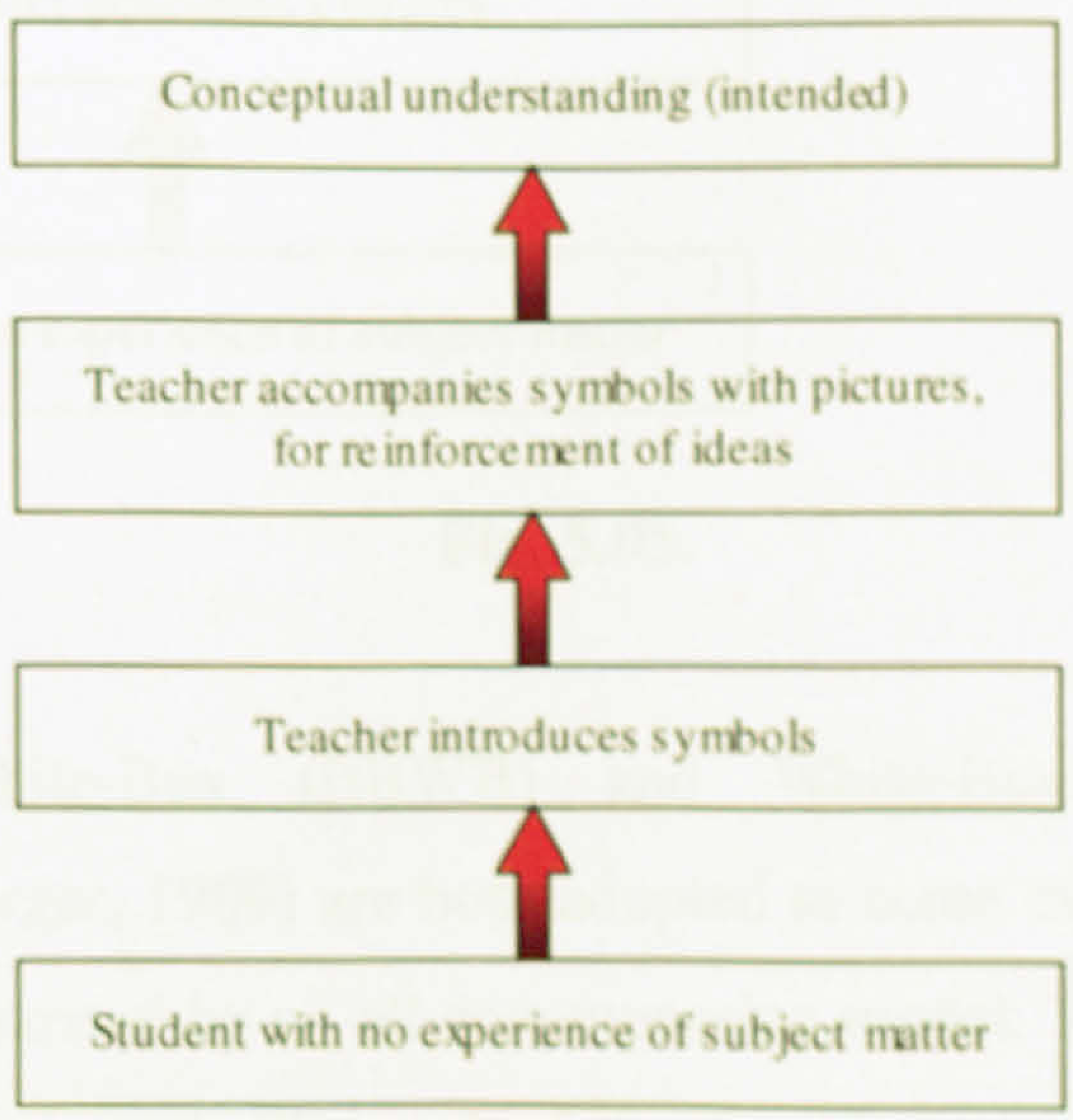


Fig. 5.04.

The preferred non-linear approach, supported by the appropriate use of the software, illustrated in Fig. 5.05 below, is much more dynamic. The software encourages a constructivist approach to learning, and facilitates the important interactions between student and teacher, and between symbols and pictures. The human interaction is facilitated by the Socratic method of strategic questioning (see Chapter 3), and the computer is employed as a vehicle for generating scenarios in which the interaction between symbols and pictures can take place. Both of these forms of interaction are important for the progression from instrumental to relational understanding.

Preferred non-linear approach

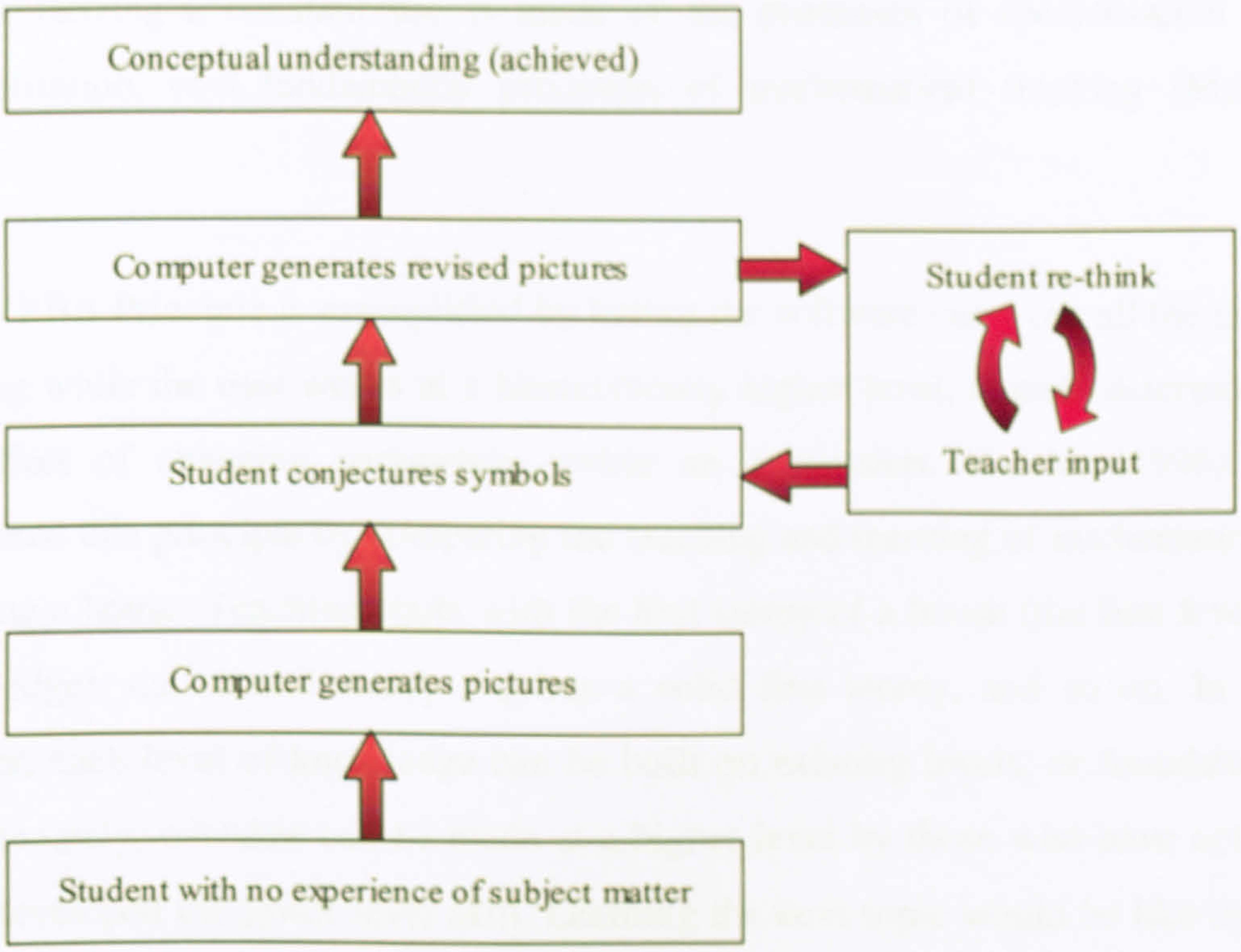


Fig. 5.05.

The Black-Box/White-Box (BBWB) and White-Box/Black-Box (WBBB) approaches (Buchberger, 1989) are both adopted to some extent, thus the software use cannot be summarised by an all encompassing model. The use of the software is neither the beginning nor the end of the learning process - the intermediate 'results' provide the user with new questions. The interaction is taking place on two levels - sometimes the software does 'lower level' tasks, such as plotting the graphs, while at other times the software raises questions for the user. It stimulates the user to investigate, to conjecture, to explore, and to justify, using a mixture of graphical and algebraic techniques, performed by either the software or the user.

The BBWB approach is adopted, as emphasis is placed on explorations and investigations into the effect of symbolic changes to expressions on a graph. Another example of this investigative approach (Herring, 1995) involves the use of DERIVE as a tool in the specialisation stage so that insight can be quickly gained in order that a generalisation of the solution may be obtained. Both in this work and in Herring's, constant use is made of the processes of specialisation and generalisation, two fundamental processes of mathematical thinking (Mason, 1984).

The WBBB Principle is exemplified by letting the software carry out all the graph plotting while the user works at a hierarchically higher level, namely determining the effect of changing parameters within an expression. Kutzler (1996) has illustrated this principle by comparing the teaching and learning of mathematics to building a house. Teaching starts with the first storey of a house (the first level of knowledge), the second storey requires a solid first storey, and so on. In this manner, each level of knowledge can be built on existing levels, or foundations. Unfortunately, mistakes can be made at a higher level by those who have not yet fully developed the lower level skill. Learning the next topic would be like trying to erect a new storey on top of one that was still incomplete. The software here can overcome this problem. The first skill, the plotting of graphs, can be taught in the traditional way, but when the second, higher order, skill is taught, e.g. the interpretation of function translations, the computer can be allowed to take care of all the tasks that require the first skill. Thus the computer serves as a scaffolding between the two storeys. The software eliminates the possibility of mistakes as a result of carrying out the lower level task incorrectly. The student can now concentrate fully on the higher level task at hand. This approach enables students to proceed with more advanced topics without handicap if they are poor at the lower level skill.

Although the use of the software does not fit neatly into either the WBBB or BBWB models, the sorts of operations described are essential in any mathematical software usage. The most interesting and valuable aspect of the software is the interaction between user and software, which does not fit into the Buchberger model. This two-way communication stimulates the student, and thus enhances the learning process. The use of the software can therefore not be fully explained by either model, as they are too rigid in their hierarchical view of mathematics. Quoting J. F. Cloutier, "*the model is like lasagne, but mathematics is like spaghetti*" (Drijvers, 1995). In order to maximise understanding and, perhaps more importantly, to assist in the retention of knowledge, a mixture of the two approaches is required.

As per the communication model (Laurillard, 1990), during software usage control is given to the user who takes responsibility for any learning that takes place. The student constructs knowledge while the teacher facilitates the process. This is quite different to the use of more instructivist computer-based teaching (CBT) packages, such as MATHWISE. Usage of such packages does not allow for the accommodation of interaction, which is required in order for it to constitute a communication model. Note that not all CBT software is instructivist in nature – hypermedia materials allow students to interact with other materials, and promote interaction with teachers and peers. CBT packages, such as MATHWISE, have assisted in the development of distance learning activities, but with this, unfortunately, the direct communication between teacher and student decreases. The value of the communication model is that it puts greater emphasis on the student's view of the world. When teachers play only an indirect or absent role in the teaching process, as they do with more instructivist packages, they relinquish the opportunity for the kind of negotiation that face-to-face communication provides. Educational material such as this would appear, therefore, to operate within Laurillard's didactic model. A preferable scenario is that described by the software usage described here, where communication, in the form of strategic questioning, plays a key role in the teaching and learning process. The power of the computer should be harnessed to provide a closer approximation to the communication model. The software illustrates that computer-based educational materials can be significantly less didactic, and more constructivist than they have been in the past.

Coherent articulation of different representations is important for the construction of conceptual understanding. Through dedicated software usage, students pass through the five levels identified by Hitt (1998b) by building up the relationship between these representations, as well as effectively switching between them, thus leading to the construction of concepts. They progress in a similar fashion through the four stages in the learning process identified by Dreyfus (1991).

The software was designed in order to help in the development of understanding, not just processes, which is the key to success in mathematics. Students can construct properties of the concept for themselves, and reflect upon what they have learnt. The software is not just concerned with the final product, but has been specifically designed to promote discovery-based learning activities involving multiple representations. Visualisation is used to force the mental image making process (Dreyfus, 1991), which assists in the process of generalising, where students can move from the particular to a more general case. As with Laurillard's communication model, the responsibility for learning is transferred from teacher to student.

The interplay between the symbolic expression and the graph acts as a scaffolding as students construct their personalised 'correct' concept images (Vinner, 1991). In this case, the symbolic expression acts as Vinner's concept definition, which helps to form, reinforce, or redefine the concept image. The concept image is not formed by the symbolic expression alone, but instead via constructivist explorations with the software, facilitating interactions between the two forms.

Students come to view the graph as a conceptual entity (Harel and Kaput, 1991). When considering the translation of a function graph, it is much more efficient to consider the function graph as an entity, and not as a series of pointwise operations, i.e. processes acting on individual elements of the domain. Effective use of the software requires the encapsulation of function as an entity. This helps with the focus of attention. For example, let us consider the function $f(x+a)$. If students can view $f(x)$ as an entity, then they can focus on the effect of a, i.e. view a as a shift operator taking $f(x)$ 'as an argument'. This enables students to focus on the property of the function that is most relevant to the solution of the particular problem, namely the effect of the introduction of a into the symbolic notation.

In terms of the horizontal and vertical growth discussed by Harel and Kaput (1991), horizontal growth is supported by the interaction between the different representations, and the constructed conceptual entities can be transferred to new situations where they can be operated on further, thus assisting in vertical growth.

Meaningful learning takes place as the students are mentally active. The different methods of construction in reflective abstraction (Dubinsky, 1991) are now discussed in terms of this software usage. **Interiorisation** is the translation of a series of plots, linking symbols and pictures, towards the creation of mental images. The interiorised concept has been constructed by making sense out of the perceived phenomena. **Coordination** is the symbol manipulation and plotting of graphs, i.e. the coordination of the different representations, in order to construct a concept image. **Encapsulation** is the conversion of the constructive processes that result in viewing the function as a static object, i.e. a conceptual entity. In this manner, the actions become entities which can be utilised in the further vertical growth of ideas. **Generalisation** refers to the situation in which, after having completed the first few examples in the software, the student can apply newly acquired knowledge to functions of an unfamiliar type. The constructive processes have been encapsulated into an object. Reversal is switching between the traditional process of 'symbols to graph' and the innovative process of 'graph to symbols'. The familiar process has thus been reversed. The new process has to be constructed by reversing the start and end points of the traditional one.

When functions are considered in the software, it is necessary to be able to think of the functions as objects, so that they can be treated as conceptual entities. Users of the software must therefore perform an encapsulation in order to consider the functions as objects instead of just processes. The user needs to alternate between thinking about the same entity as a process and as an object. Encapsulation also involves taking specific concrete situations and using the resulting observations to stimulate the development of further general properties. For many students, the concept of function is all about a process, and therefore the connection with the function graph has no meaning. The student needs to be able to coordinate a function's process and the properties of its graph. The student can then relate to the power of the relationship between the function as process and the function as object. The function needs not only to be an interiorised process, but as a result of encapsulation this process can be treated as an object. The graphical form can help with this encapsulation. In the example of the addition of two functions, the student must view this as an operation which takes two objects and transforms them into a third object. In order to do this, the original two objects must be viewed as processes. These processes can then be coordinated and the resulting process encapsulated into an object, i.e. the new function as the result of the addition of the two original ones. The cognitive interpretation of function therefore necessarily switches between process and object.

The activities with the software therefore foster reflective abstractions. Via the software, the student interiorises a process. As that same process can then be treated by the software as an object on which operations can be performed, the student is likely to encapsulate the process. It therefore provides the opportunities for reflective abstraction necessary for the construction of concepts.

It is conjectured that the software facilitates the acquisition of relational understanding (Skemp, 1976), as opposed to merely instrumental understanding, which can be adapted to new problem solving situations.

5.3.3.3. Dialogue to Illustrate the Learning Process

Having described the functionality of the software, together with a detailed rationale for its design, this section provides an example of how the software is actually used, which illustrates the interactive processes that occur in the zone of proximal development (as discussed in Chapter 3).

The following is a typical dialogue between student and teacher whilst the student attempts one of the questions from the software (Fig. 5.06) during the case-study. It shows how a student's ability can be determined as it matures, as a result of how a student responds to mediation, rather than simply measuring the student's factual knowledge. It illustrates the role of the teacher whilst the student constructs knowledge in the zone of proximal development.

The teaching style adopted is the Socratic method of strategic questioning, as described in Chapter 3 (Rowlands et al., 1997). Working within the students' zone of proximal development, props and hints are used to challenge misconceptions and lead the student to the construction of the target concept. Probing questions are used in an attempt to assess the student's ability to accommodate a different style of learning. The student's ability to respond appropriately is of interest, rather than the ability to regurgitate information. The intentional aim is to increase the size of the zone.

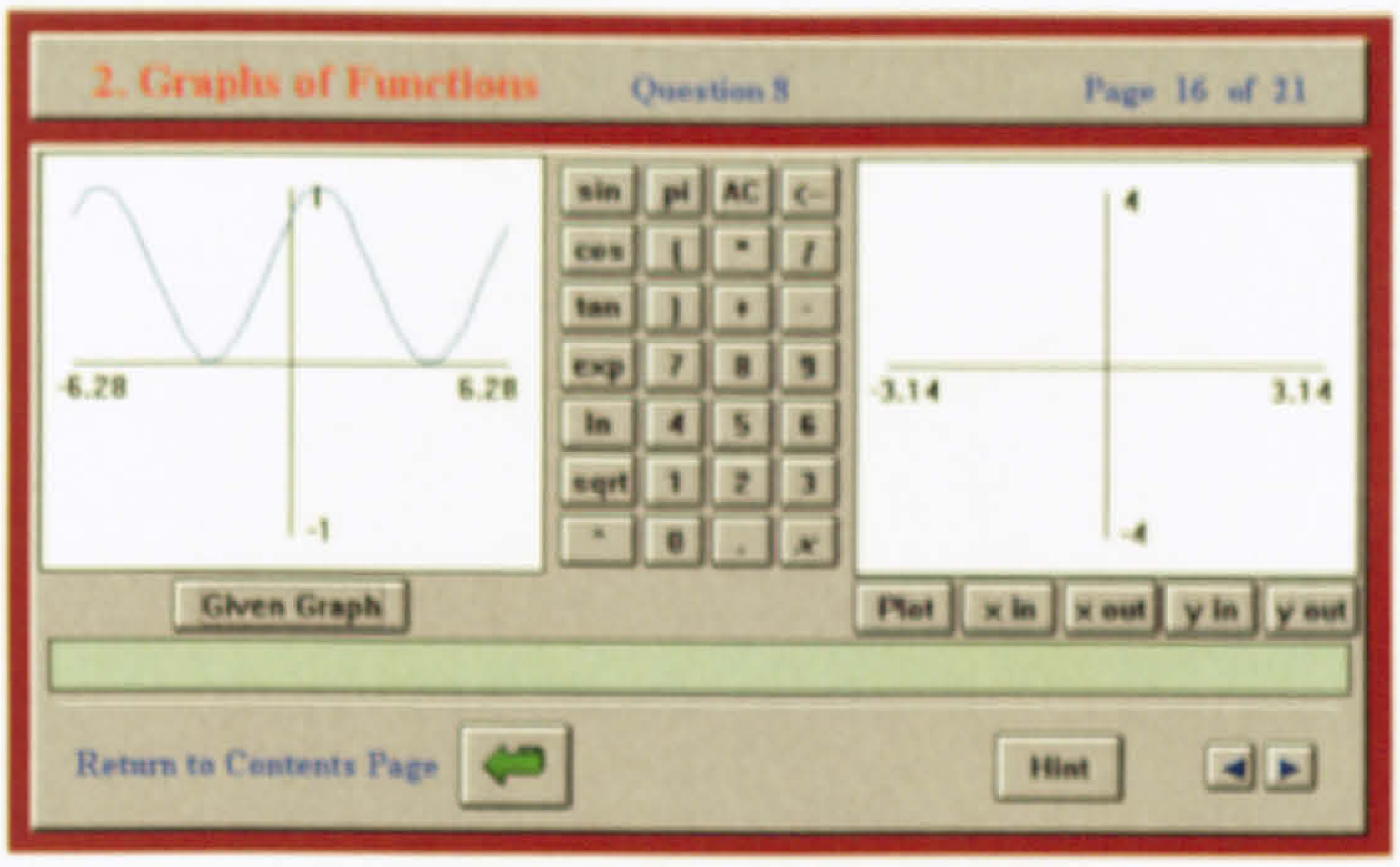


Fig. 5.06.

['Reversal' of start point]

Student: I don't even know where to start with this one.

Teacher: Well, what does the shape of the graph remind you of? What family of curves might it belong to?

[Viewing graph as a 'Conceptual Entity']

Student: It looks like a cosine graph.

Teacher: Or perhaps even a sine graph?

Student: Well, yes. They're kind of the same aren't they?

The teacher briefly comments on the relationship between sine and cosine graphs.

['Relational Understanding']

Teacher: OK. So why don't you plot $\cos(x)$ to start with, and we'll take it from there. We'll see how it looks compared to the given graph.

The student plots the graph of $\cos(x)$, as in Fig. 5.07.

[Combination of 'BBWB' and 'WBBB']

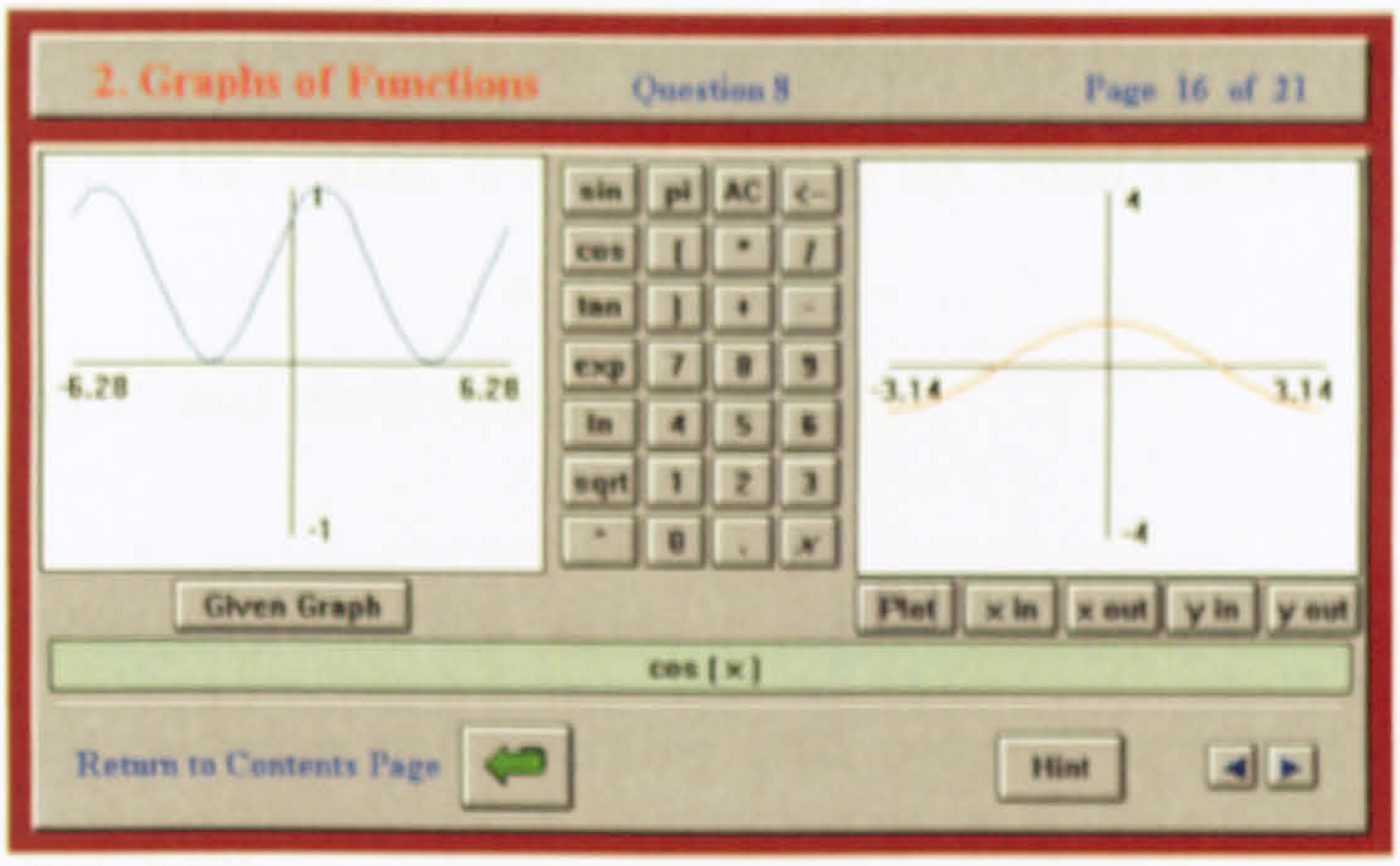


Fig. 5.07.

Student: Well that doesn't look the same.

Teacher: Why not? What's different?

Student: It's a different shape.

The teacher doesn't respond, and lets the student think about it for a moment. After a long pause, the student notices that the scale is different, and says that she will make the scale the same as on the other pair of axes to see what effect that has on the shape of the graph, as in Fig. 5.08.

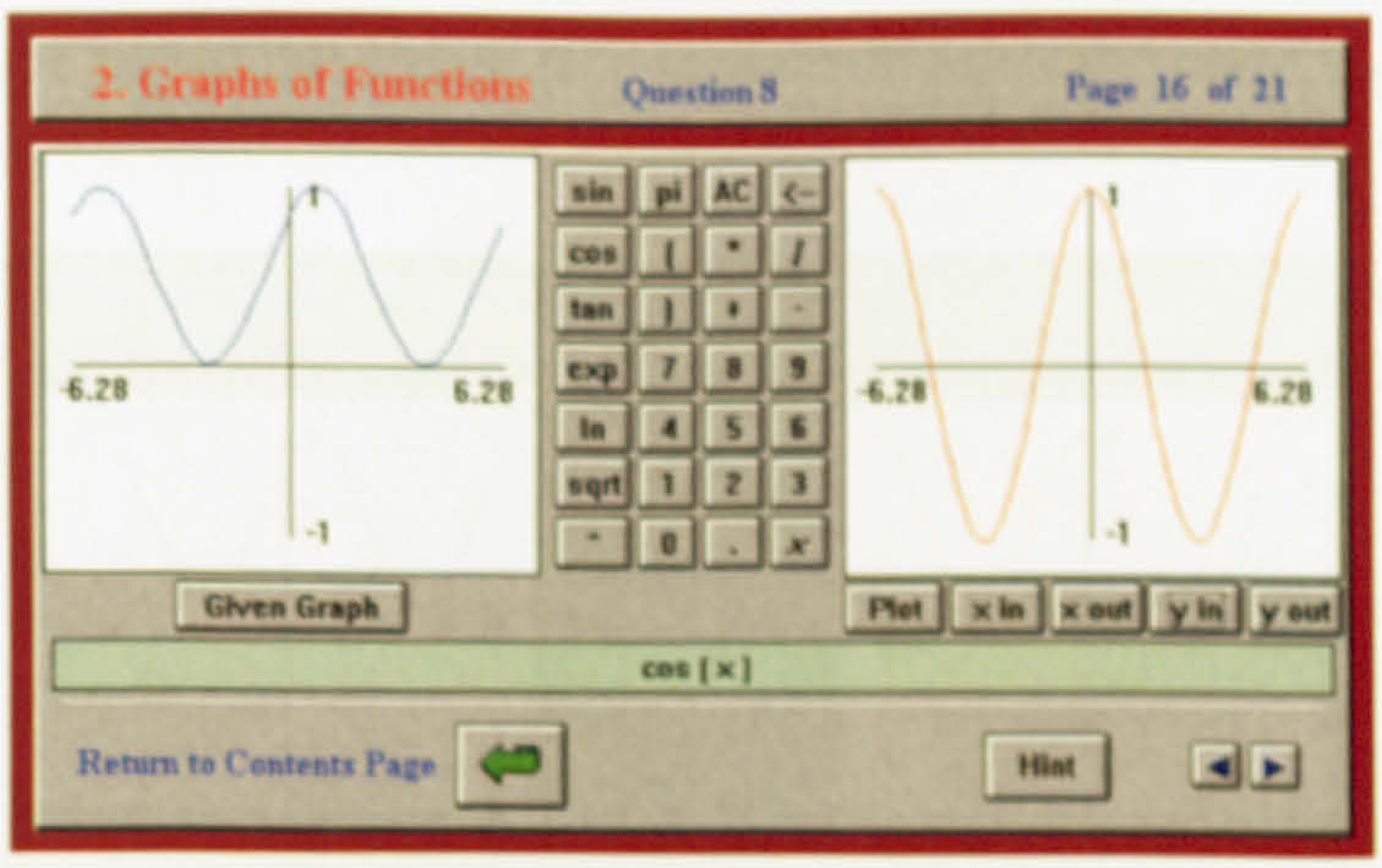


Fig. 5.08.

Student: It looks more like the given graph, but now it's going above and below the x-axis. It needs shifting up. Shall I add 1 to it?

Teacher: Try it, and observe what happens.

The teacher deliberately lets the student make mistakes, as it is through the resolution of such mistakes that learning will occur. The graph of $\cos(x)+1$ is produced, as in Fig. 5.09.

['Coordination' of different representations]

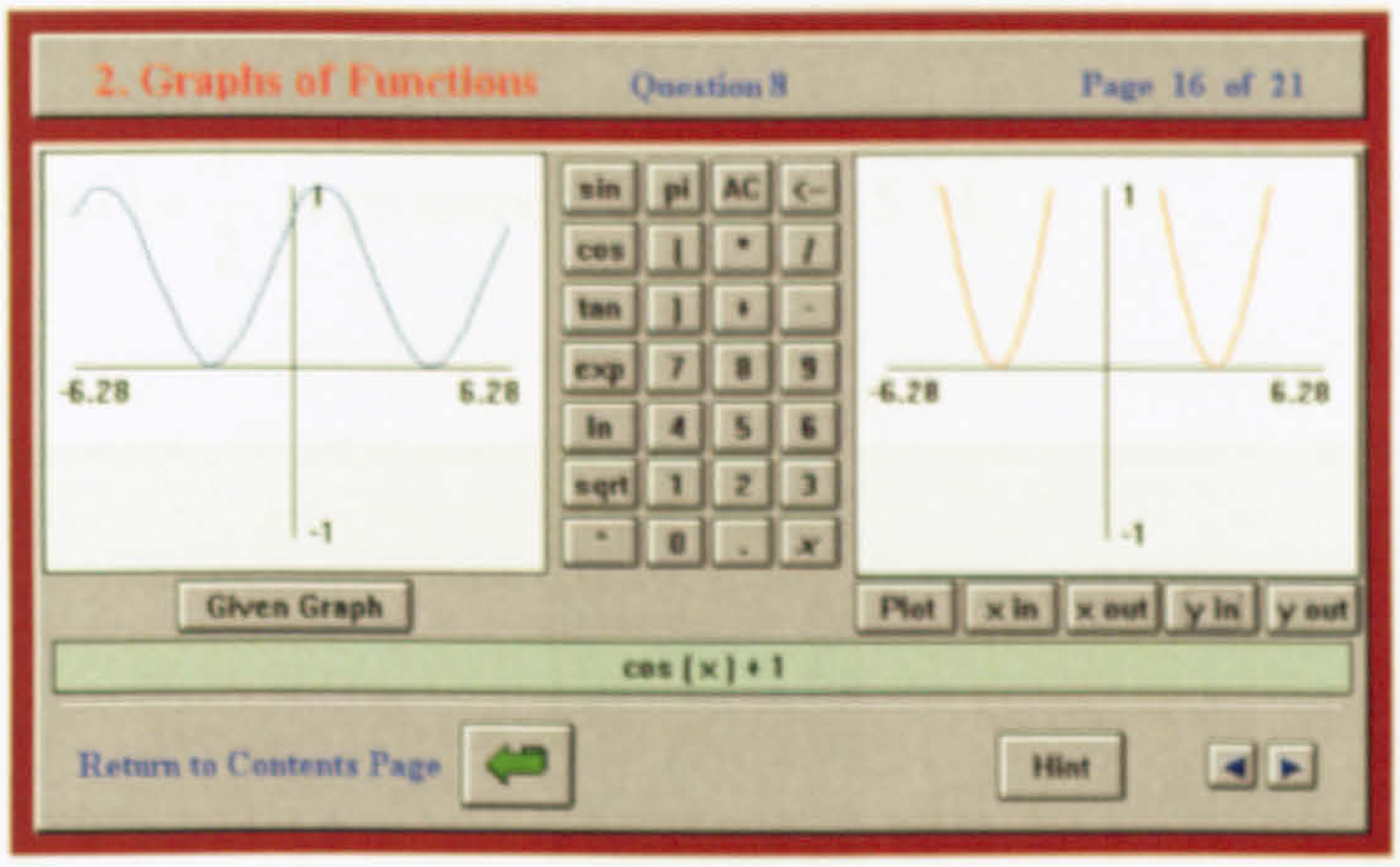


Fig. 5.09.

Student: That's no good. It's gone up twice as much as I wanted it to. It's off the top of the screen (the student changes the scale, as in Fig. 5.10).

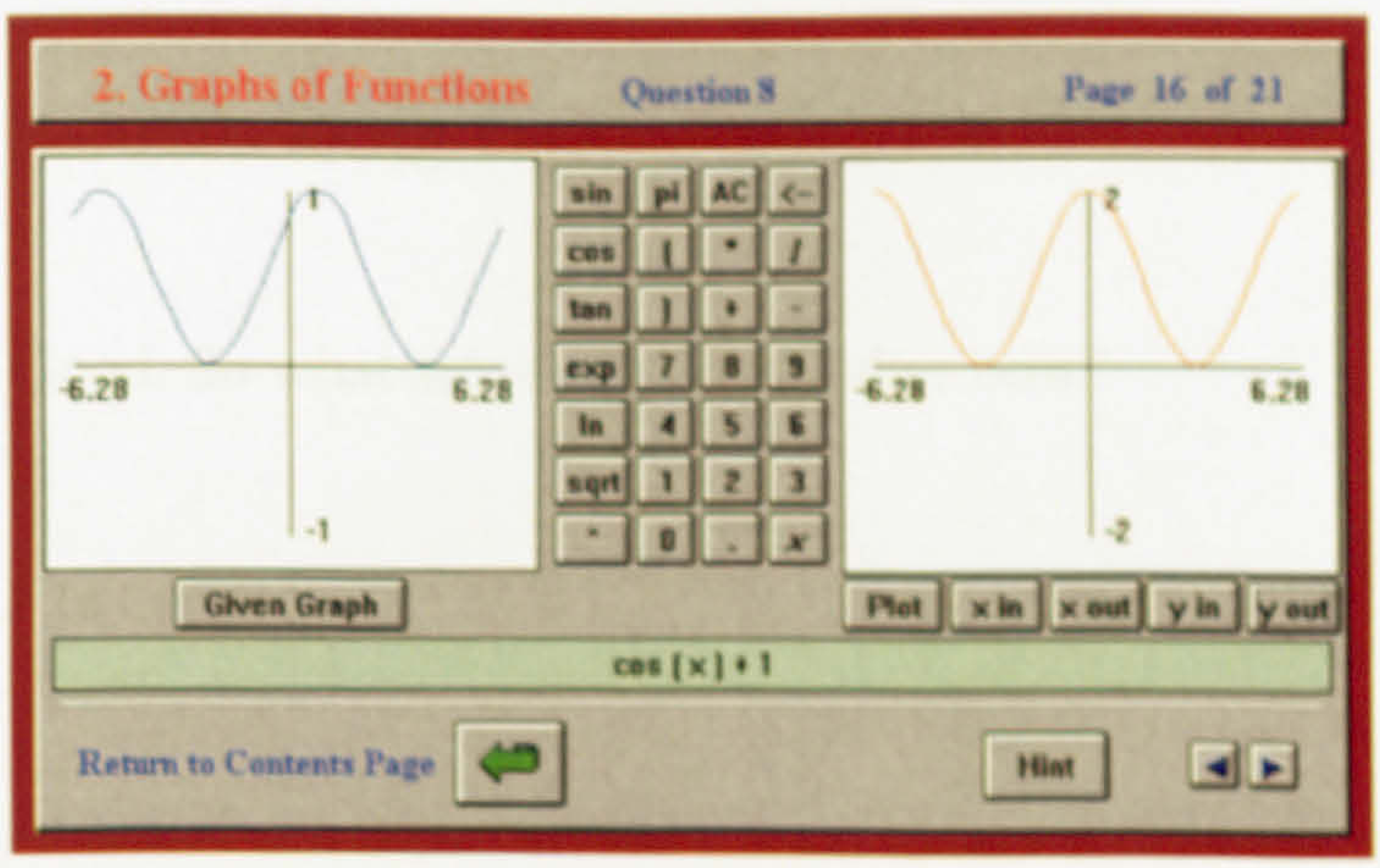


Fig. 5.10.

Student: I want it between 0 and 1, not 0 and 2.

The teacher tries to introduce some formal language at this stage, and explains that making the graph 'less tall' is making the 'amplitude' smaller.

Teacher: What do you think you need to do to make the amplitude smaller?

Student: I should have shifted it up by 0.5, instead of 1.

Teacher: Try it.

The graph of $\cos(x)+\frac{1}{2}$ is produced, as in Fig. 5.11.

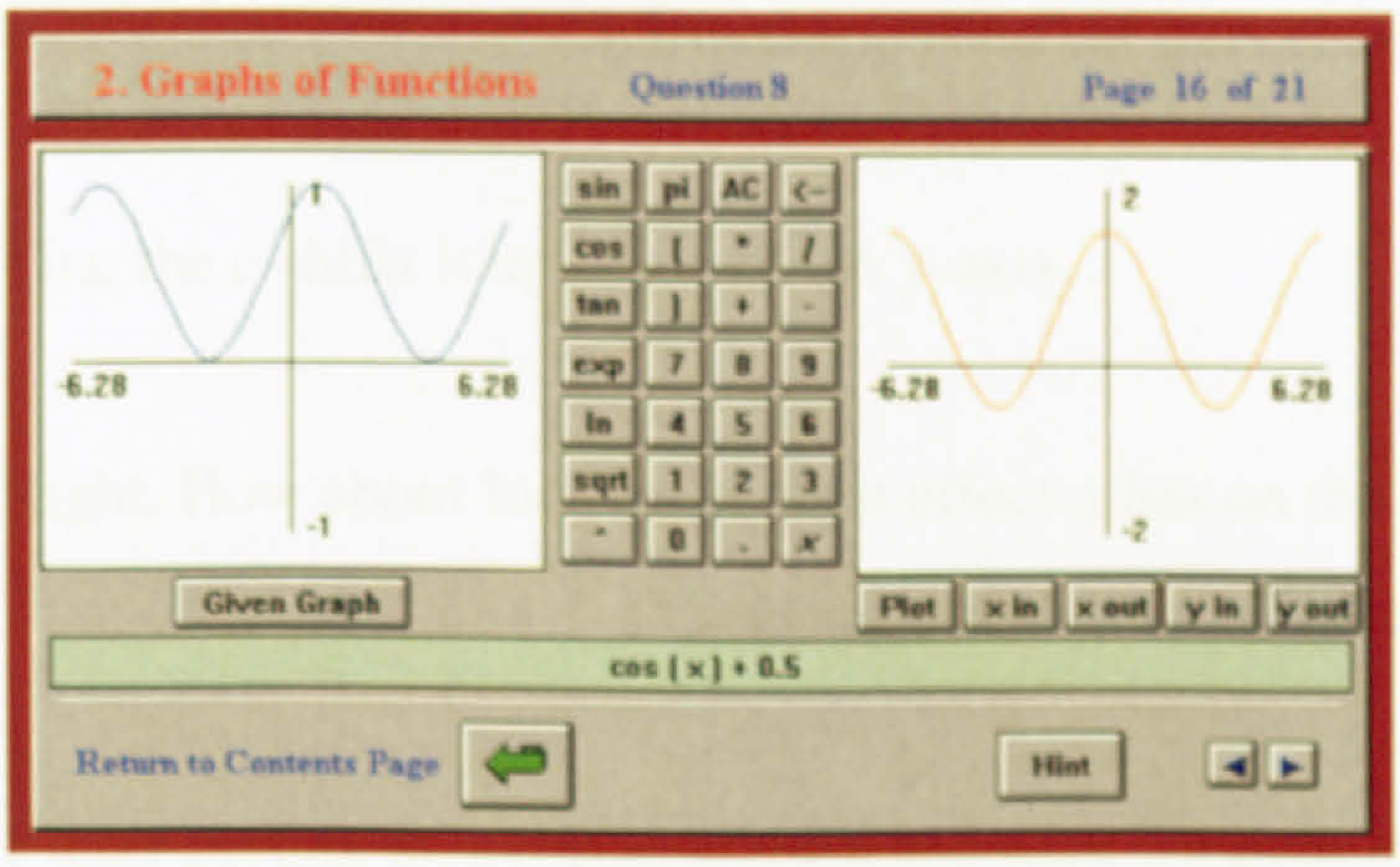


Fig. 5.11.

Student: No, that's wrong. I need to squash it in half.

Teacher: Good. So what are you going to alter now?

Student: I'm not sure.

The teacher asks the student to click on the 'Hint' button for assistance, as in Fig. 5.12.

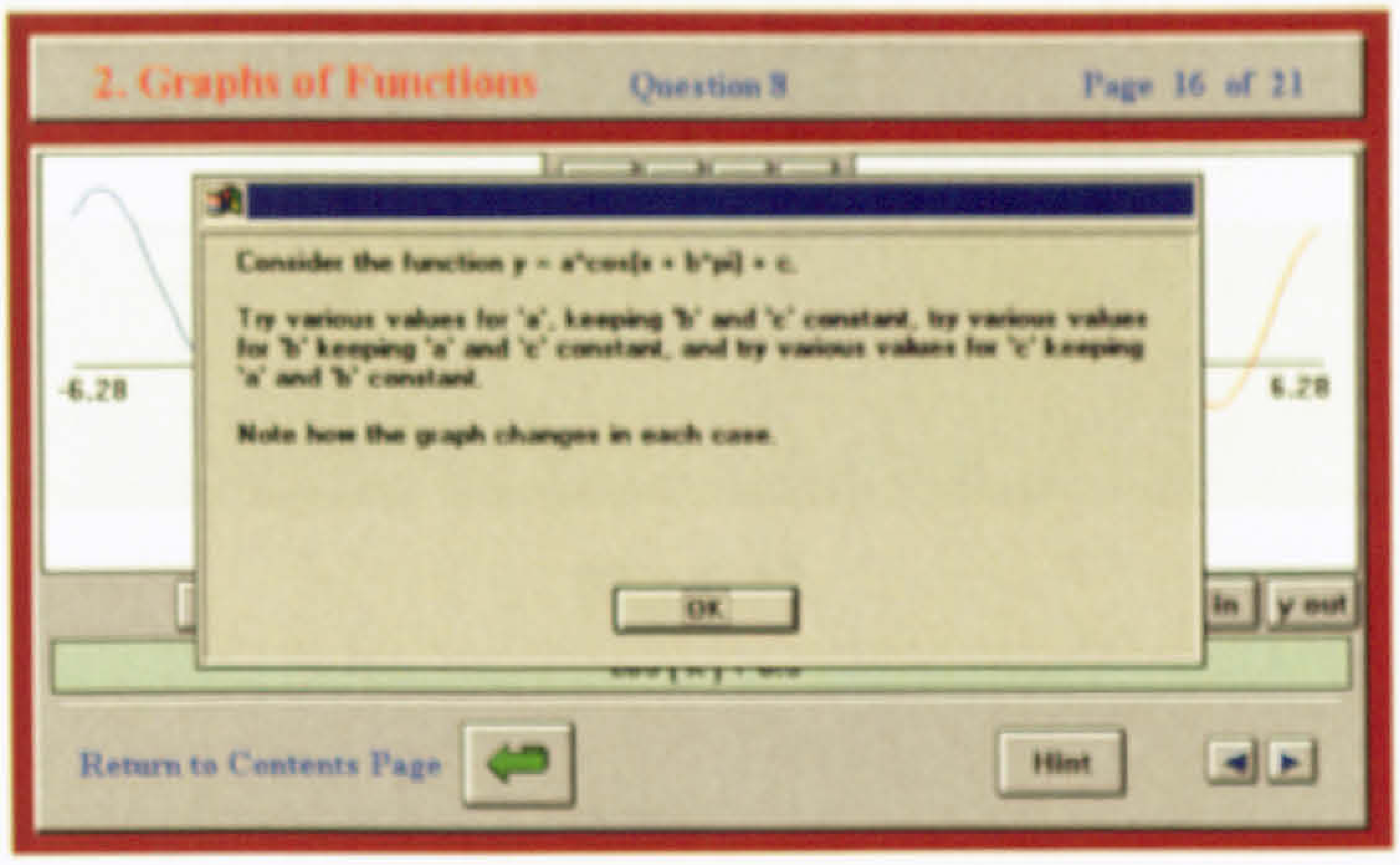


Fig. 5.12.

Teacher: Well, what have you learnt so far about *any* of the parameters.

['Interiorisation' of perceived phenomena]

Student: That the c shifts it up or down the y-axis.

Teacher: Right. How about looking at what effect a has on the graph, now.

[Treating graph as a 'Conceptual Entity']

Student: I'll try putting in $2\cos(x)+1$. I'll see what happens and we'll take it from there.

The graph of $2\cos(x)+1$ is produced, as in Fig. 5.13.

(the student has gone back to the original value of $c=1$, as this looked better).

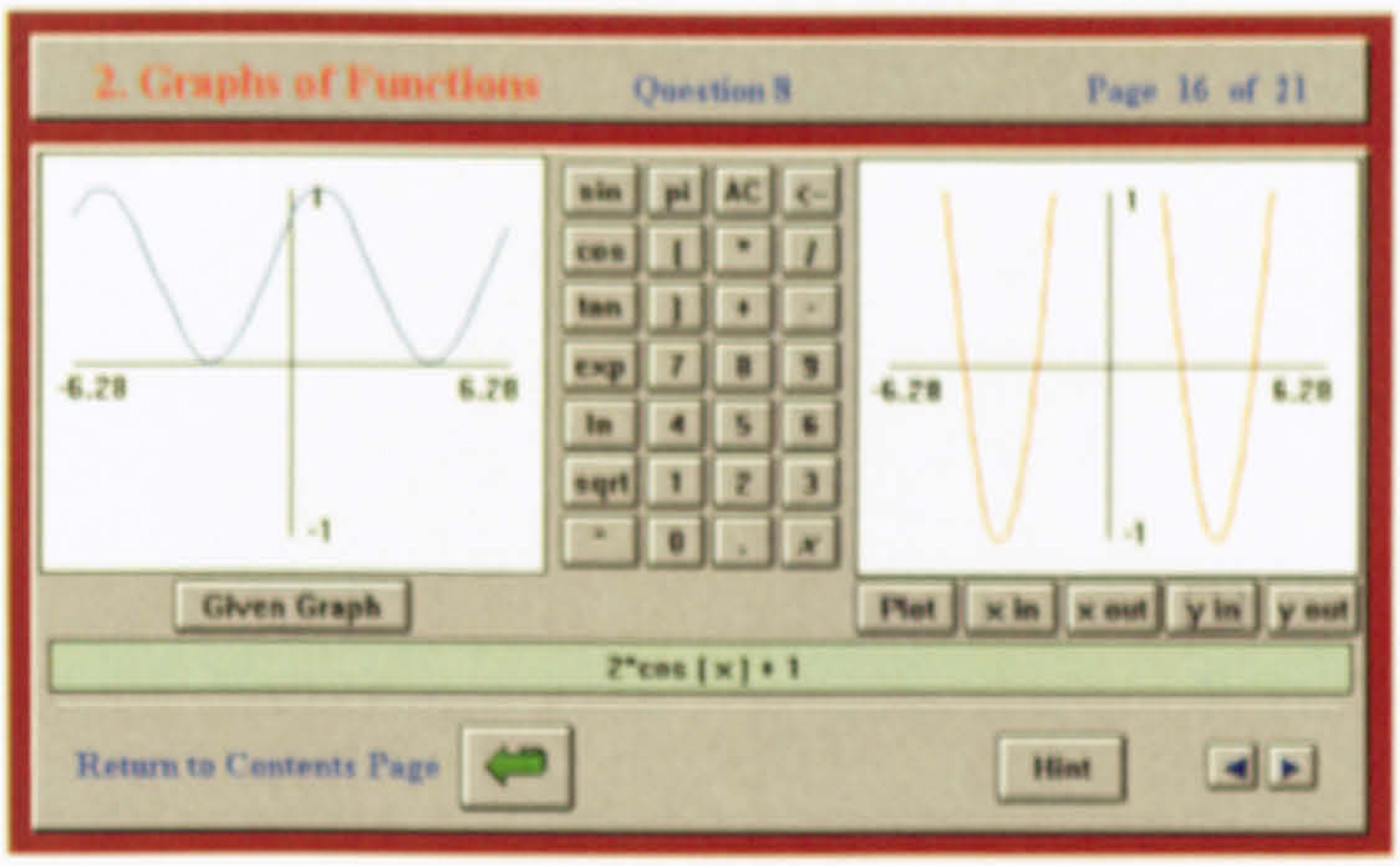


Fig. 5.13.

The student looks confused with the result.

Teacher: So what's happened? What effect has that '2' had on the graph?

The student adjusts the scale of the *y*-axis a couple of times to take in fully what has happened.

Student: It's stretched it even more.

Teacher: Great. So what should you try now, do you think?

Student: $\frac{1}{2}\cos(x)+1$.

Teacher: Good idea. See what happens.

The graph of $\frac{1}{2}\cos(x)+1$ is produced, as in Fig 5.14.

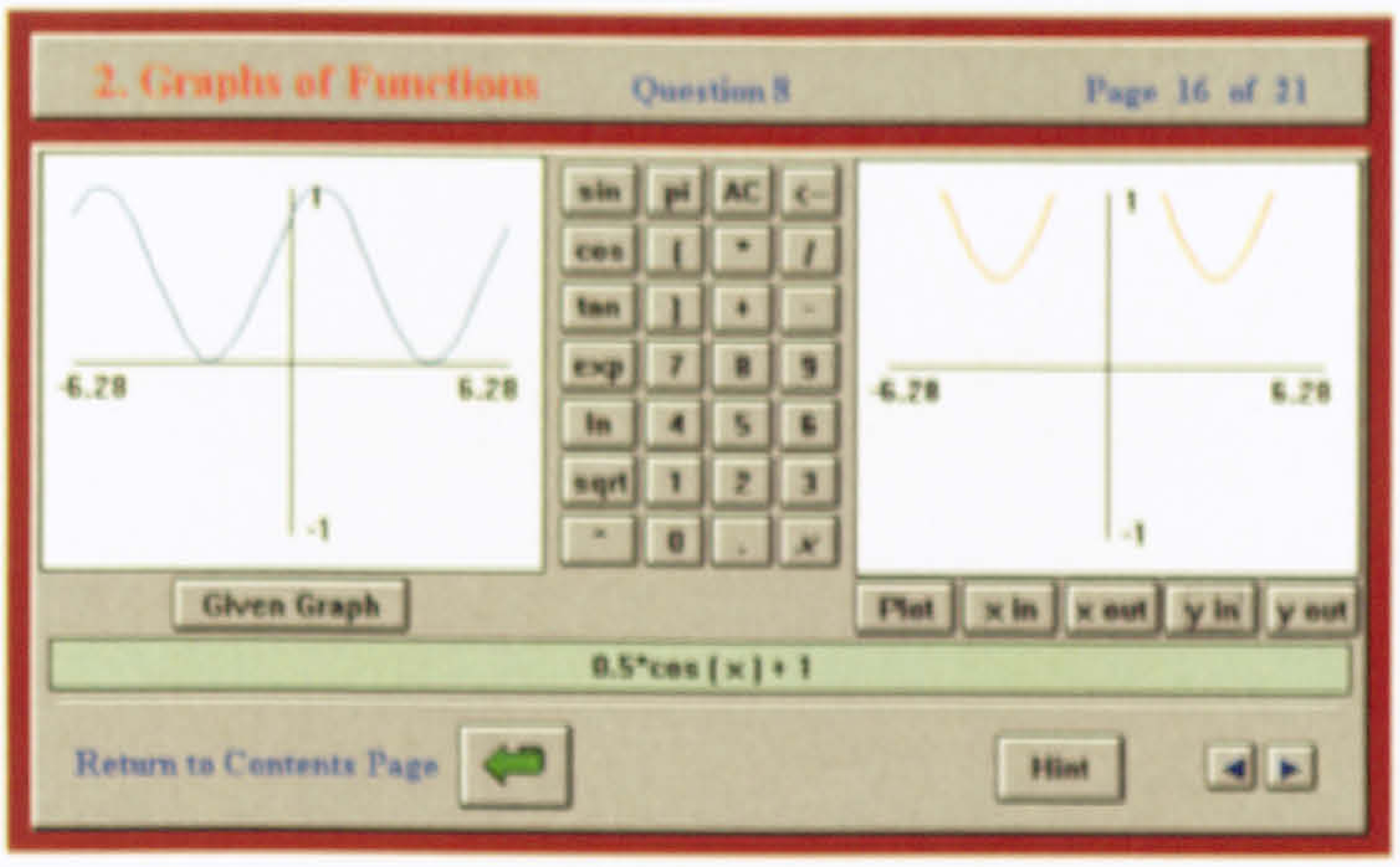


Fig. 5.14.

The student experiments again a couple of times with the scale of the *y*-axis.

Student: That's better … I think.

Teacher: How is it better?

Student: It's now got the right shape. It's just not in the right place. It needs shifting down a bit.

Teacher: Good. So you have the amplitude correct now (formalising the exploration process). I'm sure you know how to move the graph down.

['Generalisation' of acquired knowledge]

The student tries $\frac{1}{2}\cos(x)+\frac{1}{2}$, as in Fig. 5.15.

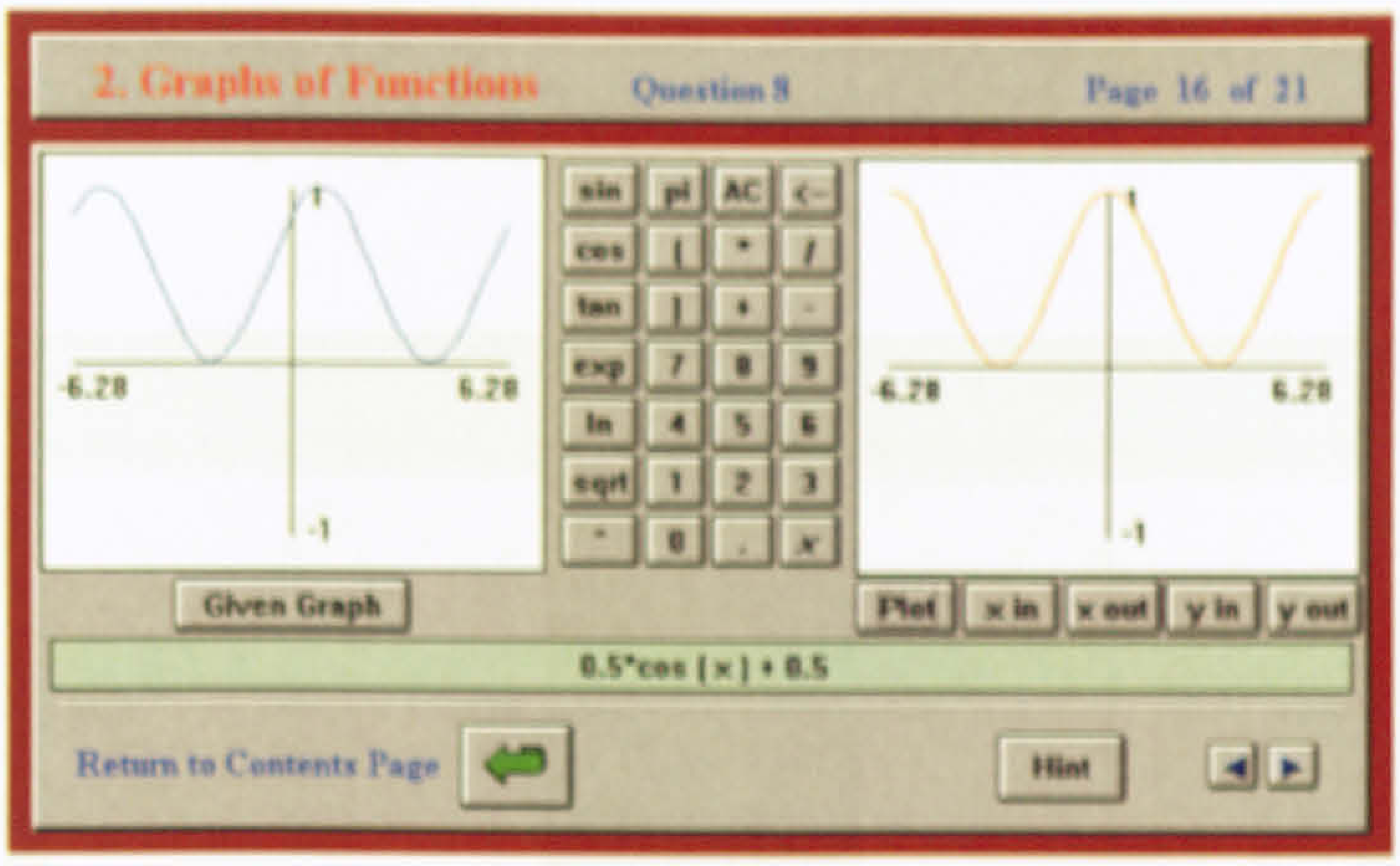


Fig. 5.15.

Student: Great, I'm getting there now.

Teacher: You've now got the mean position of the wave correct.

Student: But doesn't it need to go over to the right a bit? I assume that's what the *b* does (looking at the 'Hint' again).

[Treating graph as a 'Conceptual Entity']

Teacher: Good. Well you know that, don't you, from previous examples?

['Generalisation' of acquired knowledge]

Student: Oh yes, of course.

Teacher: That's known as phase difference (again formalising).

Student: I'll try $\frac{1}{2}$ for *b* to move it to the right a bit.

Teacher: OK, try it (the teacher wants the student to get it wrong, so that resolution will lead to a firmer understanding).

['Encapsulation' of constructive processes]

The graph of $\frac{1}{2}\cos(x+\frac{\pi}{2})+\frac{1}{2}$ is produced, as in Fig. 5.16.

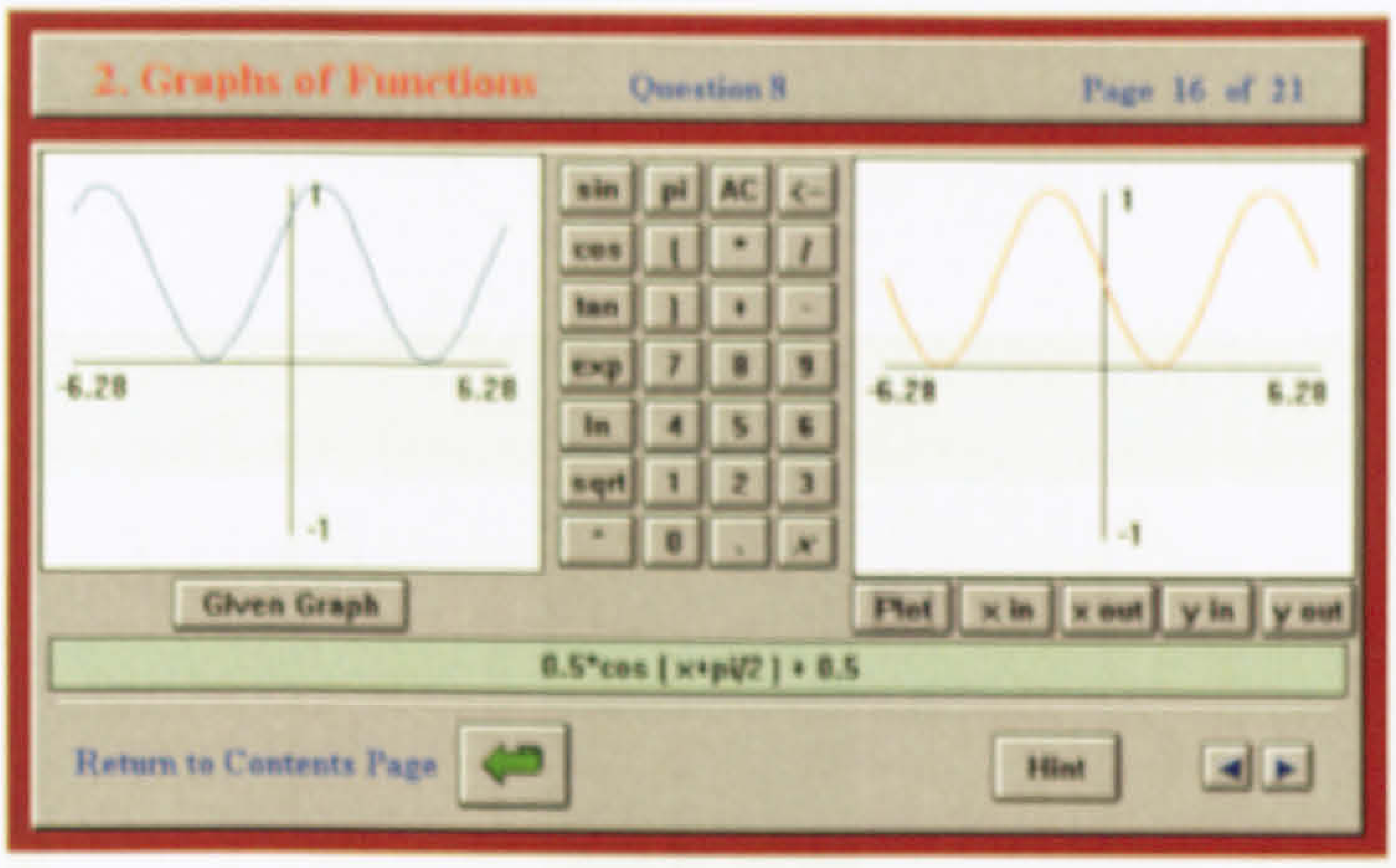


Fig. 5.16.

Student: That's obviously not right (slight pause). Oh, it should be $-\frac{1}{2}$. I should have known that.

Teacher: Good.

The student changes the sign and plots the graph of $\frac{1}{2}\cos(x-\frac{\pi}{2})+\frac{1}{2}$, as in Fig. 5.17.

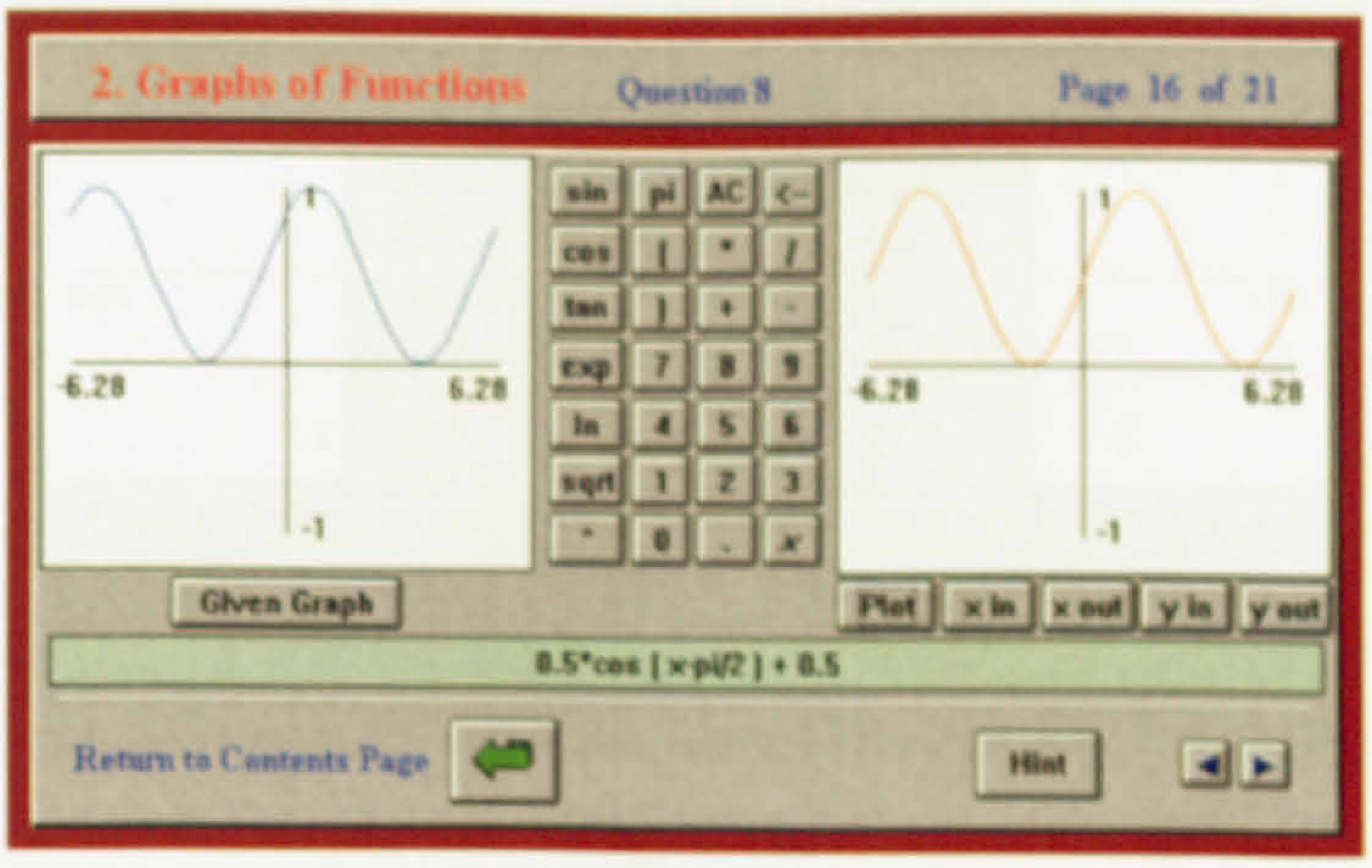


Fig. 5.17.

Student: That's too much. It must be $-\frac{1}{4}$.

Finally, the graph of $\frac{1}{2}\cos(x-\frac{\pi}{4})+\frac{1}{2}$ is produced, as in Fig. 5.18.

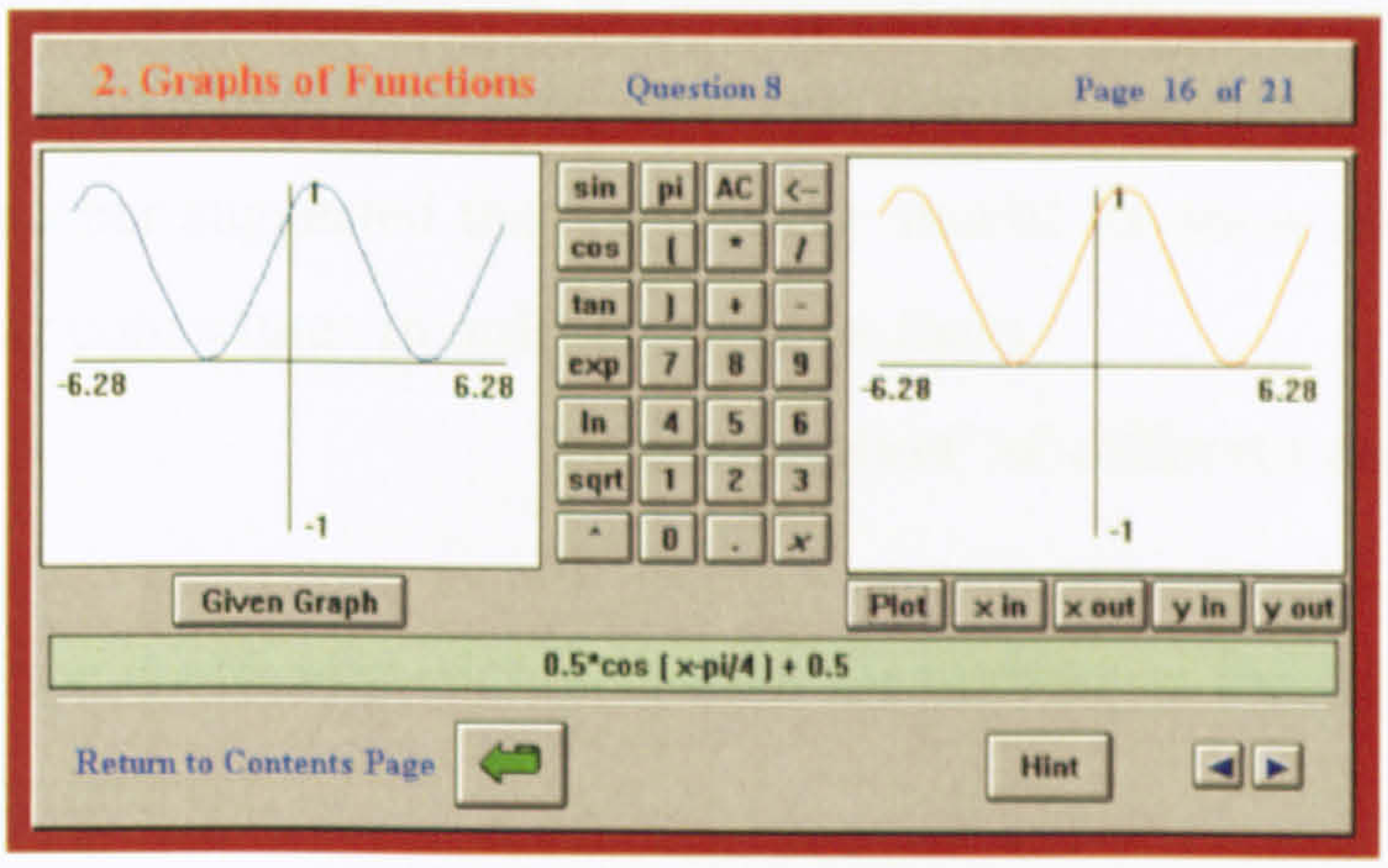


Fig. 5.18.

Student: Eureka!!

Teacher: Great, well done.

A window appears congratulating the student on her efforts, as in Fig. 5.19.

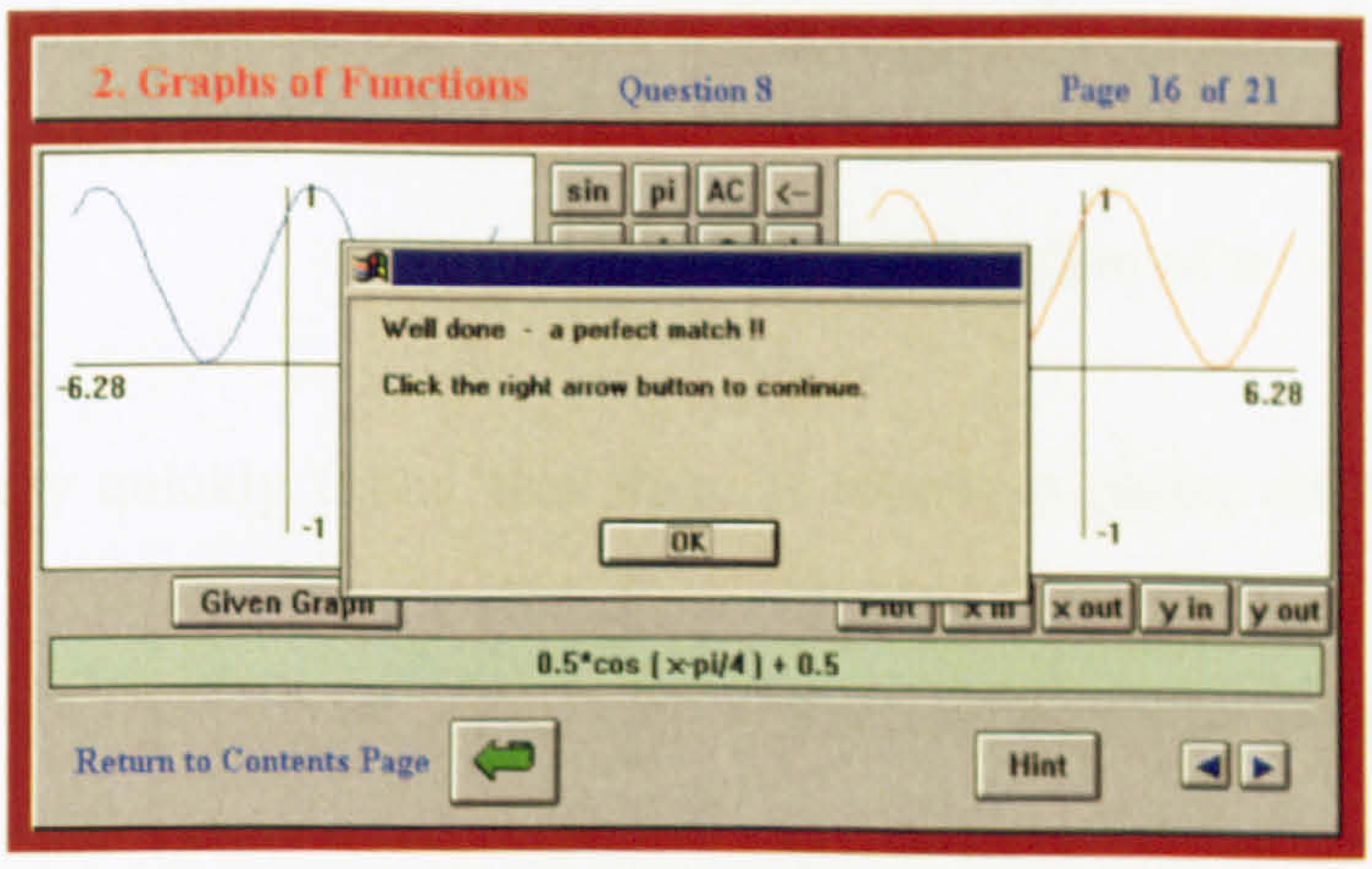


Fig. 5.19.

The teacher then led a short conversation to confirm all the stages of the exploration in terms of amplitude, phase difference, and mean position.

[Formalisation phase of 'BBWB']

Further probing also took place regarding how the student thought she would handle frequency, i.e. the number of repetitions in relation to the size of the interval. The student commented that she would need to multiply the x by a certain number. The teacher suggested that the student should try some plots to confirm, or otherwise, her conjectures in order to evaluate them.

['Coordination' of different representations]

She found that the coefficient of x represents the number of repetitions in relation to the size of the interval.

['Interiorisation' of perceived phenomena]

The teacher again formalised the process by explaining that this is what is known as frequency, and mentioned that the phase difference must now be divided by the frequency.

[Formalisation phase of 'BBWB']

The teacher then challenged the student to match the given graph again, but this time starting with a sine curve, in order to appreciate the relationship between the sine and cosine functions.

[Creation of a 'Concept Image']

The student very quickly found that there is merely a phase difference between them.

['Vertical Growth' of understanding]

The student now felt a real sense of satisfaction and achievement. The student felt that she had owned the problem, resolved it, and was now confident to tackle the next one. The student clicked on to the next page, and was greeted with the 'friendly' screen in Fig. 5.20 before continuing.

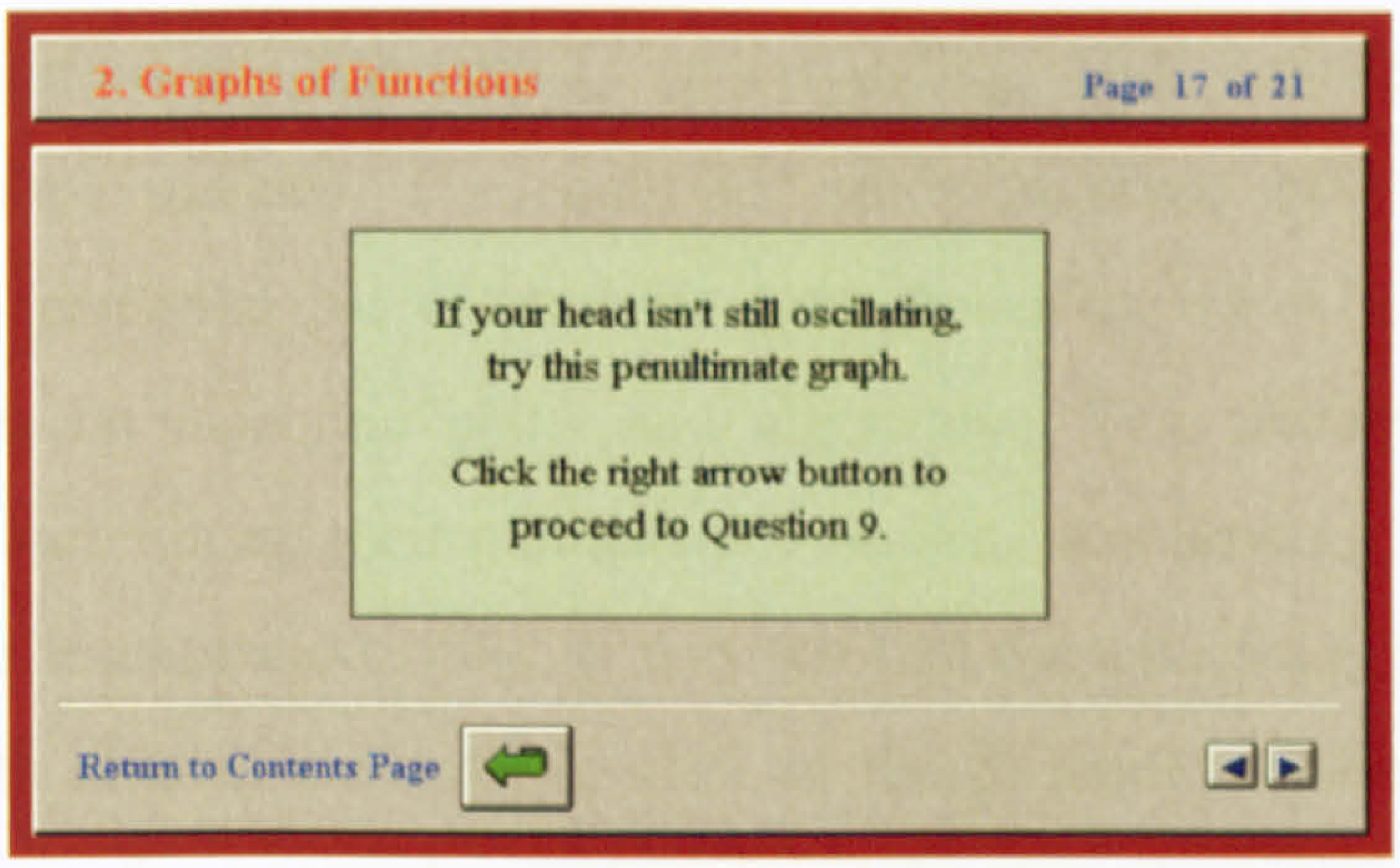


Fig. 5.20.

5.3.4. Evaluation Process

Educational software can only be evaluated in terms of the development of understanding or skills, or preferably both. The activity in which knowledge is developed is not separable from cognition; it is an integral part of what is learnt (Squires and McDougall, 1996). The learning environment helps to develop knowledge through activity. Knowledge is constructed in the context of activity by interacting with the environment, and it is this knowledge that is evaluated. All the components of the learning environment interact and contribute to the learning process; the evaluation takes into account the interactions between the students, the teacher, and the software.

This section looks at the logistics of the experiment, which considers the experimental design and the questions used to assess enhancement of student understanding and skills.

5.3.4.1. Experimental Design

After the software had been designed and developed, experimental trials to attempt to quantify improved conceptual understanding and enhanced skills via visualisation were planned. The controlled study involved 245 16-19 year old students. This particular set of students was chosen as the A-level curriculum mainly focuses on algebraic skills and algorithms. The students did not use computer software during their mathematics studies, thus limiting the opportunity to demonstrate visualisation skills. It was felt that the case-study would be more significant with A-level students, in view of the difficult transition, in terms of skills expectancy, from school to university. A-level mathematics classes tend to conform to an instructivist pedagogy, whereas many university educators are trying to incorporate more constructivist activities into their mathematics classes. The chosen age range was therefore appropriate for testing students' skills at the school/university interface.

A written test was prepared for evaluation purposes (see Section 5.3.4.2), which assessed a student's ability to switch between algebraic and pictorial representations of functions, as well as testing procedural skills. The test was given to upper-sixth (year 13) mathematics students who had done a considerable amount of work with functions during the first year of their A-level course (they did not use the interactive software). Lower-sixth (year 12) mathematics students, who had done very little work with functions, used the constructivist visual software. The upper-sixth students therefore acted as the control group (**136 students**) who had been taught 'functions and graphs' by traditional instructivist methods, without any software use, and the lower-sixth students acted as the experimental group (**109 students**) who had learnt 'functions and graphs' via the interactive software (issues concerning the experimental design are discussed further in Section 7.2.2 of Chapter 7). After having used the software, the experimental group was given a summary sheet covering the key issues dealt with in the session (see Appendix B) in order to formalise their explorations. They also sat the same written test. The usefulness of the software could thus be quantified in terms of mathematical skills (those of the MATH taxonomy), compared with traditional teaching techniques.

Students from three schools (referred to as Schools A, B, and C) took part in the case-study, in order to compare different sets of students in terms of the different combinations of A-level subjects studied. Most of the School A and School C mathematics students follow a very traditional course of study, i.e. three science subjects, whereas School B students study a wide variety of subjects. Male and female students are compared to see whether either of the teaching approaches favours one particular gender. Students studying 'Mathematics with Mechanics' and students studying 'Mathematics with Statistics' are compared to observe the effect of certain disciplines. Students achieving an A* or A grade compared to a B or C grade at GCSE are considered to compare students in terms of prior performance.

5.3.4.2. Test Questions

Each group was tested using a traditional pen-and-paper multiple choice test, but with pictorial information included in both the questions and the responses (the test questions, together with a description of the skills that they assess, based on the MATH taxonomy, are provided later in this section). The test consisted of 10 questions of a procedural type, i.e. questions that cued the students to respond with an answer that involved the systematic application of basic knowledge or procedures, and 10 questions of a visual type that sought a more interpretive and constructive understanding, requiring the students to successfully switch between algebraic and pictorial representations of functions (certain questions were taken from Galbraith and Haines (1995), as they assessed particular skills of interest here). The order of information transfer is significant - some questions tested a symbolic to visual transition, some visual to symbolic, and some tested the ability to transfer information in both directions. The visual questions were more demanding in terms of higher order skills. Any differences in ability in answering the two types of questions would thus be detected. Also, some insight would be gained into the effect of the interactive software on student learning. Of primary interest was any enhancement of visualisation ability after having used the dedicated software.

A factor of interest is the importance of 'another point of view' in confirming the outcome of any mathematical exercise and thus developing the students' mathematical expertise. In practice, visualisation may fulfil this role for many procedural exercises. The set of procedural questions, therefore, contained no duals of the visual questions, and thus the two sets of questions were independent.

The standard of individual questions within each set (procedural or visual) was not necessarily the same, but it was intended that both sets of questions were of equal difficulty overall.

The test questions are provided below, together with a description of the skills that each question assesses (see Section 2.3.1.3 of Chapter 2 for the skills categories).

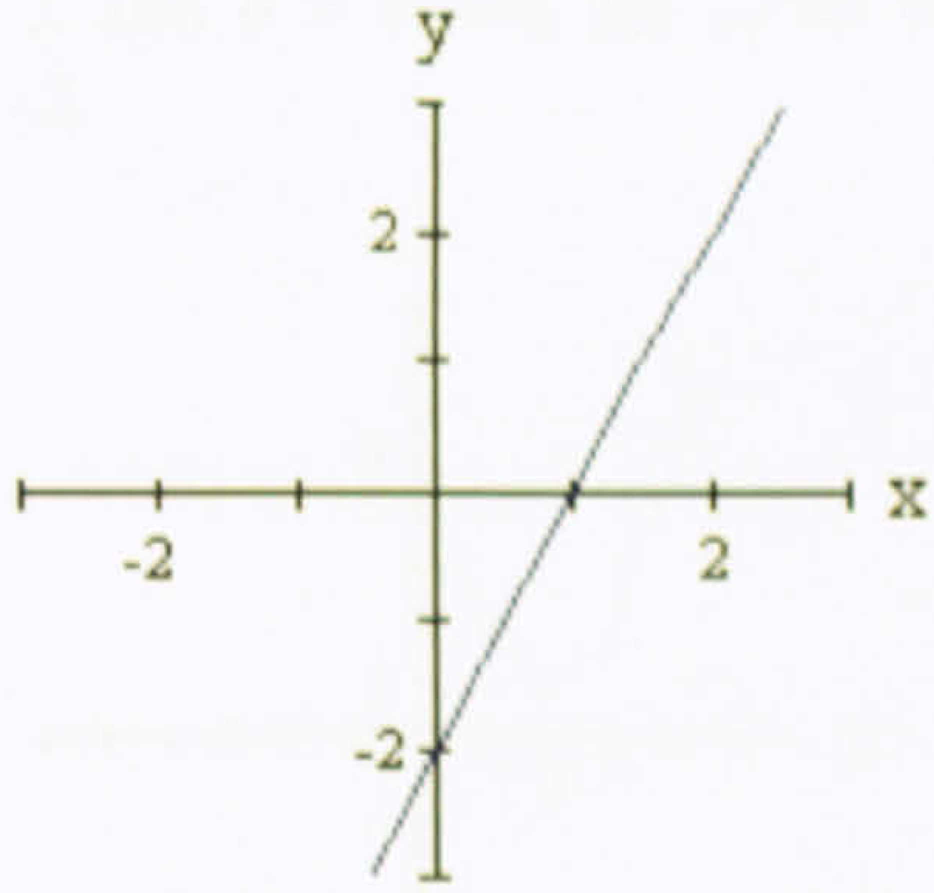
1. $x^2 - ax + 12 = 0$ represents a family of equations. Four members of the family are obtained by giving a the values 5, 6, 7 and 8. For what values of a can the equation be solved by factorising the left-hand side?

A none	**B** 5 only	**C** 6 and 7
D 8 only	**E** 7 and 8	

Skills assessed:

Comprehension of the generic form of the equation that has four possible specific forms (Group A), and **routine use of procedures** by factorising each case (Group A).

2.



The straight line graph above is represented algebraically by the equation

A $y = 2 - 2x$ **B** $y = 2x - 2$ **C** $y = 2x + 2$
D $y = -2x - 2$ **E** none of these

Skills assessed:

Information transfer from the pictorial to the symbolic (Group B), and **interpreting** the graph in terms of slope and intercept (Group C).

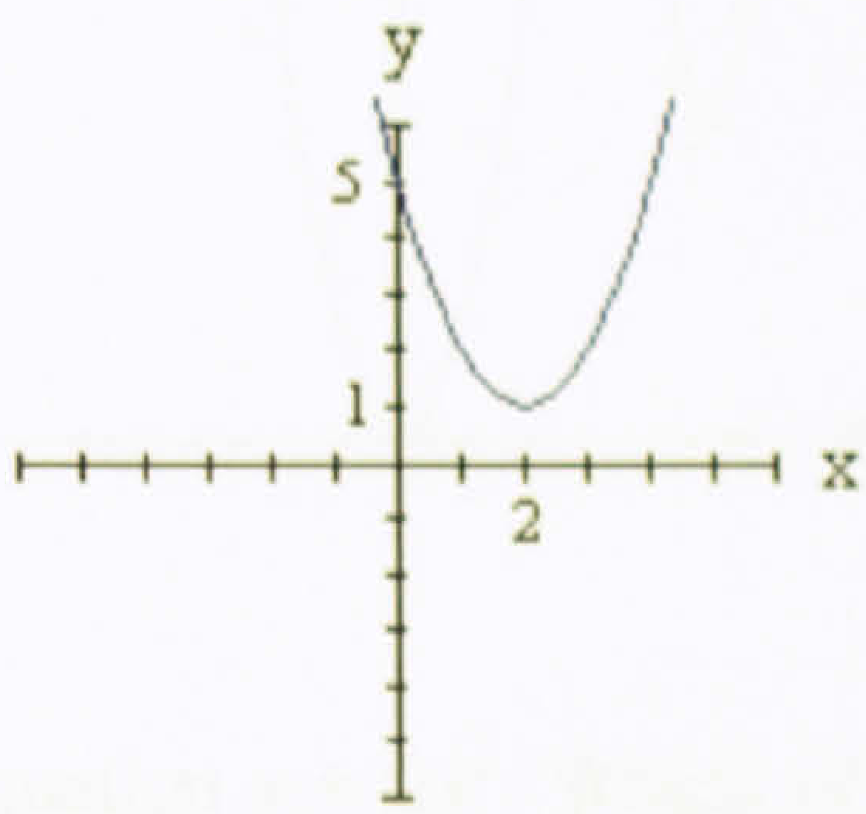
3. Given the equation $(x - 2)^2(x + 2)^2 = 0$, which of the following ways of rewriting this equation is incorrect?

A $(2 - x)^2(2 + x)^2 = 0$ **B** $(x^2 - 4x + 4)(x^2 + 4x + 4) = 0$
C $(x^2 - 4)^2 = 0$ **D** $x^4 - 8x^2 + 16 = 0$
E all of them are correct

Skills assessed:

Routine use of procedures to expand each expression (Group A).

4. Consider the family of graphs with equation $y = (x - a)^2 + b$. The graph in the diagram has $a = 2$ and $b = 1$. On the same diagram draw the graph for which $a = 1$ and $b = -2$.



Skills assessed:

Information transfer from the symbolic to the pictorial (Group B), **interpreting** the effect of the parameters in the expression (Group C), making **comparisons** of the two graphs in terms of their algebraic structure (Group C), and understanding the **implications** of altering the symbolic expression in terms of the effect on the graph (Group C).

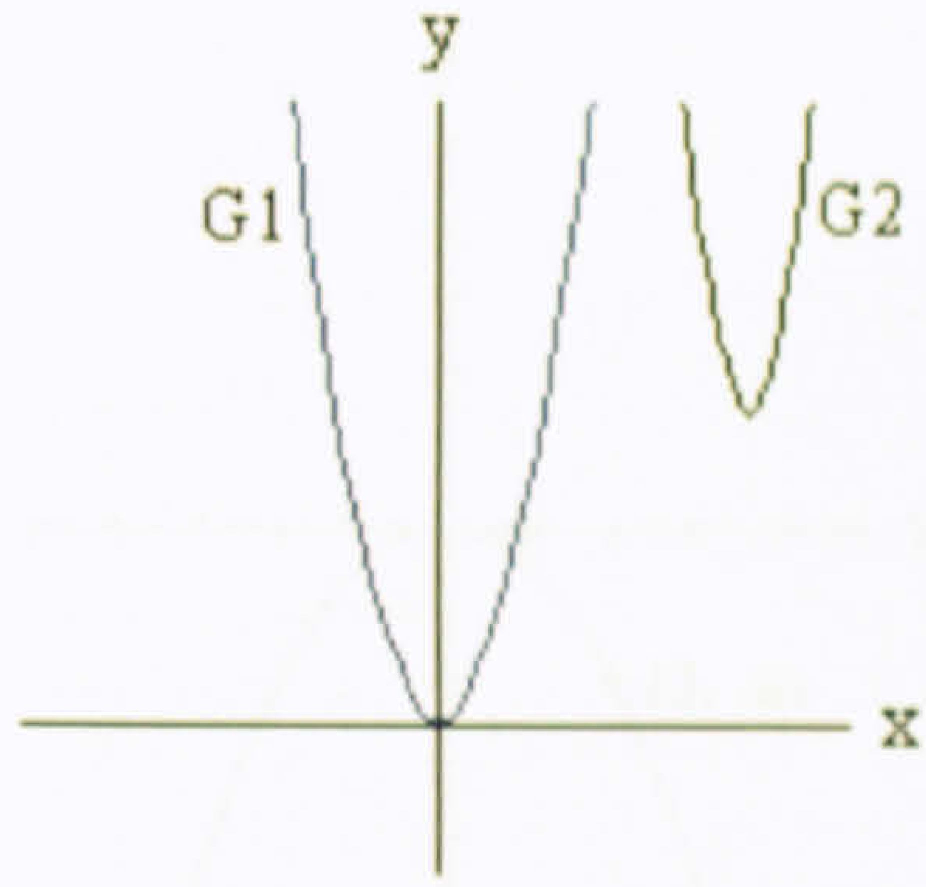
5. The two points (-1, 3) and (5, 12) lie on the graph whose equation is $y = mx + c$. The values of m and c are

A $m = 3, c = 9$	**B** $m = 9, c = 3$
C $m = 3/2, c = 9/2$	**D** $m = 9/2, c = 3/2$
E none of the above are correct	

Skills assessed:

Factual knowledge of the formula for calculating the gradient (Group A), and **routine use of procedures** to find the gradient and use substitution to find the intercept (Group A). Alternatively, **routine use of procedures** to plot the points, draw the line, and measure the gradient and read off the intercept (Group A).

6.



Graph G1 has the equation $y = 2x^2$. Which of the equations below might describe graph G2?

A $y = (x - 3)^2 + 2$ **B** $y = 2(x - 3)^2 + 2$
C $y = 4(x - 3)^2 + 2$ **D** $y = (x - 3)^2/4 + 2$
E none of these

Skills assessed:

Information transfer from the symbolic to the pictorial and vice versa (Group B), **interpreting** the graphs in terms of the parameters in the expressions (Group C), making **comparisons** of the two graphs in terms of their algebraic structure (Group C), and understanding the **implications** of altering one functional form in terms of the effect on the other (Group C).

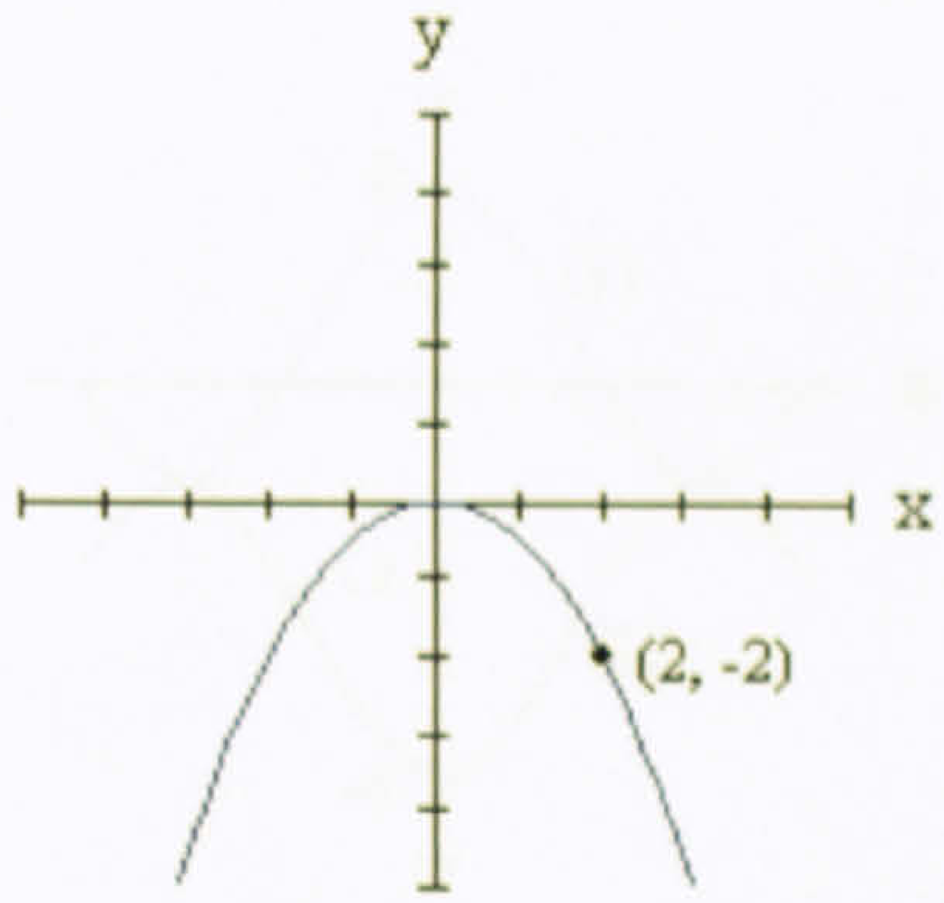
7. The graph with equation $y = 2x^2 - bx^3$ cuts the x-axis at $x = 4$. The value of b is

A 4 **B** 2 **C** 0 **D** 1/2
E none of these

Skills assessed:

Comprehension of the scenario (Group A), and **routine use of procedures** by using substitution (Group A).

8.



The graph in the diagram has equation $y = -ax^2$. If the graph is translated 3 units to the left and 1 unit upwards, sketch the resulting graph on the same axes giving its equation and the value of any constants used.

Skills assessed:

Information transfer from the pictorial to the symbolic and vice versa (Group B), **interpreting** the graph in terms of the parameters in the expression (Group C), making **comparisons** of the two graphs in terms of their algebraic structure (Group C), and understanding the **implications** of altering one functional form in terms of the effect on the other (Group C).

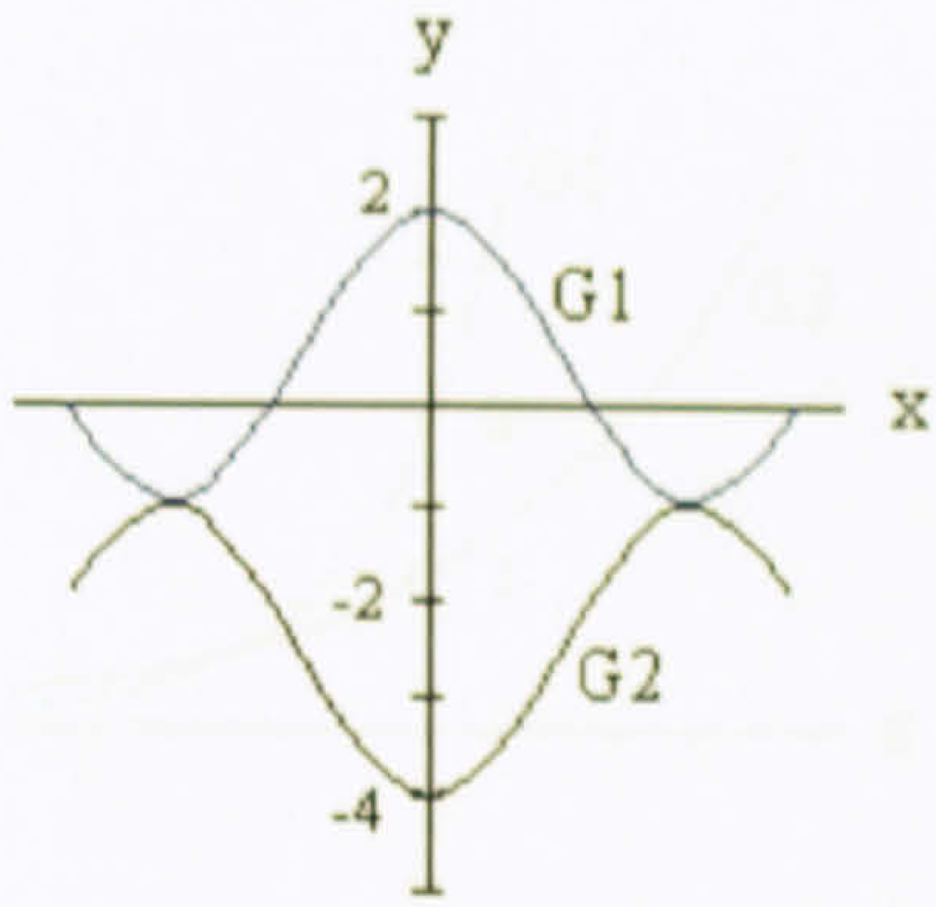
9. Write the expression $2ab - 6ad + 2b^2 - 6bd$ as a product of factors.

Skills assessed:

Conjectures of possible factors (Group C), **routine use of procedures** to carry out the factorisation (Group A), and **justifying** by multiplying out the brackets (Group C).

[Although Question 9 does not relate to functions specifically, it has been included as a non-visual question that assesses higher order skills.]

10.



Graph G1 has the equation $y = f(x)$, and graph G2 has the equation

A $y = -f(x) - 2$ **B** $y = -f(x)$ **C** $y = f(x) - 6$
D $y = f(-x)$ **E** none of these

Skills assessed:

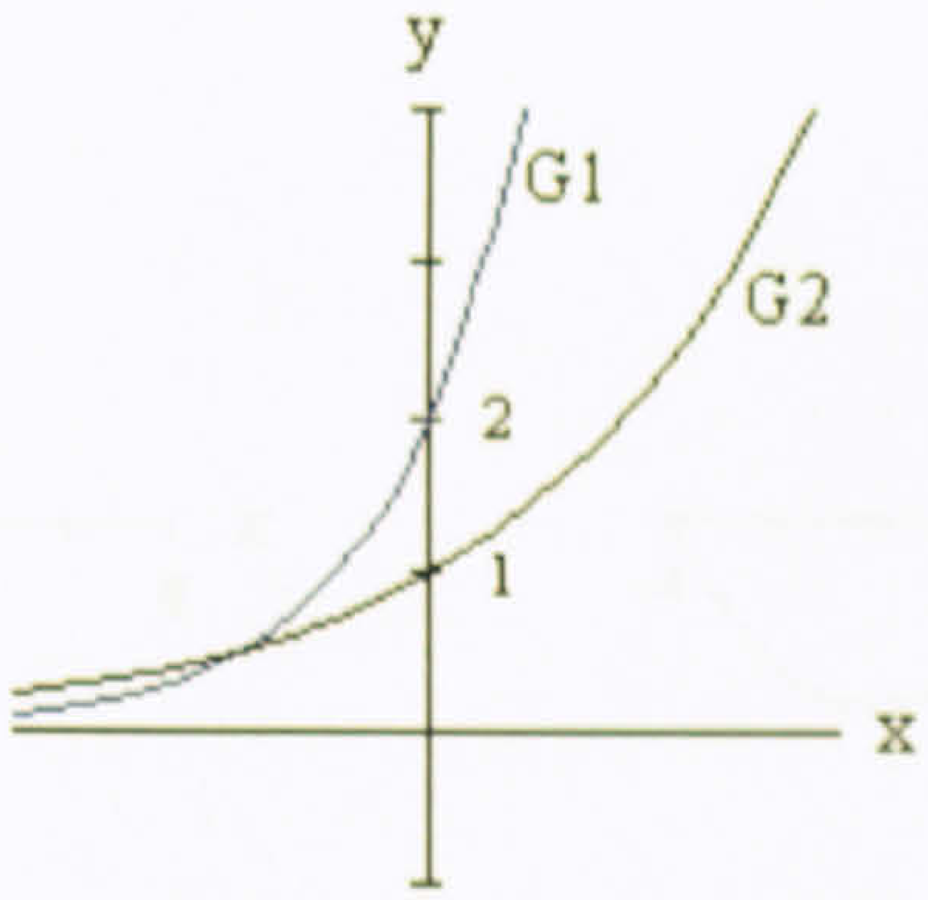
Information transfer from the pictorial to a generic symbolic form and vice versa (Group B), **interpreting** the graphs in terms of the parameters in the expressions (Group C), making **comparisons** of the two graphs in terms of their generic algebraic structure (Group C), and understanding the **implications** of altering one functional form in terms of the effect on the other (Group C).

11. Divide $x^3 + 4x^2 - 17x + 28$ by $x^2 - 3x + 4$, and thus provide one of the roots of the equation $x^3 + 4x^2 - 17x + 28 = 0$.

Skills assessed:

Routine use of procedures to carry out the division (Group A), and **comprehension** of what the result of the division means in relation to the roots of the equation (Group A).

12.



The diagram shows two graphs, G1 and G2, of the form ae^{bx}. For G1, $a = 2$ and $b = 1$. The values of a and b for G2 might be

A $a = 1, b = 2$ **B** $a = 1, b = 0.5$ **C** $a = 2, b = 0.5$
D $a = 0.5, b = 1$ **E** $a = 0.5, b = 2$

Skills assessed:

Information transfer from the symbolic to the pictorial and vice versa (Group B), **interpreting** the graphs in terms of the parameters in the expressions (Group C), making **comparisons** of the two graphs in terms of their algebraic structure (Group C), and understanding the **implications** of altering one functional form in terms of the effect on the other (Group C).

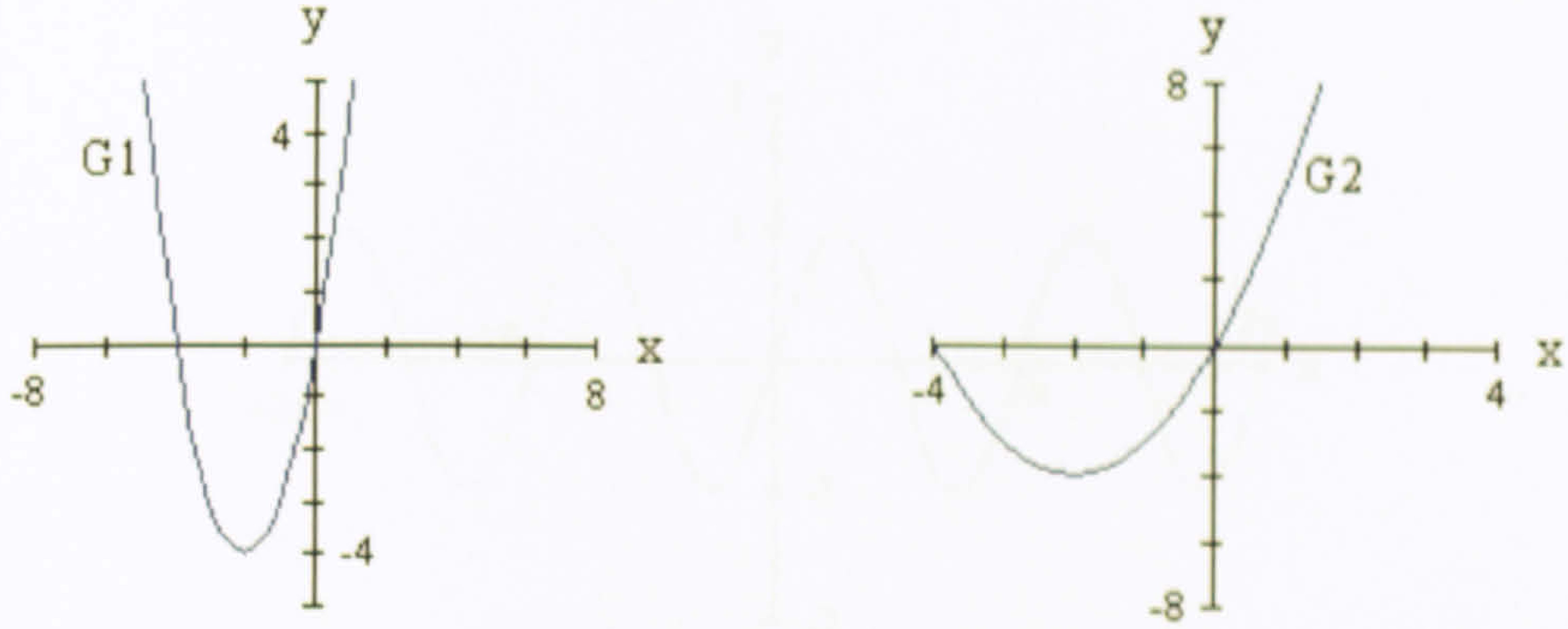
13. The straight line with equation $y = c - 2x$ cuts the x-axis at $x = -4$. The value of c is

A 2 **B** -4 **C** 8 **D** -8 **E** none of these

Skills assessed:

Comprehension of the scenario (Group A), and **routine use of procedures** by using substitution (Group A).

14.



If the graph G1 has the equation $y = f(x)$, then the graph G2 has the equation

A $y = 2f(x)$ **B** $y = 1/2 f(x)$ **C** $y = f(x)$
D $y = f(2x)$ **E** $y = f(x/2)$

Skills assessed:

Information transfer from the pictorial to a generic symbolic form and vice versa (Group B), **interpreting** the graphs in terms of the parameters in the expressions (Group C), making **comparisons** of the two graphs in terms of their generic algebraic structure (Group C), and **justifying** that both are indeed the same function, just on a different scale (Group C).

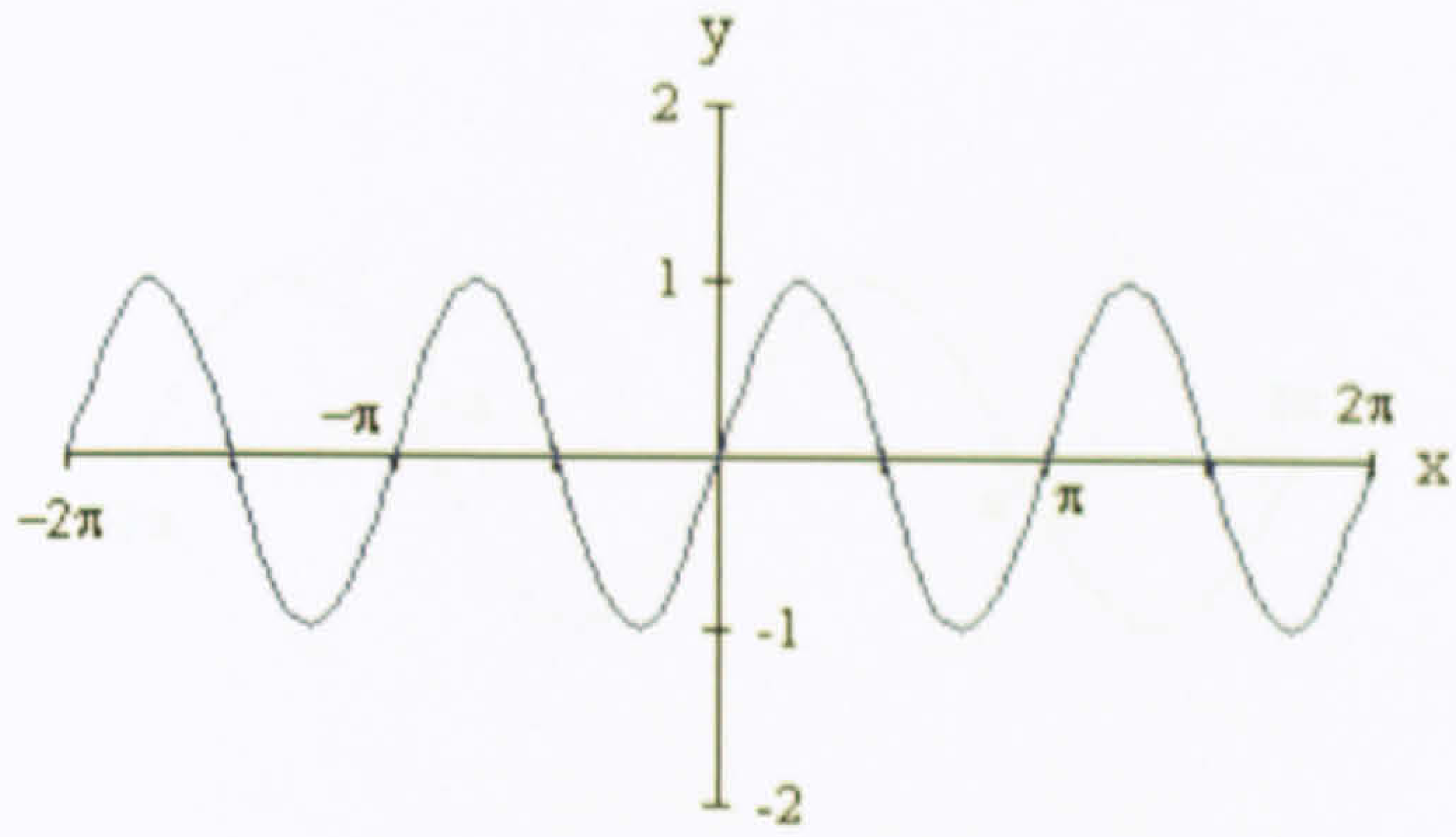
15. Which of the following is the correct way of rewriting the polynomial $x^4 + x^3 - 5x^2 + 13x - 6$:

A $(x^2 - 2x + 3)(x^2 + 3x - 2)$ **B** $(x^2 - 2x - 3)(x^2 + 3x + 2)$
C $(x^2 + 2x + 3)(x^2 - 3x - 2)$ **D** $(x^2 + 2x - 3)(x^2 - 3x + 2)$
E all of them are correct

Skills assessed:

Routine use of procedures to expand each expression (Group A).

16.



The diagram above shows the graph of $a\sin(bx)$ where $a = 1$ and $b = 2$. Draw on the same axes the graph of $a\sin(bx)$ where $a = 2$ and $b = 1$.

Skills assessed:

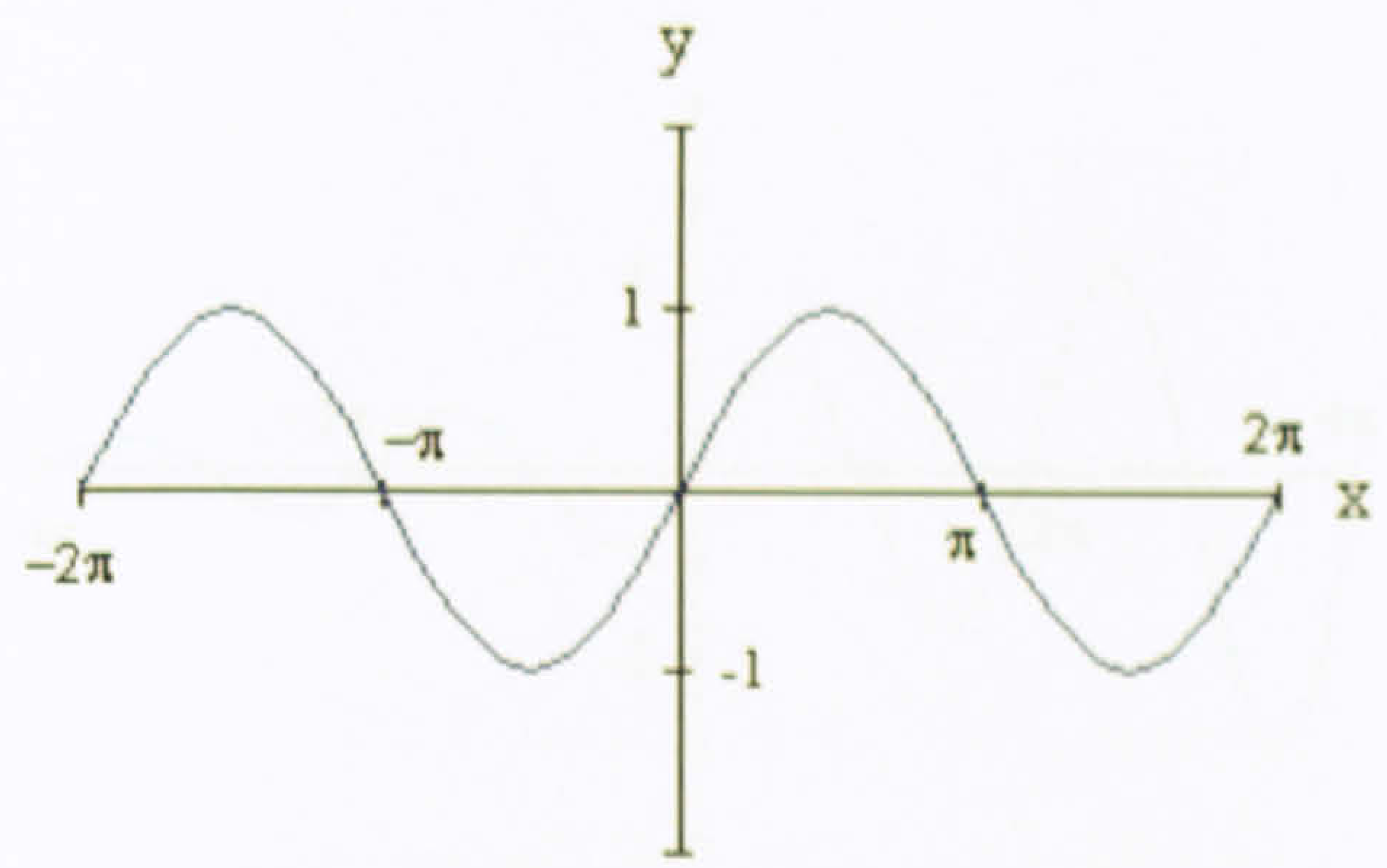
Information transfer from the symbolic to the pictorial (Group B), **interpreting** the effect of the parameters in the expression (Group C), making **comparisons** of the two graphs in terms of their algebraic structure (Group C), and understanding the **implications** of altering the symbolic expression in terms of the effect on the graph (Group C).

17. Give a quadratic equation, containing no fractions, which has the roots $x = 2/3$ and $x = 3/4$.

Skills assessed:

Factual knowledge of the relationship between the roots of a quadratic and its factorised form (Group A), and **routine use of procedures** to carry out the expansion (Group A).

18.



Which function does the above graph represent?

A $y = \sin(x)$ **B** $y = \cos(x - \pi/2)$ **C** $y = \sin(x + 10\pi)$
D all of these are correct **E** none of these are correct

Skills assessed:

Information transfer from the symbolic to the pictorial (Group B), **interpreting** the effect of the parameters in the expressions (Group C), making **comparisons** of the three expressions in terms of their graphical form (Group C), understanding the **implications** of altering the symbolic expressions in terms of the effect on the graph (Group C), and justifying that they are indeed all represented by the same graph (Group C). Alternatively, merely **factual knowledge** concerning phase difference (Group A).

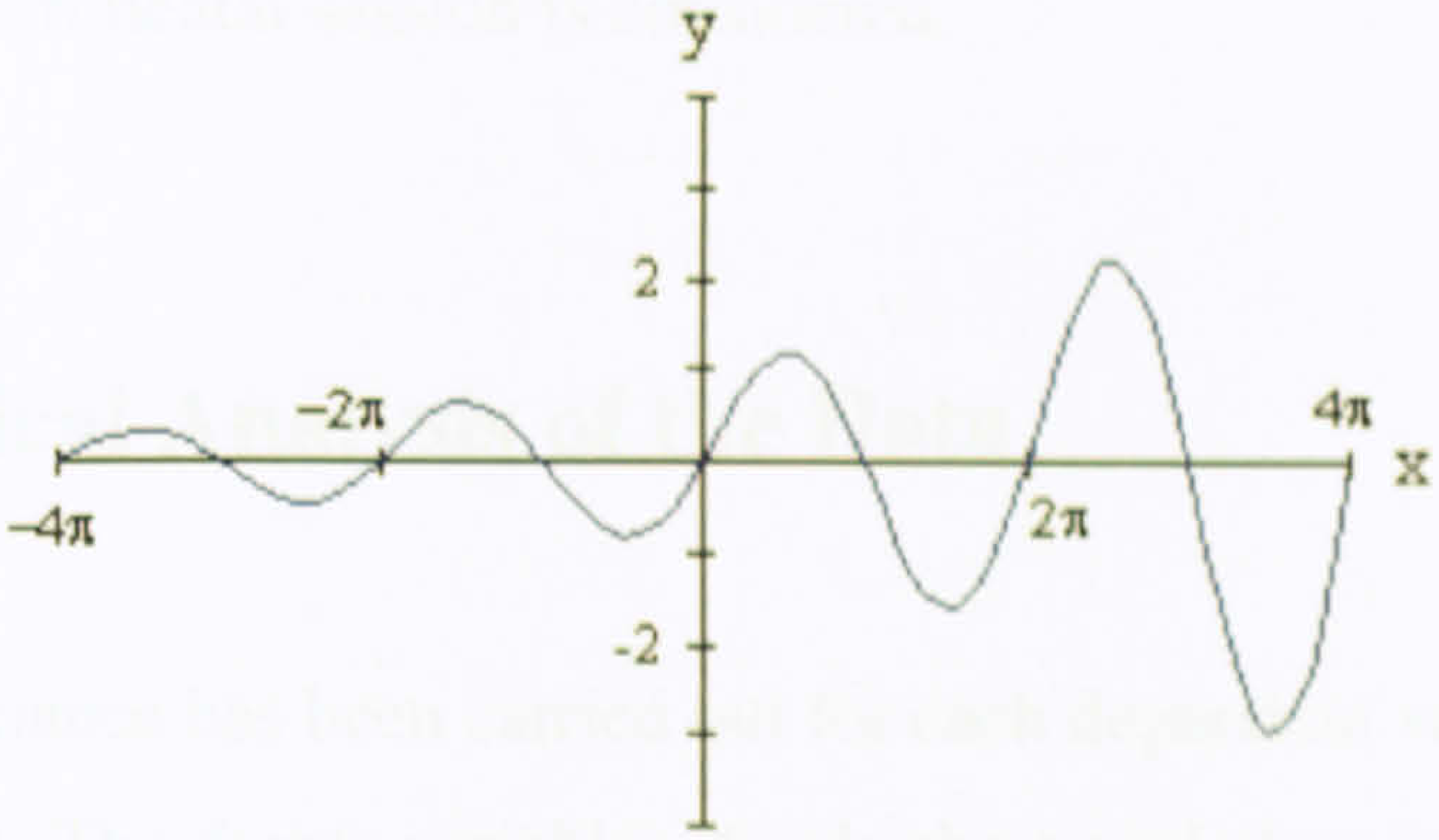
19. The graph of $y = x + 1$ cuts the graph of $y = (x - 2)^2 + 3$ at two points. The co-ordinates of these two points are:

A (0, 1) and (1, 2) **B** (1, 2) and (2, 3)
C (2, 3) and (3, 4) **D** (3, 4) and (4, 5) **E** none of these

Skills assessed:

Comprehension of the scenario (Group A), and **routine use of procedures** to find solutions that will simultaneously satisfy both equations (Group A).

20.



The above graph is a graphical representation of which of the following functions?

A $y = \sin(x) + e^{0.1x}$ **B** $y = \sin(x) + e^{-0.1x}$ **C** $y = \sin(x)\, e^{0.1x}$
D $y = \sin(x)\, e^{-0.1x}$ **E** all of the above

Skills assessed:

Information transfer from the symbolic to the pictorial (Group B), **interpreting** the effect of combining functions (Group C), and **evaluation** of each expression in the form of a graph (Group C). Alternatively, merely **factual knowledge** concerning combinations of sine and exponential functions (Group A).

An analysis of student performance on the above questions, in terms of success in relation to skills assessed, is discussed in the next section.

5.4. Results

This section provides a detailed statistical analysis of the data, which focuses on satisfying the key and secondary aims outlined in Section 5.2 earlier. This analysis is aided by scatter plots and boxplots, which serve to illustrate the findings. The statistical analysis was carried out using the statistical analysis software MINITAB and SPSS (see References for Internet addresses). Finally, student feedback relating to the experimental session is considered.

5.4.1. Statistical Analysis of the Data

An analysis of variance has been carried out for each dependent variable by several factors of interest. The factor variables divide the population into groups. Using this procedure, null hypotheses about the effects of other variables on the means of various groupings of a single dependent variable are tested, so that information regarding student performance, including specific comparisons of the factors of interest, can be obtained. All comparisons are tested at the 5% level of significance (unless otherwise stated), i.e. a significant difference is evident if the corresponding p-value is less than 0.05. The full experimental data can be seen in Appendix E.

Fig. 5.21 provides a table of mean scores (out of 10) for both procedural and visual questions, for each of the various factors across the control and experimental groups. The numbers in brackets are the sample sizes for each sub-group. 245 students took part in the case-study.

To aid the discussion of the analysis, scatter plots provide a graphical overview of the experimental data, giving a global picture for each factor of interest. Each plot is split into four 'compartments', as shown in Fig. 5.22. The grid lines have been drawn to help in the interpretation. An 'x' in the scatter plots denotes a single observation, whereas a '4', for example, denotes 4 observations sharing that particular score. The boxplots in the analysis provide quantitative illustrations. To assist with their interpretation, the factors of interest, together with their appropriate coding, are shown in Fig. 5.23.

		Control Group		Experimental Group	
		Procedural	**Visual**	**Procedural**	**Visual**
School	**1**	6.03 (39)	4.72 (39)	5.23 (44)	5.50 (44)
	2	4.61 (72)	4.40 (72)	3.76 (45)	4.87 (45)
	3	5.68 (25)	4.52 (25)	3.70 (20)	5.20 (20)
Sex	**Male**	5.64 (73)	4.75 (73)	4.35 (63)	4.94 (63)
	Female	4.71 (63)	4.24 (63)	4.33 (46)	5.52 (46)
Subject	**Mechanics**	5.67 (64)	4.95 (64)	4.14 (58)	5.31 (58)
	Statistics	4.81 (72)	4.13 (72)	4.57 (51)	5.04 (51)
GCSE	**A*/A**	5.67 (103)	4.86 (103)	5.08 (75)	5.61 (75)
	B/C	3.79 (33)	3.42 (33)	2.71 (34)	4.24 (34)

Fig. 5.21.

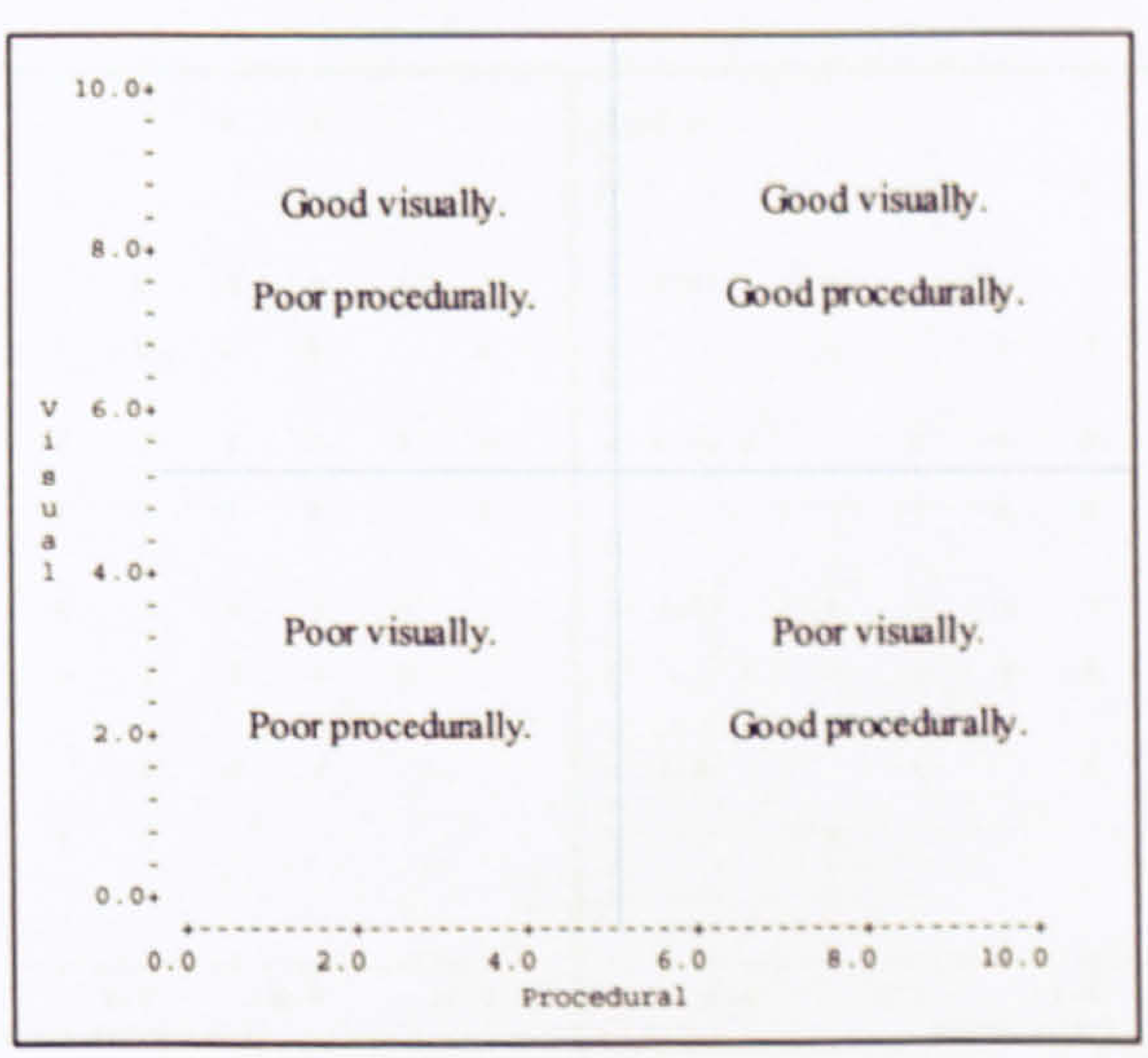


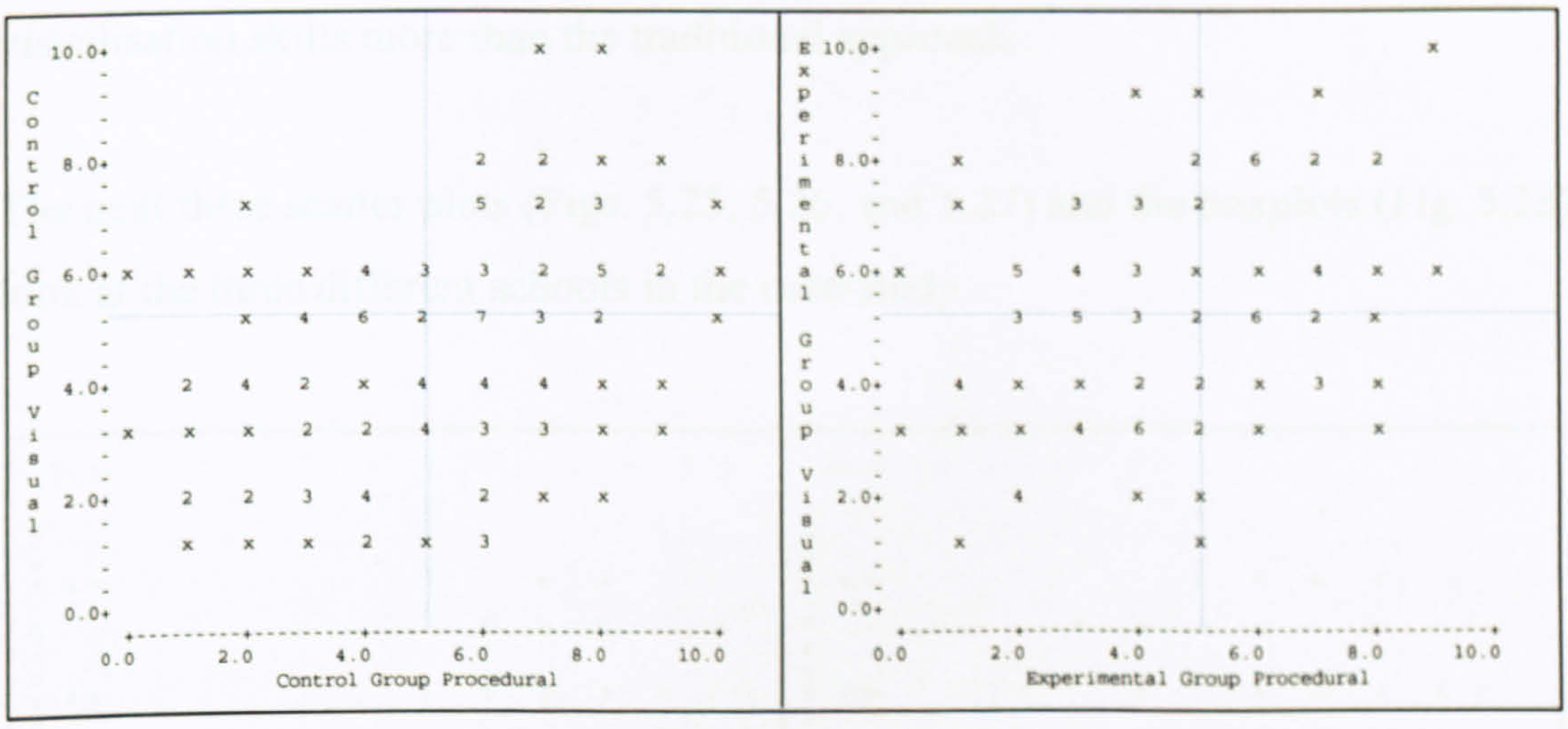
Fig. 5.22.

		Factor				
		Group	**School**	**Sex**	**Subject**	**GCSE**
Coding	1	Control	A	Male	Mechanics	A/A*
	2	Experimental	B	Female	Statistics	B/C
	3		C			

Fig. 5.23.

Firstly, the comparison of the performance of the control and experimental groups is of prime interest. For the procedural scores, there is strong evidence that the control group scored better than the experimental group ($p = 0.001$), which is to be expected as they were a year further through their A-level course. For the visual scores, there is strong evidence that the experimental group scored better than the control group ($p = 0.004$), even though they were less experienced mathematically. For the difference in scores (procedural - visual), in order to determine whether students are better at either type of question, there is strong evidence of a difference between groups ($p < 0.0001$). The control group clearly scored better on procedural questions, and the experimental group clearly scored better on visual questions.

Fig. 5.24 shows the overall pattern of scores for both groups.



(a) **(b)**

Fig. 5.24.

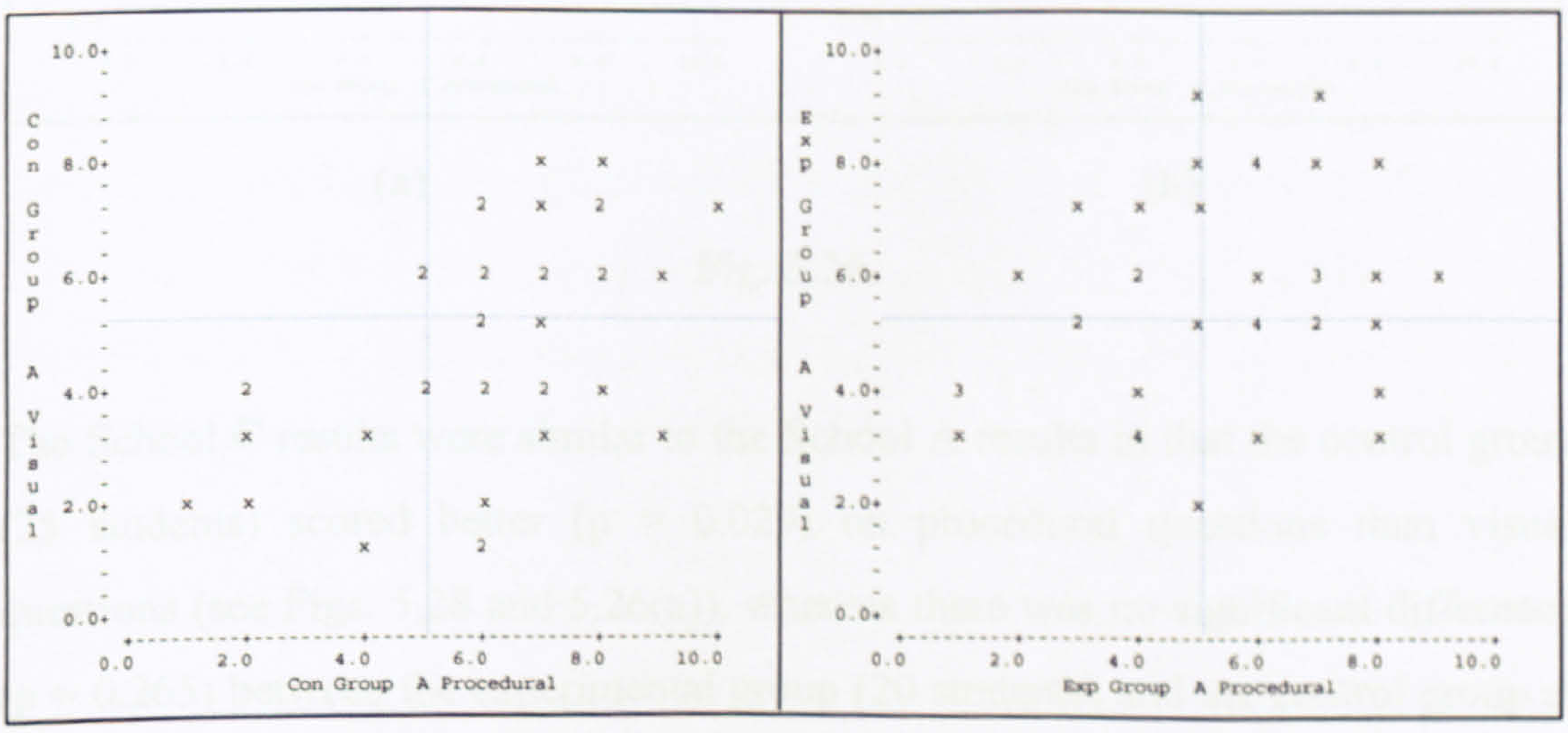
Of the 136 control group students overall, in Fig. 5.24(a), 32 students were good at both disciplines, and 32 students were poor at both. More interestingly, however, there was a considerable number, 25 students, who scored well on procedural questions and scored poorly on visual questions, but much less, only 9 students, the other way round (the remaining 38 students were borderline, i.e. on the gridlines).

There is strong evidence ($p < 0.0001$) that the traditional approach leads to better procedural skills as opposed to visual skills.

When considering the experimental group students overall, in Fig. 5.24(b), the scatter plot is somewhat different. Now there are many more, 22 students, compared to 9 above, who scored well on visual questions and scored poorly on procedural questions, with only 7 students, compared to 25 above, the other way round. This pattern of scores was to be expected in the experimental group, as they have received visual training but are one year behind the control group in terms of practising procedural skills.

There is a shift in scores across the two groups. The better visual scores in the experimental group, compared with the control group, therefore provide evidence ($p = 0.004$) that the learning style encouraged by the software has indeed enhanced visualisation skills more than the traditional approach.

The next three scatter plots (Figs. 5.25, 5.26, and 5.27) and the boxplots (Fig. 5.28) look at the three different schools in the case-study.

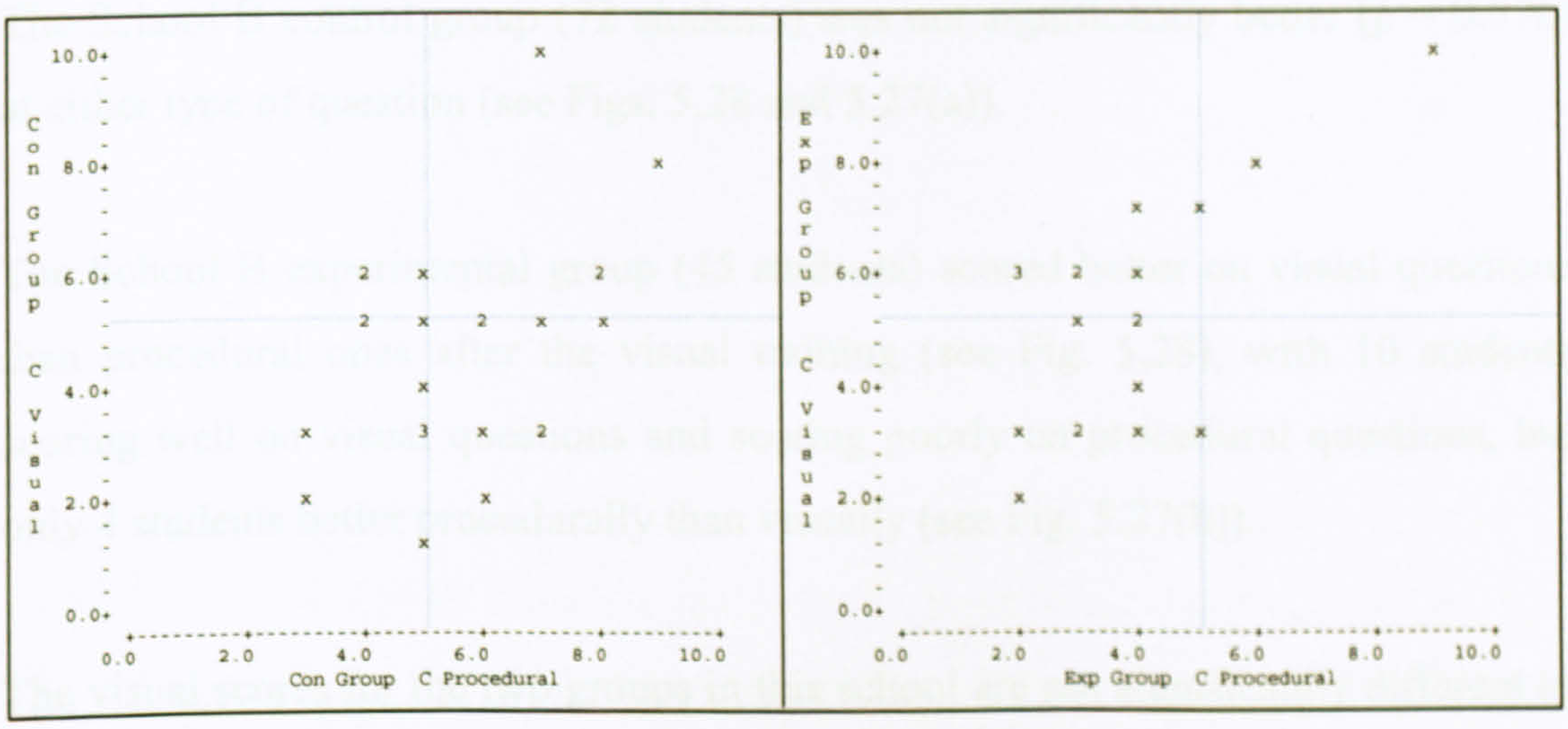


(a) (b)

Fig. 5.25.

The School A control group (39 students) scored far better (p = 0.006) on procedural questions than visual questions (see Fig. 5.28), with 10 students scoring well on procedural questions and scoring poorly on visual questions, but none the other way round (see Fig. 5.25(a)).

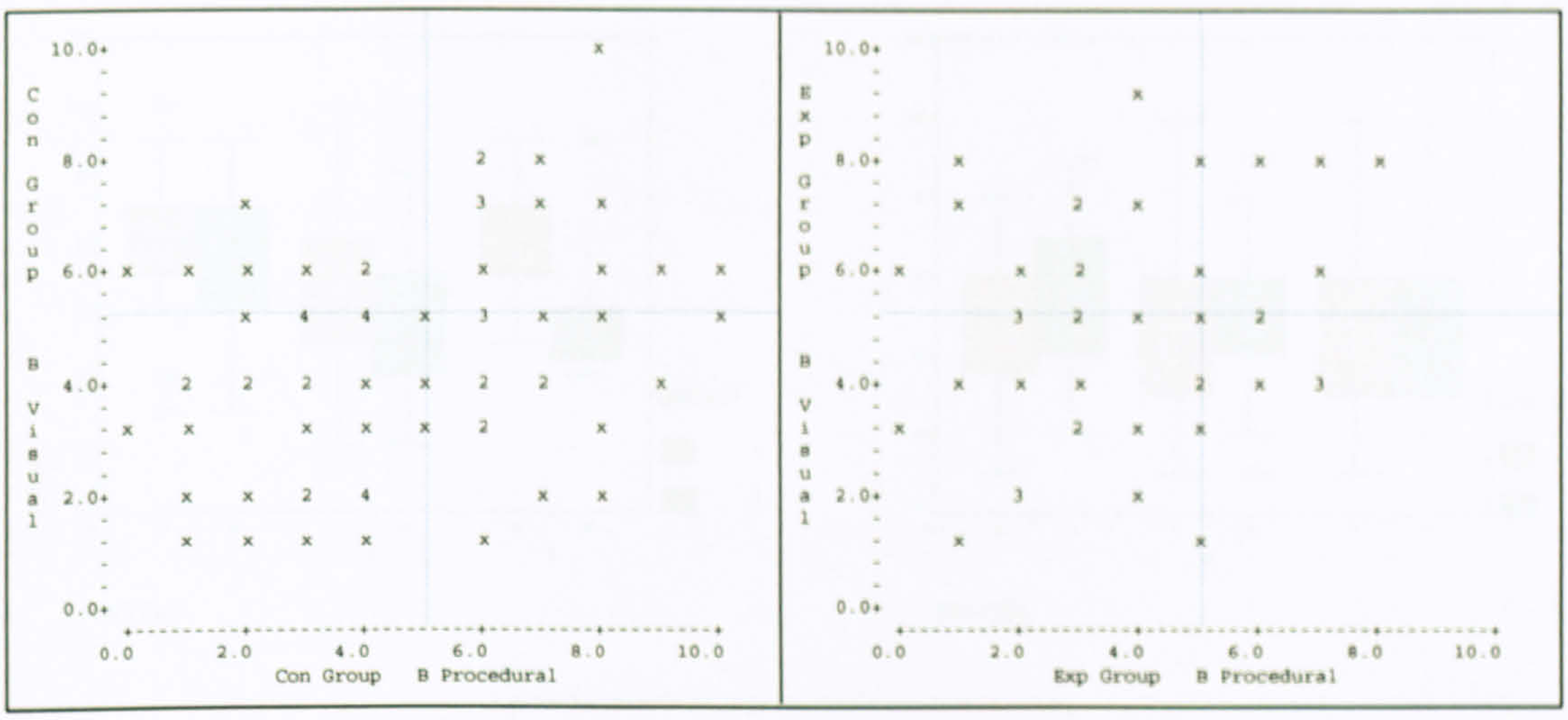
The School A experimental group (44 students) showed a slight improvement (p = 0.066, significant at the 10% level, although not at the 5% level) in visual scores compared with the control group (see Fig. 5.28), with 5 students scoring well on visual questions and scoring poorly on procedural questions, compared to none above, and only 3 students better procedurally than visually, compared to 10 above (see Fig. 5.25(b)).



(a) **(b)**

Fig. 5.26.

The School C results were similar to the School A results in that the control group (25 students) scored better (p = 0.029) on procedural questions than visual questions (see Figs. 5.28 and 5.26(a)), whereas there was no significant difference (p = 0.265) between the experimental group (20 students) and the control group at visual questions (see Fig. 5.28), even though they were very poor procedurally (see Fig. 5.26(b)).



(a) **(b)**

Fig. 5.27.

The School B control group (72 students) was not significantly better (p = 0.576) at either type of question (see Figs. 5.28 and 5.27(a)).

The School B experimental group (45 students) scored better on visual questions than procedural ones after the visual training (see Fig. 5.28), with 10 students scoring well on visual questions and scoring poorly on procedural questions, but only 4 students better procedurally than visually (see Fig. 5.27(b)).

The visual scores for the two groups in this school are not significantly different (p = 0.222), regardless of the teaching and learning approach adopted (see Fig. 5.28). It is noteworthy that the School A scatter plot for the control group (See Fig. 5.25(a)), and to a certain extent that of School C (see fig. 5.26(a)), are very different to that of School B (see Fig. 5.27(a)).

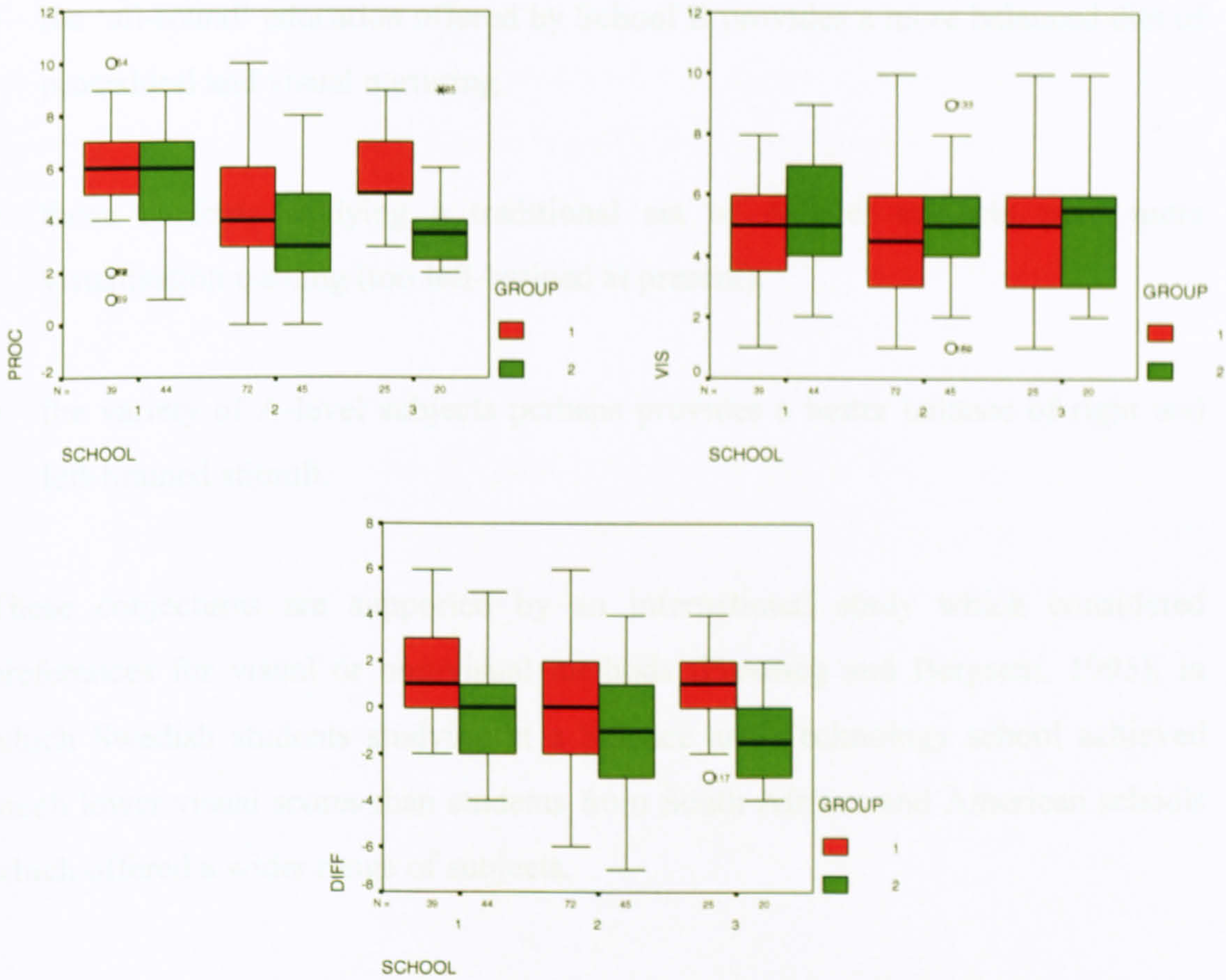


Fig. 5.28.

There is evidence of a difference in performance (p = 0.008) between School A and B pupils and School C pupils. A possible reason was mentioned in Section 5.3.4.1, namely that most School A and School C mathematics students follow a very traditional course of study, i.e. three science subjects, whereas School B students study all sorts of combinations of subjects (the former will therefore study Mechanics, and the latter Statistics – this comparison is discussed in greater detail later in this section, and further in Chapter 6). As expected in the School A and School C scatter plots, there are a number of students who are either poor at both disciplines or good at both disciplines. However, a number of School A and School C pupils are good procedurally and poor visually, and yet there are very few who are good visually and poor procedurally, whereas students' scores from School B are scattered fairly evenly across the plot. It is conjectured that:

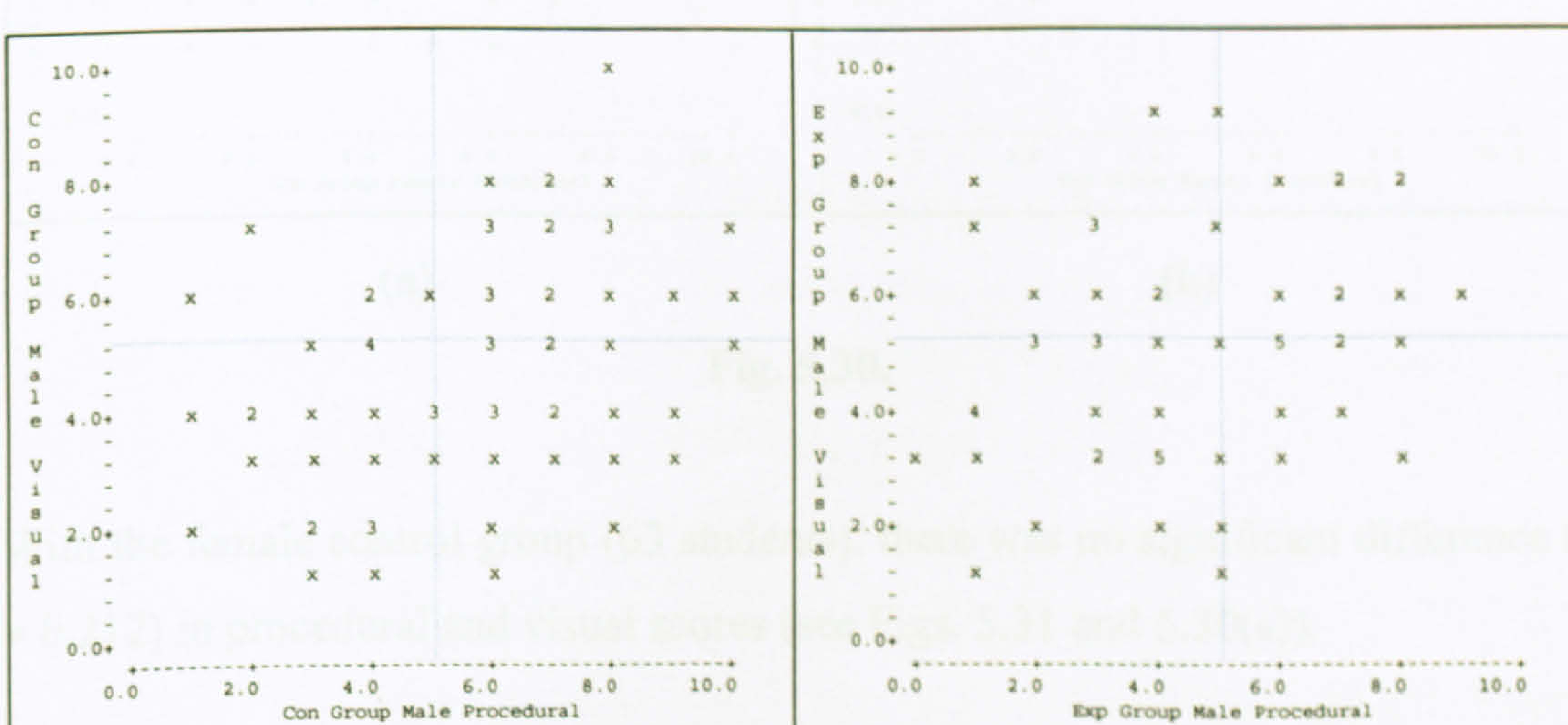
- the 'all-round' education offered by School B provides a more balanced diet of procedural and visual nurturing.

- those students studying a traditional set of A-level subjects need more visualisation training (too left-brained at present).

- the variety of A-level subjects perhaps provides a better balance of right and left-brained stimuli.

These conjectures are supported by an international study which considered preferences for visual or non-visual methods (Presmeg and Bergsten, 1995), in which Swedish students studying at a Science and Technology school achieved much lower visual scores than students from South African and American schools which offered a wider range of subjects.

The next two scatter plots (Figs. 5.29 and 5.30) and the boxplots (Fig. 5.31) look at male and female performance in the case-study.

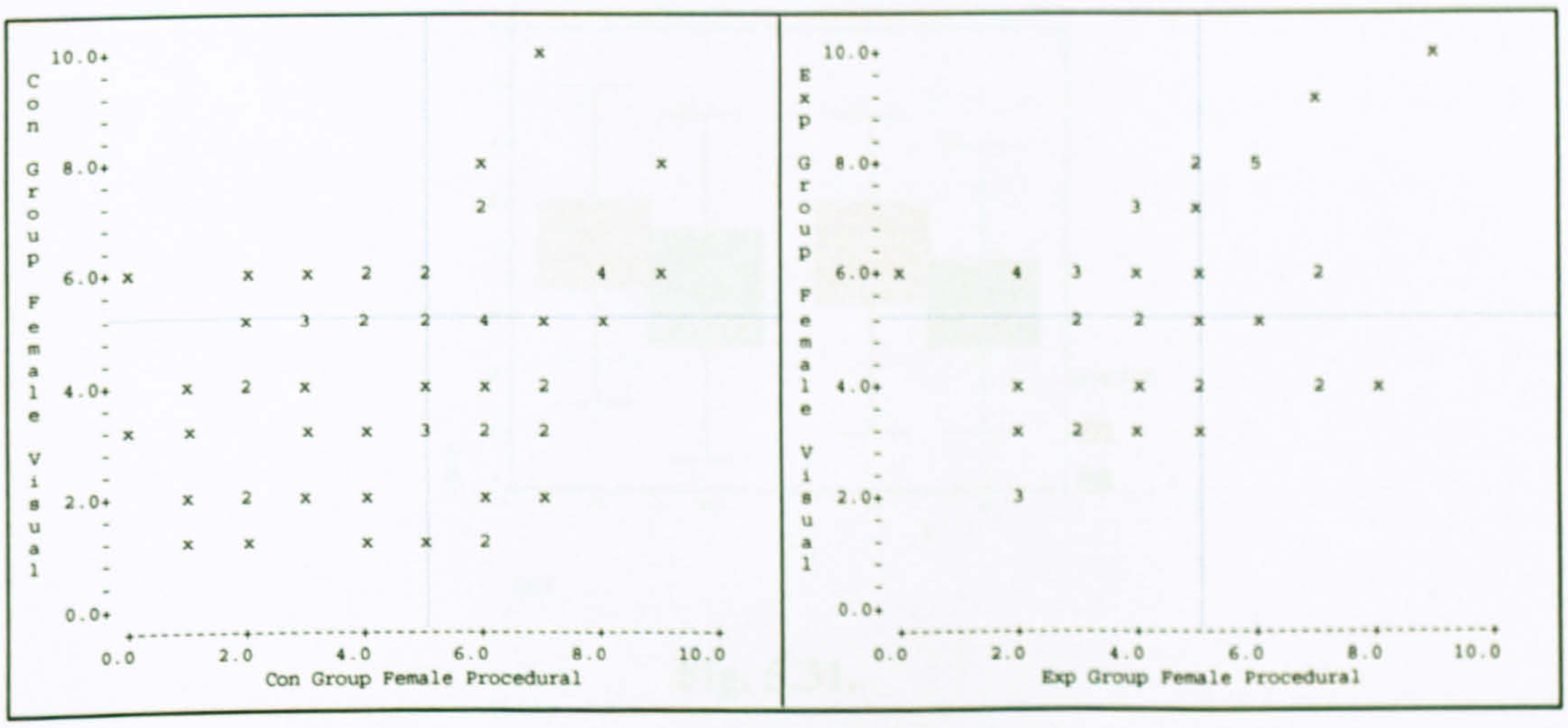


(a) **(b)**

Fig. 5.29.

The male control group (73 students) scored better (p = 0.012) on procedural questions than visual ones (see Fig. 5.31), with 14 students better procedurally than visually, but only 4 students better visually than procedurally (see Fig. 5.29(a)). They were significantly better procedurally (p = 0.027) than the female control group, but there was no significant difference (p = 0.386) in visual scores.

With the male experimental group (63 students), there was no significant difference (p = 0.584) in visual scores compared with the control group (see Fig. 5.31). They did, however, exhibit a different pattern of scores, with 10 students scoring better visually than procedurally, compared to 4 above, and only 4 better procedurally than visually, compared to 14 above (see Fig. 5.29(b)).



(a) **(b)**

Fig. 5.30.

With the female control group (63 students), there was no significant difference (p = 0.212) in procedural and visual scores (see Figs. 5.31 and 5.30(a)).

The female experimental group (46 students) scored very well visually, but not too well procedurally, with 12 students scoring better visually than procedurally, but only 3 students scoring better procedurally than visually (see Fig 5.30(b)). This was expected with the entire experimental group, given their lack of experience procedurally. The female experimental group scored much better visually ($p = 0.001$) than their control group counterparts, and also scored significantly better visually ($p = 0.041$) than the male experimental group (see Fig. 5.31).

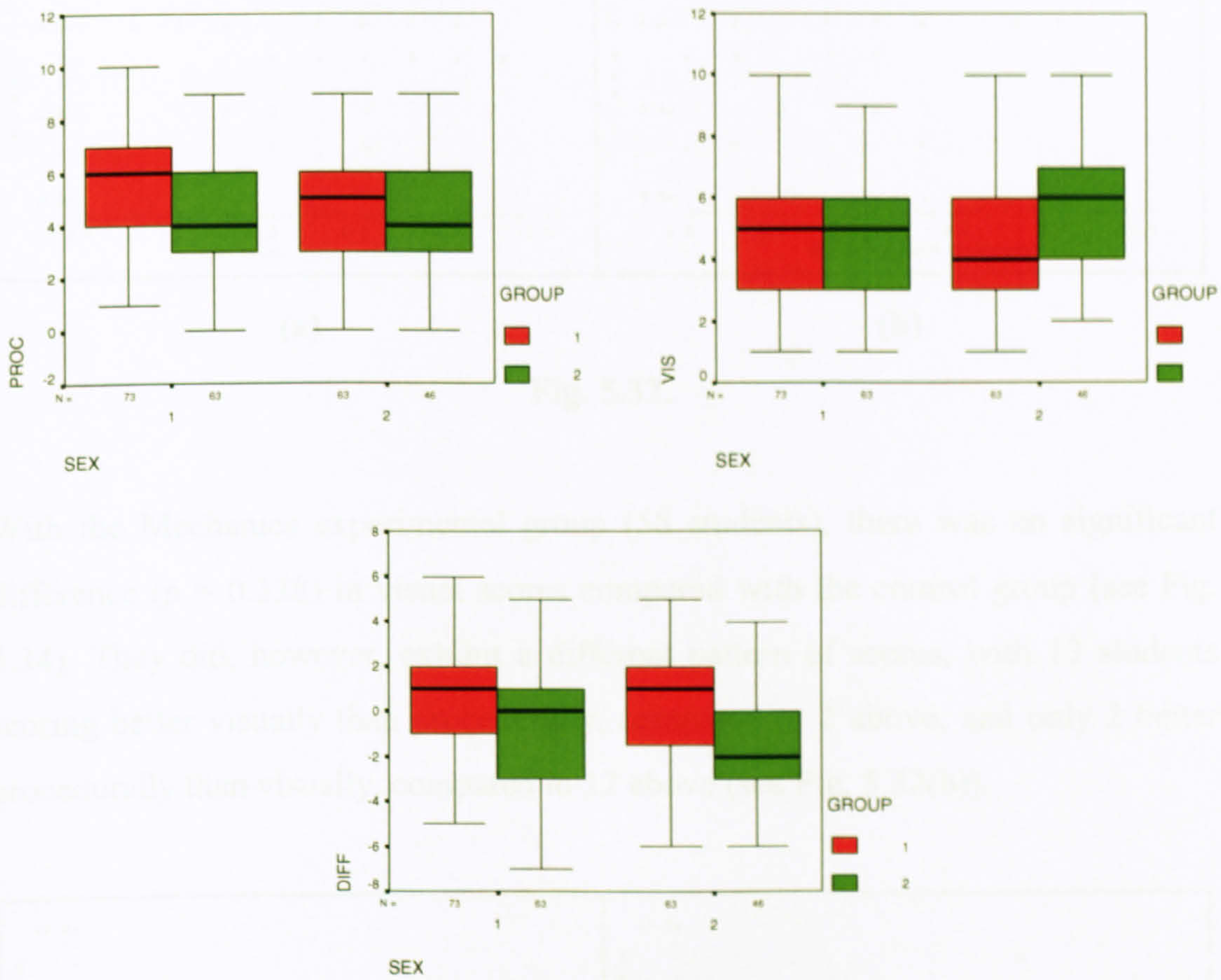
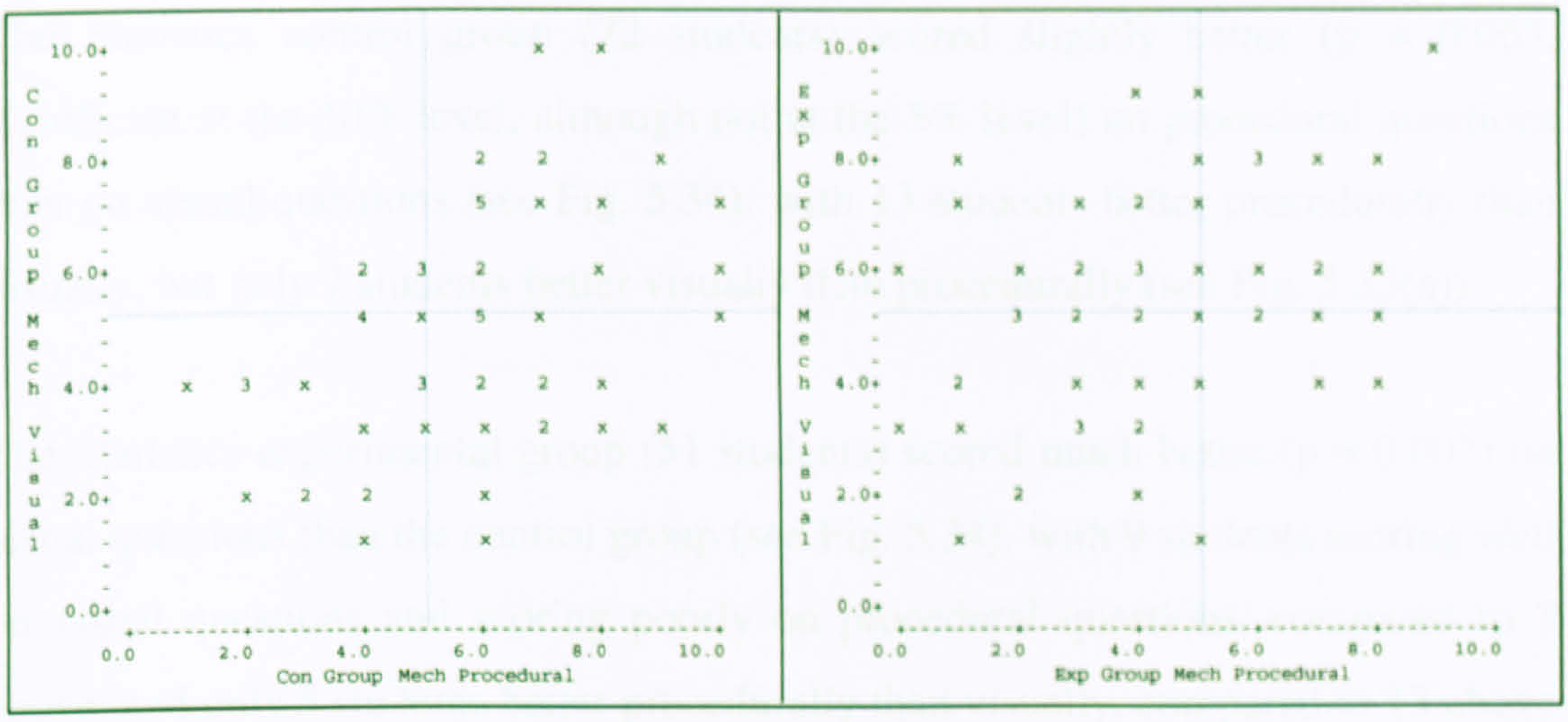


Fig. 5.31.

The next two scatter plots (Figs. 5.32 and 5.33) and the boxplots (Fig. 5.34) look at students studying Mechanics and Statistics in the case-study.

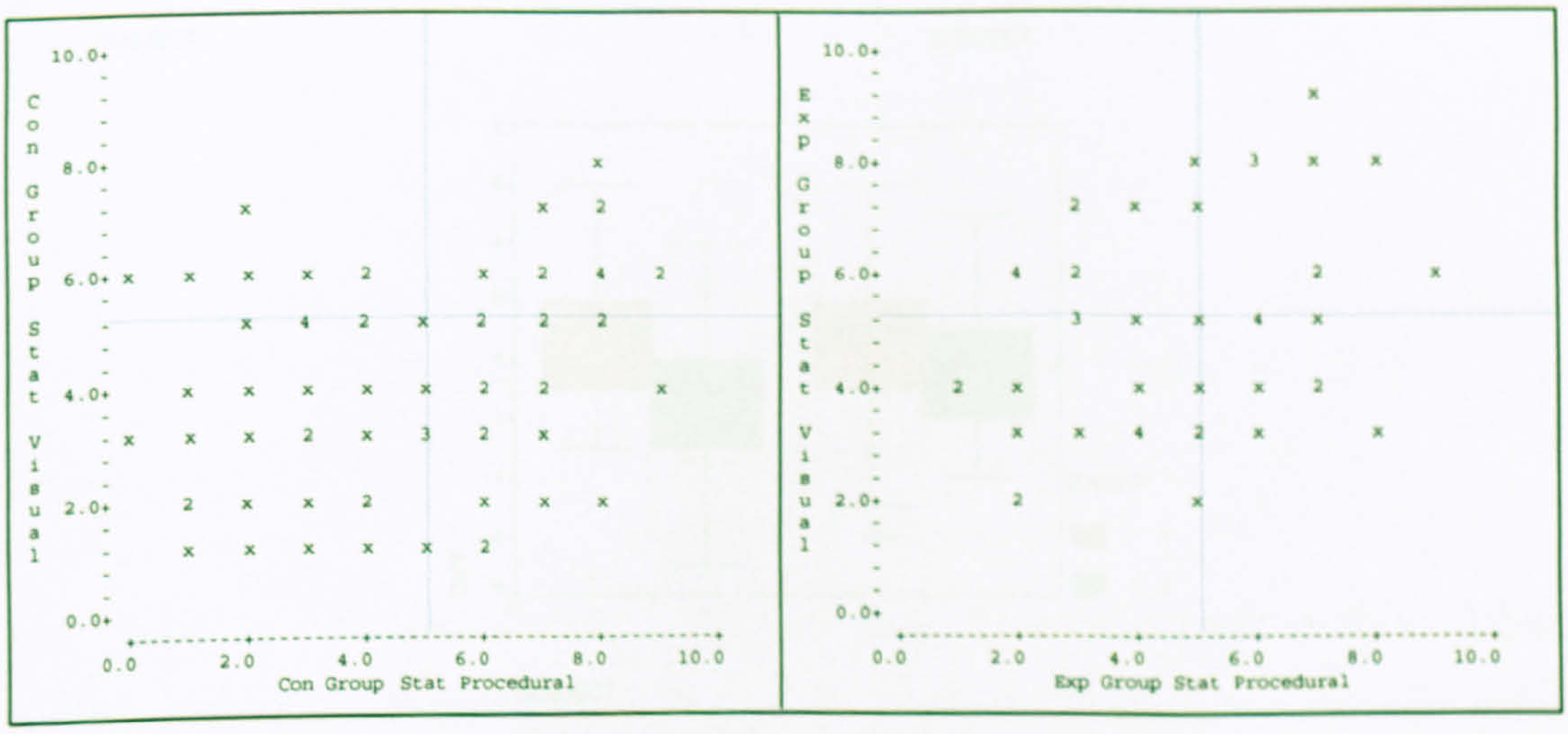
The Mechanics students control group (64 students) scored better ($p = 0.046$) procedurally than visually (see Fig. 5.34), with 12 students scoring better procedurally than visually, but only 2 students better visually than procedurally (see Fig. 5.32(a)).



(a) (b)

Fig. 5.32.

With the Mechanics experimental group (58 students), there was no significant difference (p = 0.338) in visual scores compared with the control group (see Fig. 5.34). They did, however, exhibit a different pattern of scores, with 13 students scoring better visually than procedurally, compared to 2 above, and only 2 better procedurally than visually, compared to 12 above (see Fig. 5.32(b)).



(a) (b)

Fig. 5.33.

The Statistics control group (72 students) scored slightly better (p = 0.063, significant at the 10% level, although not at the 5% level) on procedural questions than on visual questions (see Fig. 5.34), with 13 students better procedurally than visually, but only 7 students better visually than procedurally (see Fig. 5.33(a)).

The Statistics experimental group (51 students) scored much better (p = 0.007) on visual questions than the control group (see Fig. 5.34), with 9 students scoring well on visual questions and scoring poorly on procedural questions, compared to 7 above, and only 5 students better procedurally than visually, compared to 13 above (see Fig. 5.33(b)).

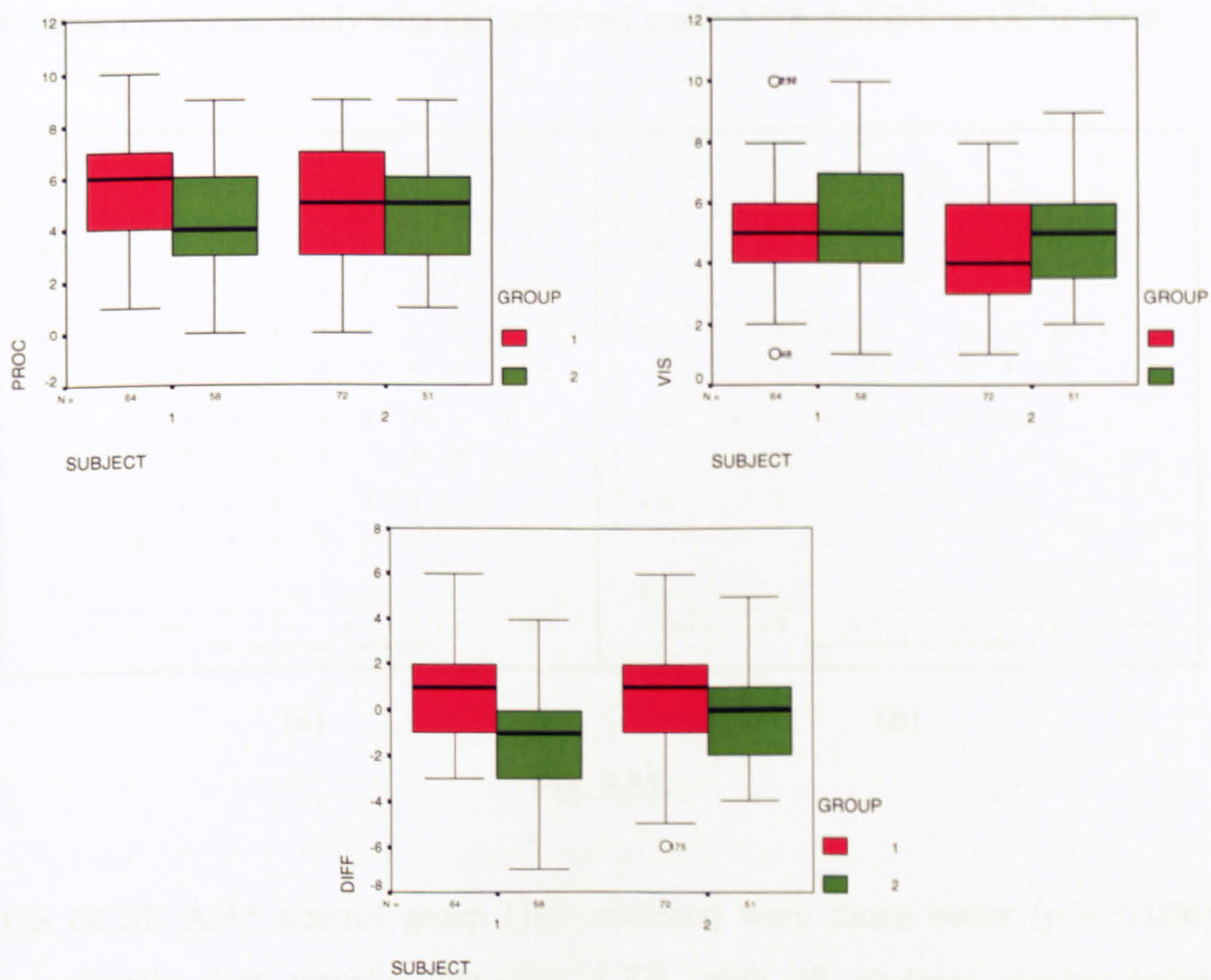
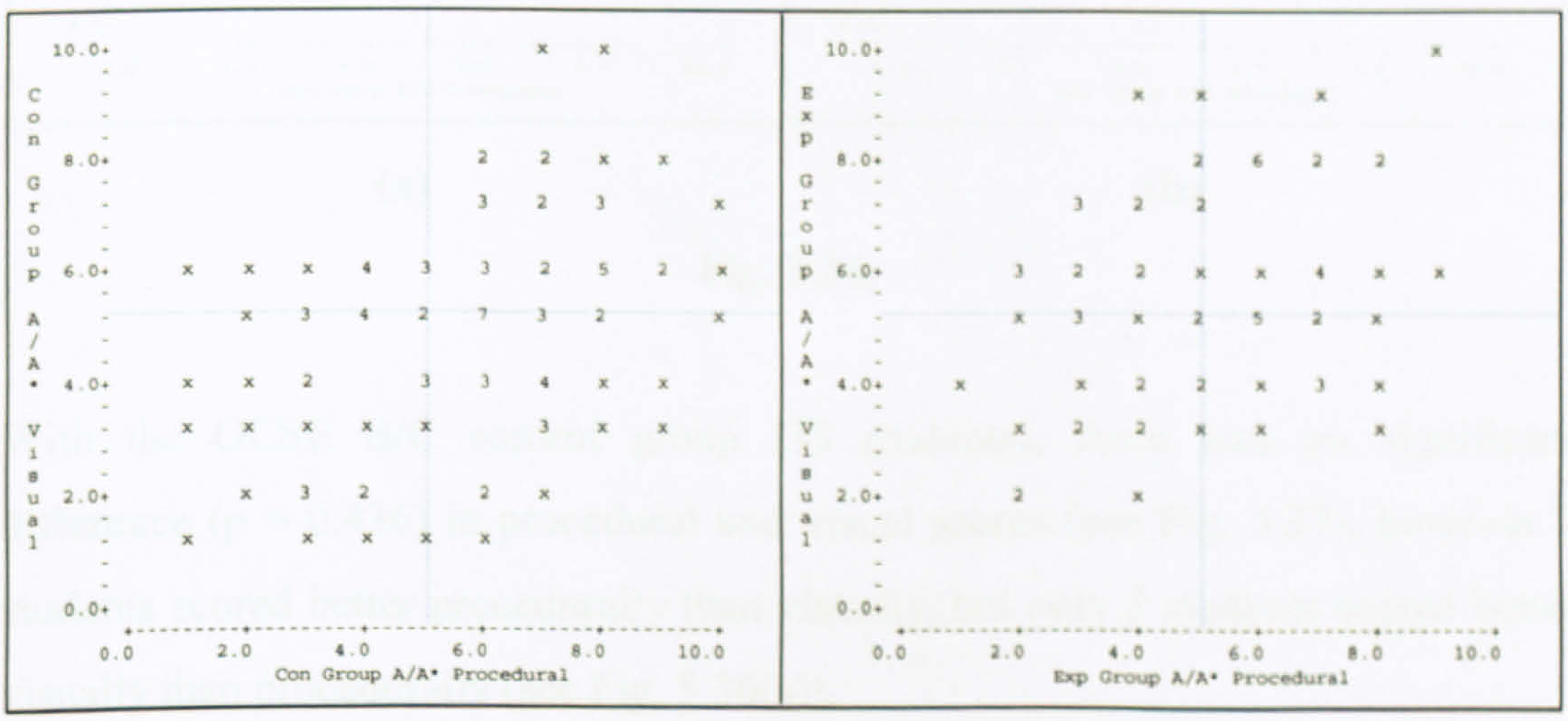


Fig. 5.34.

The Mechanics experimental group scored better visually (p = 0.05, significant at the 10% level, although not quite at the 5% level) than the Statistics experimental group (see Fig. 5.34).

It is interesting to note that both the Mechanics and Statistics control groups were more successful at the procedural questions than the visual questions, even though most Mechanics and Statistics questions involve pictorial representation (a discussion of the skills required to tackle both Mechanics and Statistics questions is provided in Chapter 6).

The next two scatter plots (Figs. 5.35 and 5.36) and the boxplots (Fig. 5.37) look at students in the case-study who had achieved grade A*/A and B/C at GCSE level.

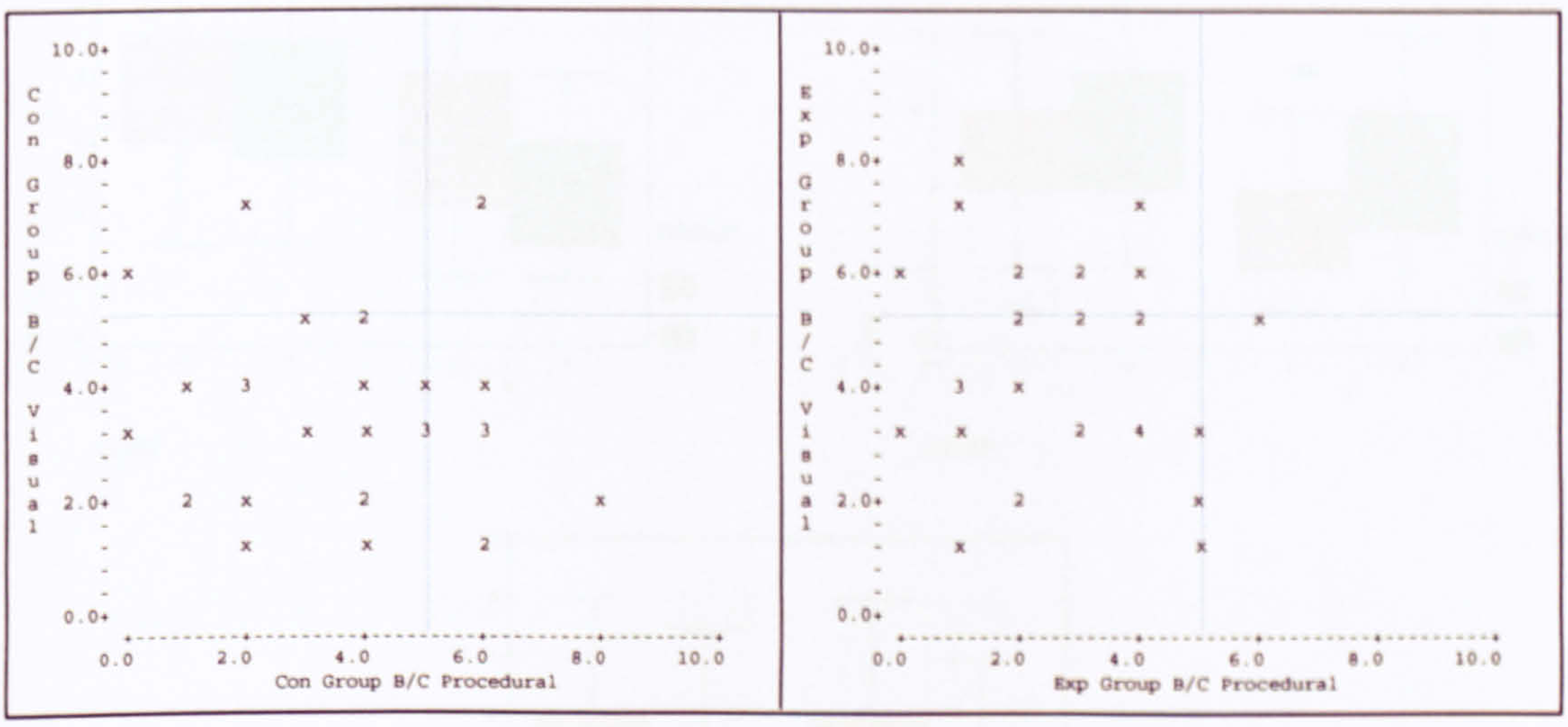


(a) (b)

Fig. 5.35.

The GCSE A/A* control group (103 students) were much better (p = 0.006) procedurally than visually (see Fig. 5.37), with 18 students scoring better procedurally than visually, but only 7 students better visually than procedurally (see Fig. 5.35(a)).

The GCSE A/A* experimental group (75 students) scored much better (p = 0.011) on visual questions than the control group (see Fig. 5.37), with 13 students scoring well on visual questions and scoring poorly on procedural questions, compared to 7 above, and only 7 students better procedurally than visually, compared to 18 above (see Fig. 5.35(b)).



(a) **(b)**

Fig. 5.36.

With the GCSE B/C control group (33 students), there was no significant difference (p = 0.436) in procedural and visual scores (see Fig. 5.37), however 7 students scored better procedurally than visually, but only 2 students scored better visually than procedurally (see Fig. 5.36(a)).

The GCSE B/C experimental group (34 students) scored slightly better (p = 0.059, significant at the 10% level, although not at the 5% level) on visual questions than the control group (see Fig. 5.37), even though they were very poor procedurally (see Fig. 5.36(b)), with 9 students scoring well on visual questions and scoring poorly on procedural questions, compared to 2 above, and none better procedurally than visually, compared to 7 above (see Fig. 5.36(b)).

For both the control and experimental groups, the GCSE A*/A group outscored the GCSE B/C group on procedural questions ($p = 0.001$ and $p < 0.0001$ respectively) and visual questions ($p = 0.002$ and $p = 0.001$ respectively).

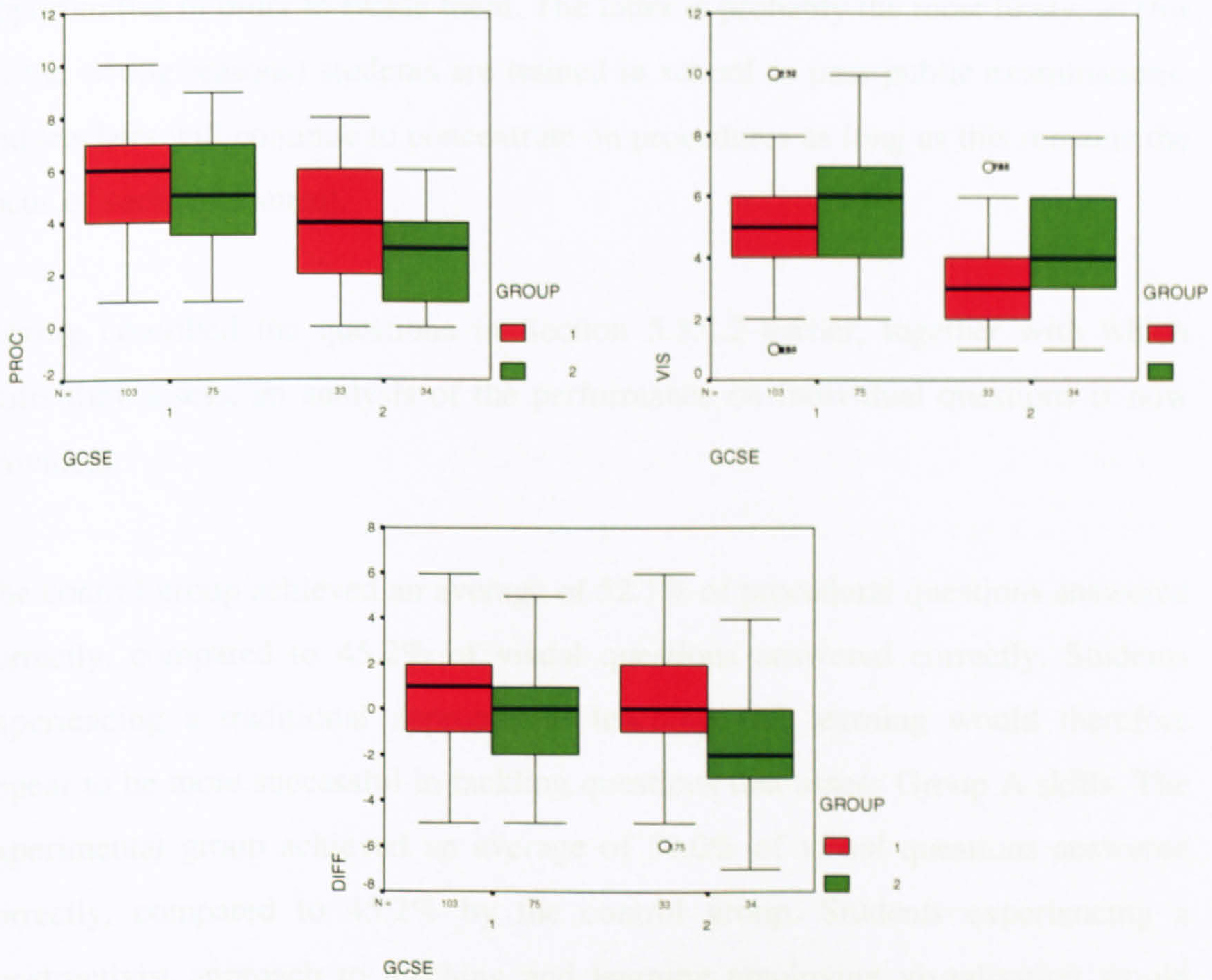


Fig. 5.37.

The statistical analysis of the data has shown that the constructivist visual approach has been more beneficial for the experimental group, School A students, females, Statistics students, and GCSE A*/A students. This would suggest that a sub-group of students matching all these criteria would achieve far superior visual scores than a corresponding control group. This was indeed true, with the experimental sub-group achieving a 40% higher mean visual score than the control sub-group.

The analysis has shown that after the appropriate training, the experimental group, even though less experienced procedurally, managed to score well on the visual questions. This was not the case with the control group, with the majority of students scoring better on the procedural questions. This would suggest that students generally cannot do questions that assess higher order skills, are not familiar with them, or have not been provided with the appropriate learning opportunities in order to tackle them. The latter is probably the most likely, as (for all the wrong reasons) students are trained in school to pass public examinations, and teachers will continue to concentrate on procedures as long as this remains the focus of such assessment.

Having described the questions in Section 5.3.4.2 earlier, together with which skills they assess, an analysis of the performance on individual questions is now provided.

The control group achieved an average of 52.1% of procedural questions answered correctly, compared to 45.2% of visual questions answered correctly. Students experiencing a traditional approach to teaching and learning would therefore appear to be more successful in tackling questions that assess Group A skills. The experimental group achieved an average of 52.0% of visual questions answered correctly, compared to 45.2% by the control group. Students experiencing a constructivist approach to teaching and learning employing visualisation would therefore appear to be more successful in tackling visual questions that assess Group C skills than students experiencing a traditional approach.

All the procedural questions (odd numbered) assess Group A skills. They all assess the routine use of procedures and some additionally assess factual knowledge and comprehension. Only Question 9 of the procedural questions assesses Group C skills (Conjectures and Justifying), and is therefore more like one of the visual questions (even numbered) in terms of the skills required to complete it successfully. Hence it was expected that Question 9 would be answered less successfully than the other procedural questions. In both the control and experimental groups, it was the third most unsuccessfully answered procedural question, with only 33.8% of control group students answering it correctly (compared to a control group average of 52.1% for procedural questions), and only 29.4% of experimental group students answering it correctly (compared to an experimental group average of 43.3% for procedural questions). Although the experimental group had developed higher order skills via software use, it appears that they did not have the necessary procedural knowledge base to interact with the acquired skills.

Question 18 was the most demanding question of all in terms of the nature and quantity of skills assessed. In both the control and experimental groups, it was the second most unsuccessfully answered visual question, with only 22.1% of control group students answering it correctly (compared to a control group average of 45.2% for visual questions), and only 29.4% of experimental group students answering it correctly (compared to an experimental group average of 52.0% for visual questions).

Both control and experimental groups found Question 11 to be the most difficult procedural question, with only 22.1% of control group students answering it correctly, and only 13.8% of experimental group students answering it correctly. The control group had a better success rate than the experimental group, which was to be expected given their extra year of mathematical development. The high failure rate would appear to be due to the fact that the routine use of procedures required are of a more complex nature than in other questions.

The control group found Question 7 to be the easiest procedural question, with 76.5% of control group students answering it correctly. Unlike Question 11, the routines required are of a simpler nature, so although Questions 7 and 11 assess the same skills, it would appear that there are different levels of difficulty within each skill.

The experimental group found Question 5 to be the easiest procedural question, with 66.1% of experimental group students answering it correctly. It would appear that this was simply the most familiar topic for their stage of mathematical development (gradient and intercept of a straight line).

In both groups, the same general pattern of success and failure emerged. Regardless of their stage of mathematical development, both groups similarly found certain questions difficult, certain questions moderately difficult, and certain questions easy.

Both control and experimental groups found Question 8 to be the most difficult visual question, with only 11.0% of control group students answering it correctly, and only 28.4% of experimental group students answering it correctly. The experimental group had a better success rate than the control group, as they appeared to be better at switching between representations in both directions, and had developed more Group C skills experience.

Both control and experimental groups found Question 2 to be the easiest visual question, with 88.2% of control group students answering it correctly, and 89.9% of experimental group students answering it correctly. As earlier, this was probably due to the fact that it was the simplest topic (gradient and intercept of a straight line).

Questions 4, 6, 8, 10, 12, and 16 all assess the same skills, however neither group answered this set of questions similarly. As mentioned earlier, this perhaps illustrates the different levels of difficulty within the same skill groupings.

Questions 4 and 16 not only assess the same skills, but also the direction of information transfer is the same. Similar success rates would therefore be expected. For the control group, a 95% confidence interval for the difference between the means for each question is (-12.20, 42.67). This includes zero, and therefore there is no evidence of a difference in performance on the two questions. The individual confidence intervals for each question overlap significantly. For the experimental group, a 95% confidence interval for the difference between the means for each question is (1.33, 57.41). This does not include zero (just), and therefore there is evidence of a difference in performance on the two questions. The individual confidence intervals for each question do not overlap significantly. The experimental group found Question 4 easier than Question 16, even though both questions assess the same skills and direction of information transfer. It would appear that they were simply more comfortable with quadratics than trigonometric functions given their experience, whereas the control group were familiar with both types of function and therefore scored similarly on both questions.

The full analysis, which provides a detailed breakdown of performance in the individual questions, can be found in Appendix D.

5.4.2. Student Feedback

Feedback was sought in order to gauge student opinion on the usefulness of the software, and to assess psychological factors such as motivation and enjoyment after experiencing a constructivist visual approach. The feedback obtained from all the experimental group students provides an overall 'feel' for what the students thought of the software and the teaching and learning approach. The following takes each question from the feedback questionnaire (Appendix C) and provides a bar chart of responses.

Question 1. Did you find the session helpful?

A. Yes, extremely helpful.
B. Yes, quite helpful.
C. Neither helpful nor unhelpful.
D. No, not very helpful.
E. No, most unhelpful.

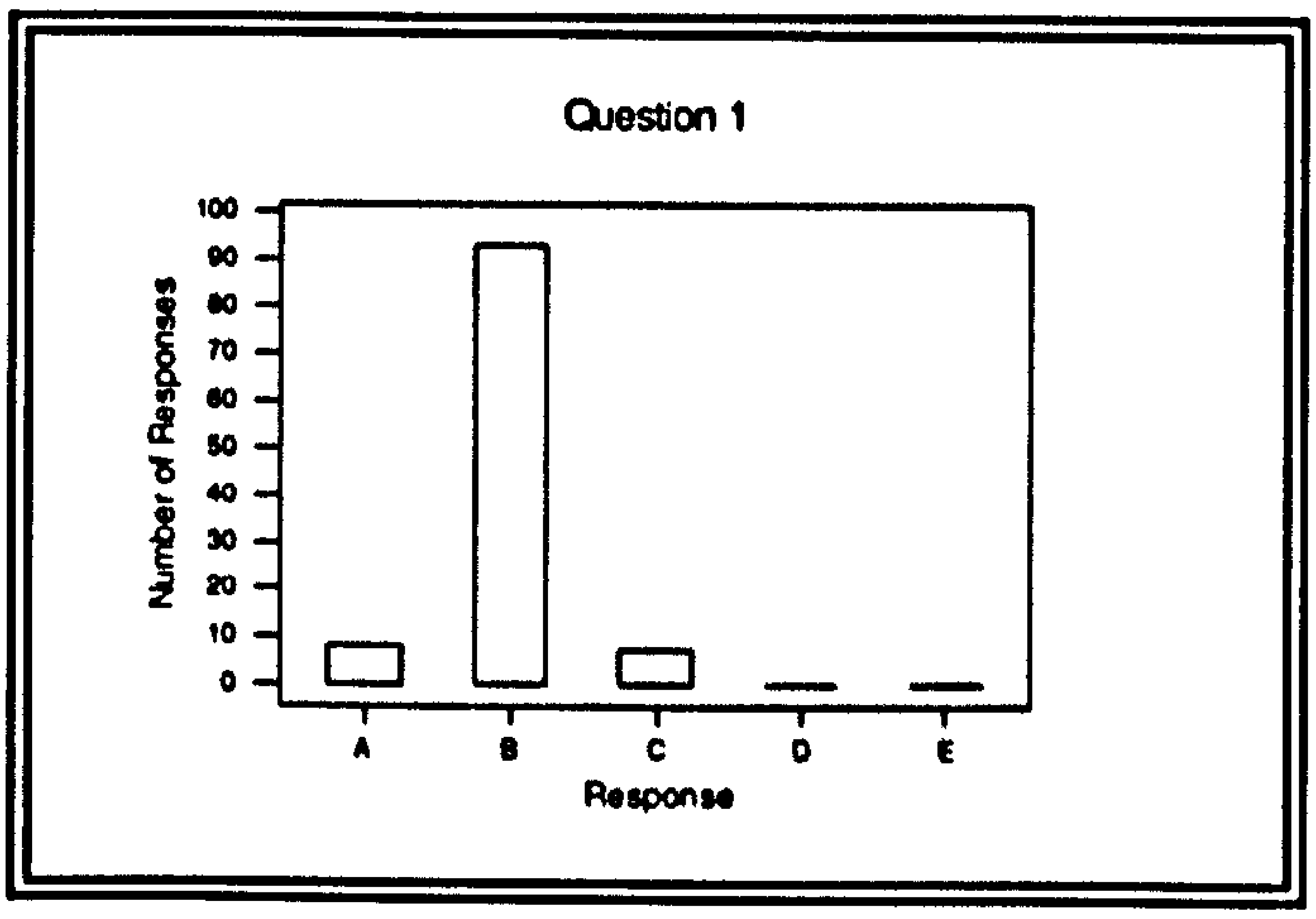


Fig. 5.38.

Question 2. Did you enjoy using the package?

A. Yes, very much.
B. Yes, a little.
C. I didn't like it nor dislike it.
D. No, I didn't really like it.
E. No, I hated it.

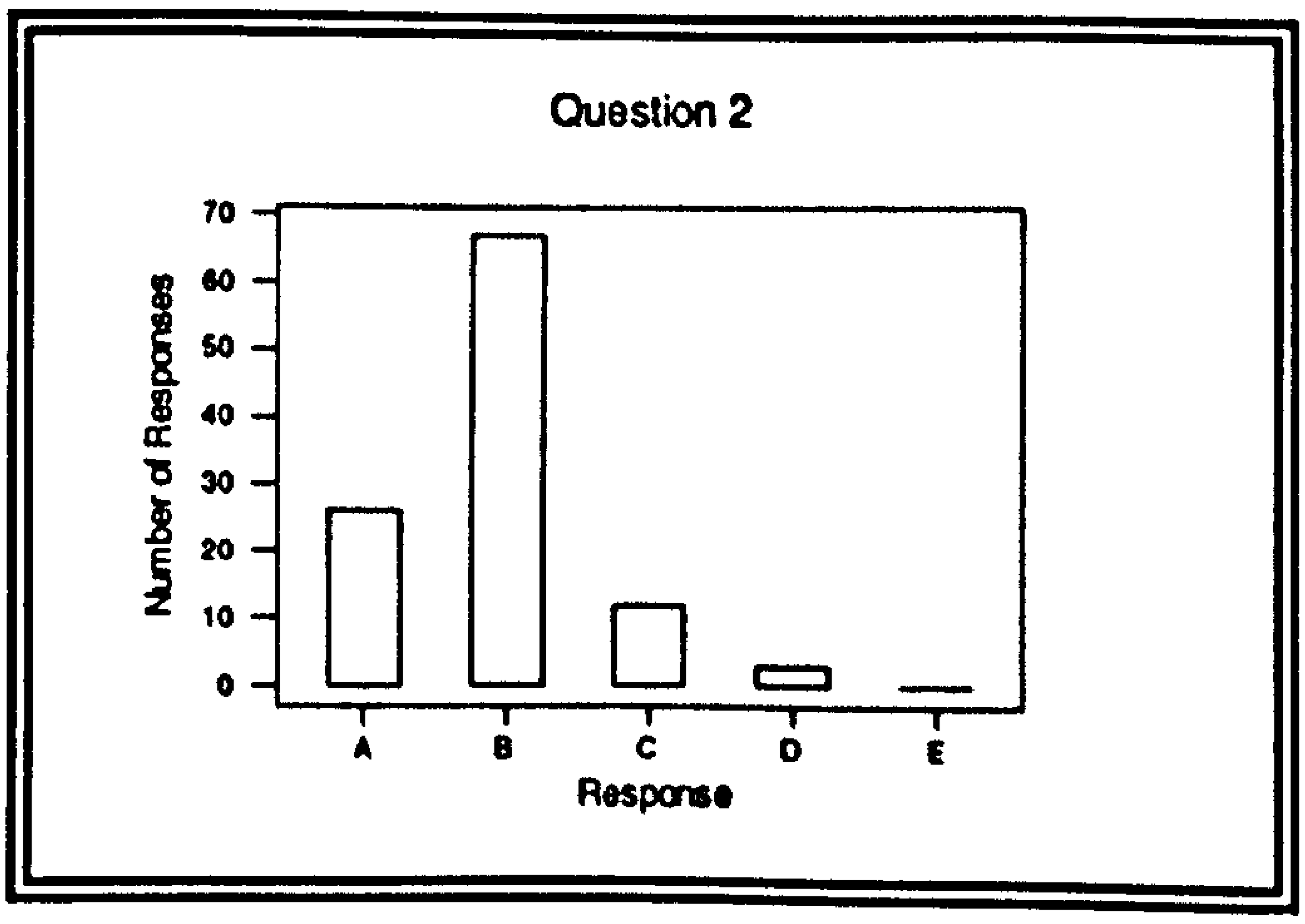


Fig. 5.39.

Question 3. Having used this software, would you be motivated to use software concerned with other mathematical topics?

A. Yes, very much so.
B. Yes, probably.
C. Maybe, maybe not.
D. No, probably not.
E. No, definitely not.

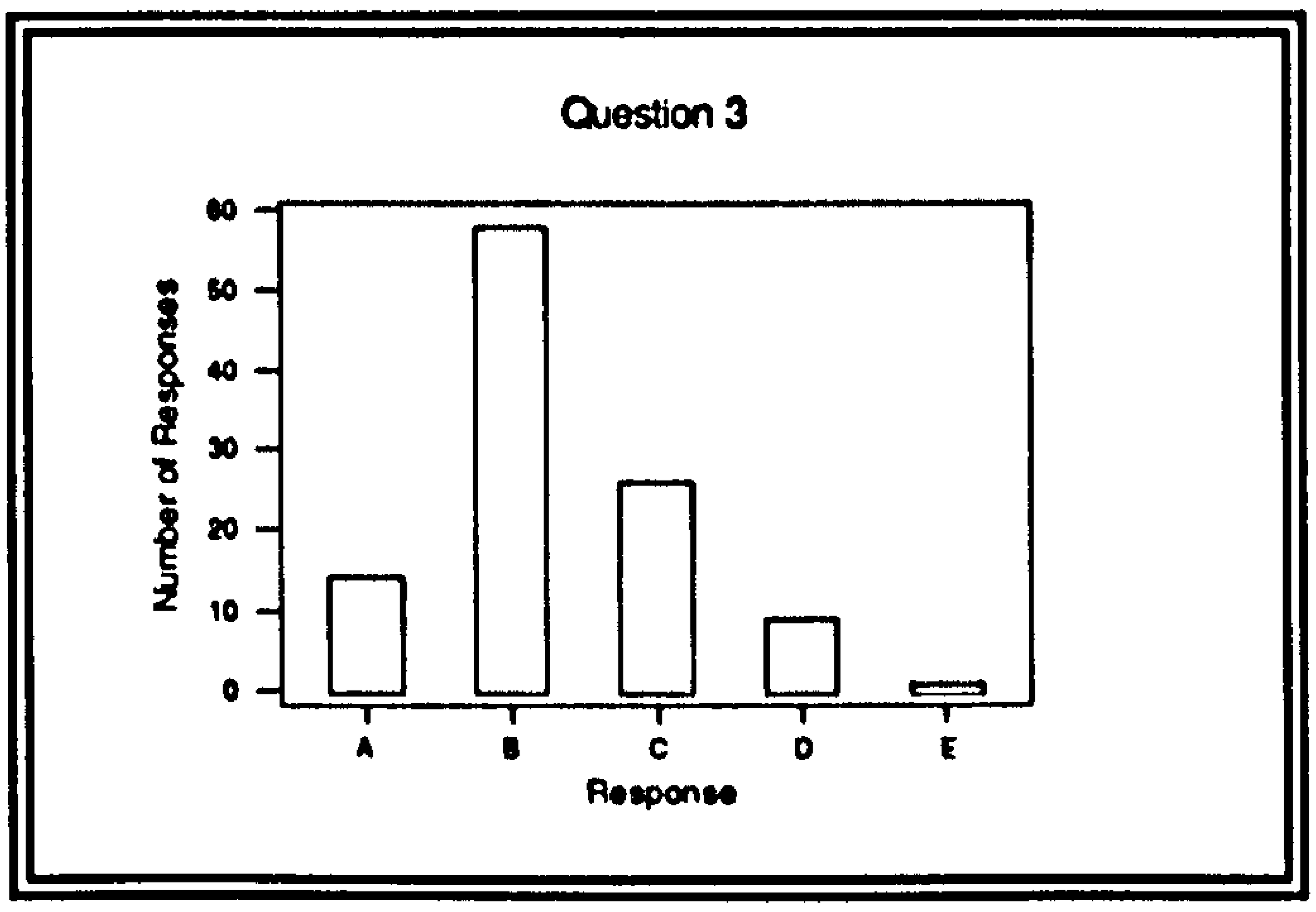


Fig. 5.40.

Question 4. How did this 2-hour computer session compare with 2 hours of traditional mathematics classroom activities?

A. I would much rather learn via the computer.
B. I would probably prefer to learn via the computer.
C. I have no preference.
D. I would probably prefer to do traditional mathematics classroom activities.
E. I would much rather do traditional mathematics classroom activities.

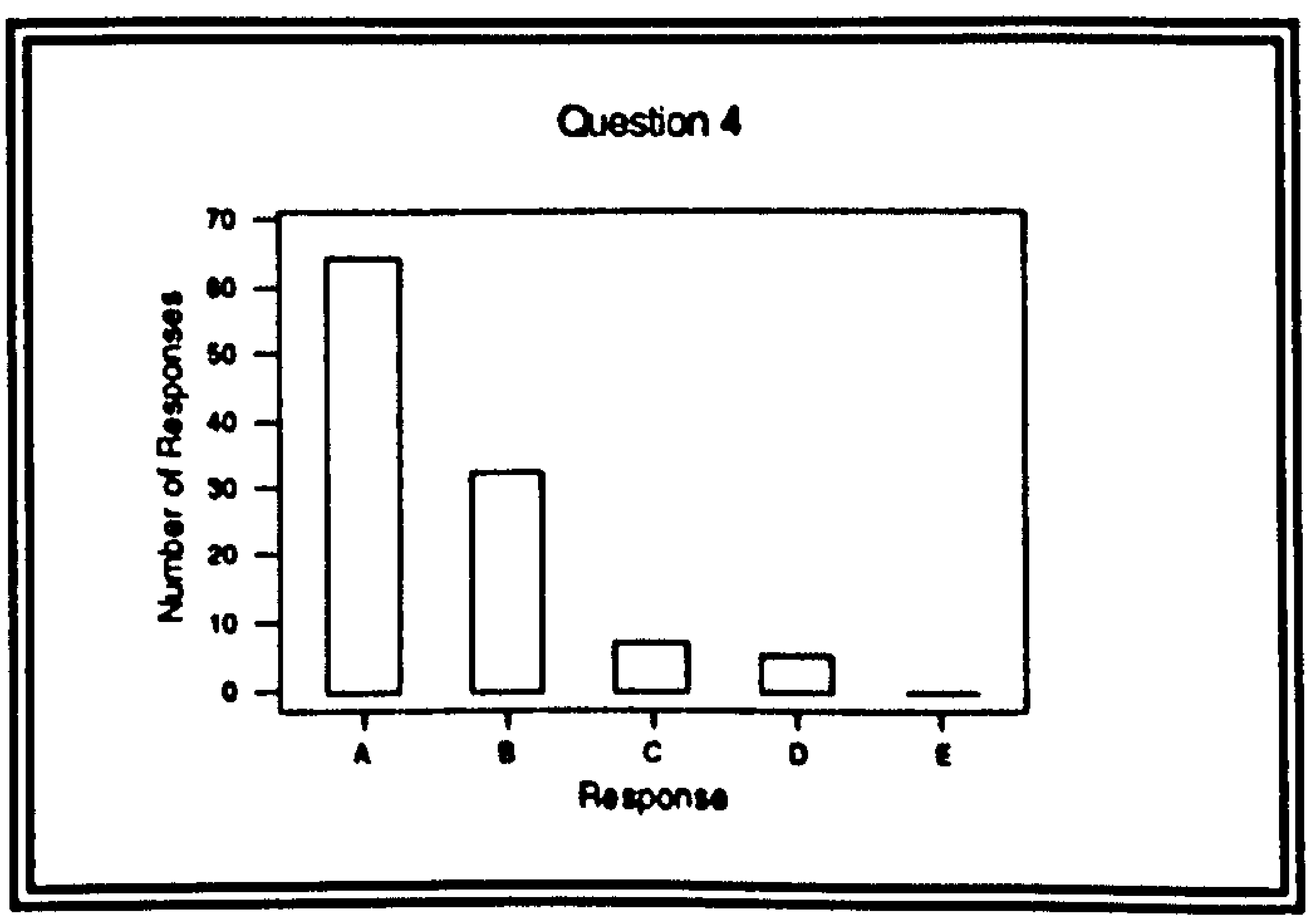


Fig. 5.41.

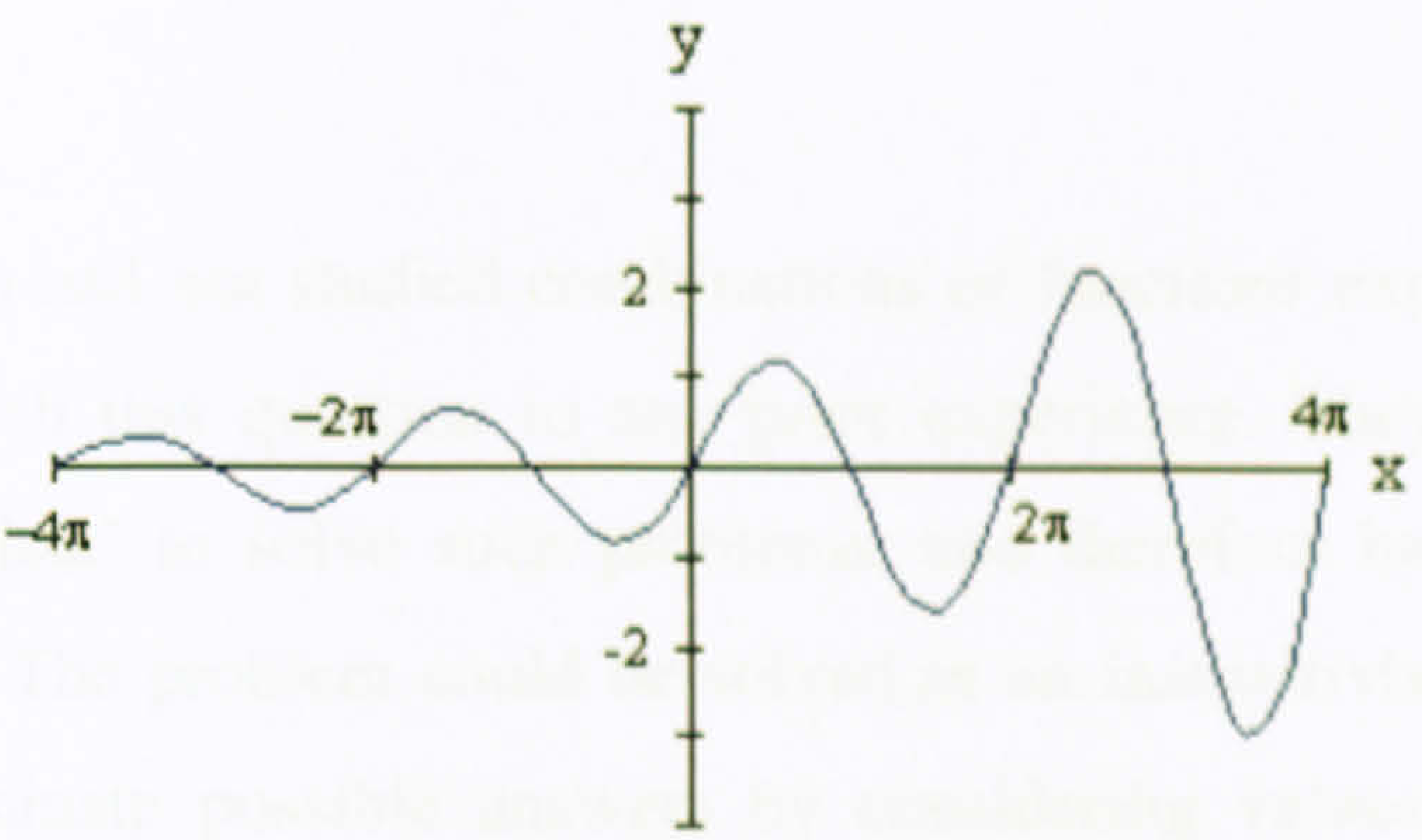
All the above bar charts clearly illustrate that the session was a success in terms of usefulness, enjoyment and motivation. As a result of an informal discussion with the students, it became clear that they appreciated that the software facilitated a constructivist style of learning, and that they had made useful mathematical discoveries. There was very little resistance to the experimental approach. General comments (those offered in Question 5) concerning how to improve the package are discussed as considerations for future developments in Section 7.2.1 of Chapter 7.

5.5. Conclusions

It was conjectured that the constructivist use of visualisation had enabled students to develop more Group C skills, whereas students undergoing the instructivist treatment were mainly limited in skills to those of Group A. There was strong evidence that the experimental group scored better than the control group ($p = 0.004$) at visual questions (requiring the application of higher order skills), and there was strong evidence ($p < 0.0001$) that the traditional approach led to better procedural skills as opposed to visual skills. This evidence, from the statistical analysis earlier, affirmed that the initial conjecture was indeed the case, but moreover there was evidence to suggest that *linkages between the skill groups were more pronounced, creating a more holistic view of mathematics*. This is best summarised by considering responses to Question 20 in particular, as shown in Fig. 5.42 (although only one example, it is indicative of the findings in general).

The question below assesses whether or not the students have been able to take the knowledge gained from local tasks and apply it globally.

20.



The above graph is a graphical representation of which of the following functions?

A $y = \sin(x) + e^{0.1x}$ **B** $y = \sin(x) + e^{-0.1x}$ **C** $y = \sin(x)\, e^{0.1x}$
D $y = \sin(x)\, e^{-0.1x}$ **E** all of the above

Fig. 5.42.

When faced with a graph which was the result of a combination of functions, the control group struggled to find the correct solution (32.4% answered correctly), whereas the experimental group used their knowledge relating to other families of graphs to arrive at the correct function (52.3% answered correctly). The experimental group had a more holistic view of the topic and were therefore not deterred by the nature of the task, and were able to employ their conceptual knowledge of combining familiar, specific functions (and the effect on the graph) to an unfamiliar (but similar) situation. The control group's sequential style, however, hindered their progress as they could not see any other way around their limited, linear methods.

The experimental group students were more successful as they had the ability to combine functions and understand the effect this would have on the graph, irrespective of the specific functions studied. Their whole approach to learning equipped them with better strategies for problem solving. The richness of global thinking proved beneficial as they checked their answers by more than one approach.

The control group had not studied combinations of functions explicitly, and were struggling to match this question to any prior experience. They did not have a 'recipe' or 'template' to solve such problems, and therefore had a very limited solution strategy. The problem could be solved in an instructivist manner, e.g. to methodically eliminate possible answers by considering values of x where the graph cuts the x-axis, then considering the substitution of different values of x into e^x and e^{-x}, etc., but the control group did not seem to have the necessary problem solving skills to tackle it, even in an instructivist way.

It would appear that the experimental group had a greater mathematical skills set with more flexibility in moving between the different skills when applying them. Results of the visual questions would suggest that they were better at problems that required more versatile thinking, and could apply the generic concepts to specific examples. The control group tended to follow rules that had been explicitly taught.

This example illustrates that understanding needs to be independent of the specific examples used; relational as opposed to instrumental understanding needs to be encouraged. For example, the bespoke teaching software looked at investigations specific to certain functions, but the newly acquired conceptual structures could be applied to *any* function.

It would have been expected for males and females to perform with the same degree of success on visual questions after software use, as the control group results indicated that there was no significant difference in visualisation ability. This, however, was not the case. The experimental group results showed that after learning via the software, females scored better on the visual questions than their male counterparts. The considerable improvement in female scores after having used the dedicated teaching software would perhaps suggest that females respond more positively to this form of visual training, which was not expected on the basis of the control group results.

These findings concerning male and female visualisation abilities concur with those of other studies in the literature. For example, Ferrini-Mundy (1987) found that females generally perform better at calculus, but males generally perform better at spatial visualisation tasks. It was believed, however, that spatial training may benefit performance in certain areas of calculus for females more than for males. Ferrini-Mundy suggests that females are more likely to respond better to visual stimuli than males, even though they are generally weaker visually.

There is perhaps a case, therefore, that educators need to determine the abilities and learning preferences of the learners, for example their visualisation abilities, and then tailor the learning environment and activities to fit the learner characteristics. However, even the males in the experimental group scored just as well on the visual questions as the males in the control group. So, although females excelled with this form of training, males were certainly not disadvantaged. Perhaps the key is to find a form of training that will lead to the males excelling, and to offer different forms of activities depending on the sex of the learner. This way, individual differences could be accommodated in the teaching and learning methods (research on learning styles is currently being carried out at the University of Plymouth (Berry, 2002; Smith and Berry, 2002)).

The results of the experimental group showed that the performance of the female students was superior to that of their male counterparts, contrary to the results of the control group. This observation is perhaps not surprising when compared with research carried out by Ruthven (1990). His experiment looked at comparing the mathematical performance of upper secondary school mathematics students for whom a graphic calculator is a standard mathematical tool, with that of students without access to graphing technology. Although the conditions and testing procedure were quite different to the case-study described in this chapter, there were some common findings in terms of the ability to translate from graphic to symbolic forms. The technology had an important influence on both the approaches employed by students and on mathematical attainment. The group that used the graphic calculators achieved markedly superior scores, with female students achieving higher scores relative to males. Ruthven offers a possible explanation for these results (the explanation is appropriate for both Ruthven's experiment and the case-study described in this chapter). He explains how males tend to display less anxiety under conditions of uncertainty, and tend to outperform females on tasks that require visual abilities. Appropriate training with dedicated software/technology, including feedback, leading to increased confidence and competence can reduce uncertainty and thus diminish anxiety. This is particularly likely to improve the performance of female students.

5.6. Summary

This experiment has considered a specific teaching and learning approach, using a specific software package, for a specific subject domain. Nevertheless, taking into account its limited application, it does offer strong evidence that the employment of computer-based visualisation coupled with a constructivist approach can have a positive effect on student achievement. As a result, students adopt a more holistic approach to problem solving, and develop a broader skills base.

A summary of the outcomes of the case-study is provided below.

Comparison of control and experimental groups:

- The control group scored better procedurally than the experimental group, as expected, but was poor visually. The traditional approach limited students to Group A skills.

- The experimental group scored better visually than the control group, and in turn demonstrated higher order (Group C) skills, even though they were relatively poor procedurally. The constructivist visual approach enhanced visualisation ability, and in turn higher order skills. It not only developed a broader skills base, but provided more flexibility in moving between the skills groups when applying them.

- Visualisation influences learning outcomes and helps in the application of knowledge to other subject domains.

- The experimental group developed a more holistic view of mathematics, and better strategies for problem solving. They demonstrated global thinking, and could check answers by more than one approach. The control group did not seem to have the necessary problem solving skills, even in an instructivist manner (the skills required for successful problem solving are discussed in Chapter 6).

- The experimental group could take knowledge gleaned from local tasks and apply it globally. Their understanding was independent of specific functions studied, so that the conceptual knowledge acquired could be applied to any function, i.e. the knowledge gleaned from local tasks could be applied globally. Their knowledge was therefore relational as opposed to instrumental. The control group's sequential style hindered progress as they had no recipe or template for the solution.

- The experimental group could apply generic concepts to specific examples, whereas the control group could only follow rules without any critical evaluation.

- The experimental group demonstrated better linking between visual and symbolic representations.

- The experimental group was motivated, and enjoyed the constructivist visual approach.

- The constructivist visual approach proved beneficial for all abilities.

- A Vygotskian perspective of constructivism incorporating strategic questioning, and promoting social interaction, has a positive effect on the learning process.

- It was observed in Chapter 4 that assessment which allows the use of a CAS should enable students to demonstrate higher order skills, but students still choose to perform procedural skills as a result of the nature of their learning experiences. However, students do demonstrate higher order skills when assessed after learning via a constructivist computer-based approach.

- The software supports a constructivist approach, and in turn develops students' skills. It allows for the conjecture and rehearsal of general relationships. Feedback regarding software use, in terms of usefulness, enjoyment and motivation, was very positive.

Other specific comparisons:

School:

- The control group results suggested that schools that encourage the studying of 3 science subjects score better procedurally than visually. Schools that encourage the studying of a variety of different subjects exhibit no difference in procedural and visual ability.

- Based on the comparison of control and experimental visual scores, the constructivist visual approach was more beneficial for School A.

Sex:

- For the control group: Males scored better procedurally than visually. Males scored better procedurally than females. Females demonstrated no difference in performance procedurally and visually.

- For the experimental group: Males were better visually than procedurally. Females were very good visually (much better than procedurally), were much better than the control group, and were much better visually than the experimental group males.

- Based on the comparison of control and experimental visual scores, the constructivist visual approach was more beneficial for females than for males.

Subject:

- For the control group: Mechanics students were better procedurally than visually. Statistics students were slightly better procedurally than visually.

- For the experimental group: Mechanics students were much better visually than procedurally. Mechanics students scored slightly better visually than the Statistics students. Statistics students scored better visually than procedurally, and much better visually than the control group.

- Both Mechanics and Statistics control groups scored better procedurally than visually, despite the visual nature of Mechanics and Statistics questions in general.

- Based on the comparison of control and experimental visual scores, the constructivist visual approach was more beneficial for Statistics students than for Mechanics students.

GCSE Grade:

- For the control group: A/A* students were much better procedurally than visually. B/C students demonstrated no difference in performance procedurally and visually.

- For the experimental group: A/A* students were much better visually than the control group. B/C students were better visually than procedurally, and were slightly better visually than the control group, even though extremely poor procedurally. A/A* students were much better than B/C students on both types of question, across both groups.

- Based on the comparison of control and experimental visual scores, the constructivist visual approach was slightly more beneficial for A*/A students than for B/C students.

The remainder of the thesis is concerned with a theoretical framework and future developments. Chapter 6 considers a teaching and learning framework to underpin and co-ordinate aspects of mathematics education with visualisation considered thus far. This framework builds on existing theories of teaching and learning in mathematics education. Chapter 7 offers some potential opportunities for future research as a result of the outcomes of the case-study.

CHAPTER 6

Visualisation and Constructivism: their Roles in Mathematical Cognition

Mathematicians are like Frenchmen: whatever you say to them they translate it into their own language, and forthwith it is something entirely different.

GOETHE

6.1. Introduction

The learning investigation reported in the previous chapter has shown evidence to support the view that a coherent combination of a constructivist approach to learning and teaching, together with increased emphasis on visualisation, not only enhances students' conceptual understanding of the mathematical topic under study, but can also enhance the students in their acquisition of higher order mathematical skills.

Clearly the case-study in Chapter 5 is based on a limited study of students at a particular stage in their mathematical development, and is restricted to one mathematical topic. More generally, questions of interest are:

- Are such (apparent) benefits to learning equally applicable to all stages of mathematical development, i.e. from primary school mathematics onwards, or are the main benefits to be found at later stages of development once a foundation of subject knowledge has been established?

- Are such (apparent) benefits to learning applicable to all subject areas within mathematics? For example, can constructivism and visualisation play a significant role in the development of students' ability with proof construction?

- Are there any learning and teaching 'blockages' in today's educational system that might prevent a constructivist and visual approach to learning taking place? For example, does the form of student assessment mitigate against such approaches?

In this chapter, a framework is proposed to link together and explain fully how constructivism and visualisation contribute to mathematical skills development, as well as conceptual understanding. The mathematical skills groupings described initially in Chapter 2 are assumed here to be desirable (by teachers) and achievable (by students) to some degree across the mathematics curriculum. Elements of existing learning theories, such as those described in Chapter 2, are used to develop this framework, but the emphasis here is on the benefits of visualisation and constructivism and the development of skills in particular, not just the development of conceptual understanding which features in many cognitive theories. Thus the framework proposed here is purposely meant to be of benefit to a student and/or a teacher in understanding and managing the learning process.

6.2. The Role of Visualisation in Mathematical Cognition

As Hitt (1998a, 1998b) and Duval (1995, 1999) have indicated, there are many forms of representation that a student of mathematics can experience. Broadly speaking, these forms can be described as:

- **Discursive** (sometimes referred to as 'symbolic', 'algebraic', etc.) forms such as the use of natural language in presenting definitions etc., and syntactical language such as algebra, logic, etc. to represent equations and formulae.

- **Visual** (sometimes referred to as 'pictorial', 'diagrammatic', etc.) forms such as graphs, geometrical figures, flow charts, etc. to represent actual physical objects such as 3-dimensional solids or mathematical objects such as a graphical representation of a complex number.

The skill of visualisation, or visual thinking, in a mathematical sense stretches beyond that of merely being able to identify aspects of a given picture, but should include being able to identify relationships between sub-components and/or geometrical transformations of the picture, identify strategies for selecting information from the picture that allows for the possibility of advancing towards a solution to a problem, and be able to link the visual representation with existing mathematical knowledge in whatever representational form it is in. In this respect, visualisation in mathematics is a more demanding skill than merely adding to knowledge by storing pictorial information (i.e. a Group A skill). Visual thinking leads to the creation of mental images as described by Dreyfus (1991) in Chapter 2.

All representations can support enhanced understanding of mathematical concepts, and can be utilised to increase mathematical skills, i.e. there is no superior representation in general. For some concepts and mathematical subjects it may be that the student does not have to switch from one form to another, or indeed has a choice of representations to choose from to solve a particular problem. The strategy of choosing appropriate representations in an optimum way is the important skill.

For a given problem there is likely to be an optimum solution strategy that may involve discursive forms, visual forms or a combination of both. What is important is that a learner is free to consider different representation strategies before choosing an optimum, and is not cognitively constrained or forced into a particular choice of representation that may not help find a solution.

The outcomes of Chapter 5 suggest that it is the switching between representations, particularly from the visual to the symbolic, that students find most difficult. It is also evident that, for a particular mathematical topic, different thought processes or higher order mathematical skills may be required for the 'symbolic to visual' transformation as opposed to the 'visual to symbolic' reverse transformation. For example, representing an equation as a graph *may* only need the process of obtaining an array of x and y values and knowledge of how to plot coordinates to achieve the conversion. Given the graph of the equation, knowledge of turning points, coordinate values of x and y intercepts, etc., and a strategy for assimilating such information, may all be necessary to complete the conversion to symbolic form. In other words, in some cases it may be possible for a student to change representations merely by following a known (to the student) algorithmic process; in other cases, and it would seem more likely in the visual to symbolic conversion, this algorithmic process may not exist and some higher order strategy is needed.

The evidence of Chapter 5 suggests that visual thinking, and the students' ability to switch naturally between representations of mathematical objects when appropriate, must be a key feature of any learning framework.

6.3. Problem Solving and the Role of Visualisation

As a student progresses towards the solution of a problem, this strategy of switching between representational forms can be influenced in a number of ways.

For example, if asked to solve a pair of linear simultaneous equations expressed in algebraic form and given that there is a unique solution, students are likely to stay in the given discursive form, solve the problem analytically, and present the results without recourse to a graph, even to check their analytical work. If the same problem had been posed instructing the students to graph the equations and then solve the problem, then students have to be able to switch between representations and convert the algebraic form to a visual (graphical) form and then interpret their graphs. If no information about a unique solution had been given initially, then students may well be expected to switch first to a graphical form to gain pictorial information relating to the number of solutions. The solution may then be obtained by working in either representation form.

As another example, 2-dimensional geometry problems often start with a visual representation of the problem to be solved. The solution may utilise geometry theorems (often presented in natural language) that can convert the problem to one of algebra. Alternatively, the solution may be possible by manipulating the given figure to produce another figure that is easier to interpret, a solution process that is now increasingly likely with the use of appropriate computer software. This latter solution does not involve a switch outside visual representational form.

The ability to solve a particular mathematical problem requires at least one of two processes to take place. From the given starting point, an **advancement** has to be made to enable the final goal of the problem to be reached. Along the way, this advancement may require a **conversion** (or conversions) from one representational form to another, as discussed in the examples above. For a given problem, a solution strategy must encompass a strategy for advancement. If this strategy does not require a change in representation then a conversion strategy is not needed. In general, one or more conversions may be required.

A schematic problem solving 'flow chart', designed to illustrate the processes involved and their interaction, is given below in Fig. 6.01. In the 'start' phase, the student forms a perception of the problem, based on their existing knowledge base, and also seeks to confirm the goals or objectives of the problem. Ideally, the next stage should be the 'conversion' phase, where the student considers appropriate representational forms and switches between them if necessary to produce choices of possible problem representations, i.e. begins to develop possible strategies for advancement (this is a more encompassing version of Vinner's work on concept definition and concept image, as discussed earlier in Chapter 2).

In the 'advancement' phase, these different strategies are inherently given a ranking and tested one-by-one. In the event of failure with one strategy, alternatives may include trying to solve a simpler problem first and then generalising the solution later, and/or solving an analogous problem that may indicate a refinement of an existing strategy. Hence, hopefully, the solution process is advanced.

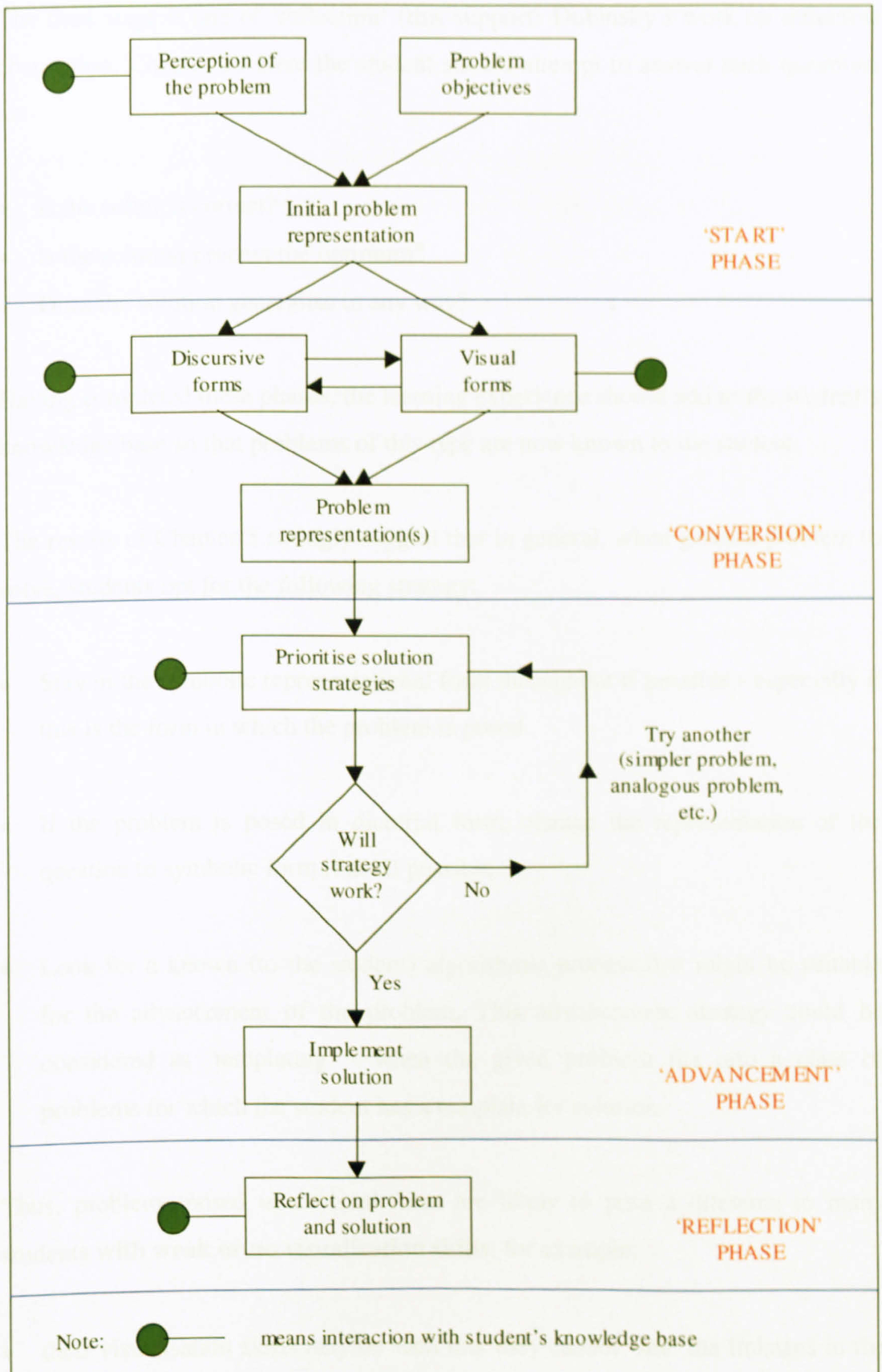

Perception of the problem
Problem objectives
Initial problem representation
'START' PHASE
Discursive forms
Visual forms
Problem representation(s)
'CONVERSION' PHASE
Prioritise solution strategies
Try another (simpler problem, analogous problem, etc.)
Will strategy work?
No
Yes
Implement solution
'ADVANCEMENT' PHASE
Reflect on problem and solution
'REFLECTION' PHASE
Note:
means interaction with student's knowledge base

Fig. 6.01.

The final stage is one of 'reflection' (this supports Dubinsky's work on reflective abstraction, Chapter 2). Here the student should attempt to answer such questions as:

- Is the solution correct?
- Is the solution process the optimum?
- Does the solution generalise in any way?

Having completed these phases, the learning experience should add to the student's knowledge base so that problems of this type are now known to the student.

The results of Chapter 5 strongly suggest that in general, when given a problem to solve, students opt for the following strategy:

- Stay in the symbolic representational form throughout if possible - especially if this is the form in which the problem is posed.

- If the problem is posed in pictorial form, change the representation of the question to symbolic form if at all possible.

- Look for a known (to the student) algorithmic process that might be suitable for the advancement of the problem. This advancement strategy could be considered as 'templating' - when the given problem fits into a class of problems for which the student has a template for solution.

Thus, problems posed in a visual form are likely to pose a dilemma to many students with weak or no visualisation skills, for example:

- their visualisation skills may be such that they cannot 'see' the linkages in the pictorial information to their own established mathematical knowledge base to enable them to proceed with a solution process without changing representational form, i.e. they cannot assimilate such information to add to their existing cognitive structure.

- their desire to move from a visual representation to a symbolic form may add another possible source of student error and confusion, and is not necessarily seen as advancing the problem towards a solution.

- the move to a symbolic form may produce a problem representation that does not fit into a student's known template for a solution process, i.e. a **perturbation** to the established cognitive structure, and hence again the student is not advancing towards a solution.

- if the posed problem asks for some interpretation or conjecture which is most easily made in visual form, then the switching of representations from a symbolic back to a visual form may introduce even more error.

In essence, students with weak visualisation skills are potentially limiting the number of alternative strategies that they can generate in the advancement phase because of a poor conversion phase. They have not developed the necessary higher order skills to be able to generate alternative solution strategies, and therefore if they cannot 'fit' the problem into a familiar template, then they will not be able to advance (i.e. they will be stuck in the loop of the advancement phase). Equally, the reflection phase may be limited for such students, which will mean their knowledge base will not expand significantly.

Additionally, there is the apparent perception by many students (and maybe teachers as well) that reasoning and problem solving can ONLY be done in symbolic and/or numerical form 'as it is more accurate (and will earn more marks in an examination)'. Hence a visual approach 'is not to be favoured as it only illustrates something, but does not prove anything'.

The case study in Chapter 5 showed clearly that the favoured problem (from an instructivist teaching perspective) was one that was posed in symbolic form, required an answer in symbolic form (and hence no representation switching was needed) and for which there was a well-known and well-practiced template. Reflecting on the answer, for example discussing its implications or generalisation, was not liked. Ability to 'do' such problems is also seen generally as showing good mathematical ability, and having significant rewards in public examinations, at least up to A-level.

6.4. Mathematical Skills and Visualisation Revisited

The evidence from the case-study in Chapter 5 reinforces the view held by many (Etchells and Monaghan, 1994; Monaghan et al., 1998; Porkess, 2002) that the majority of mathematics students entering university have mainly Group A skills, and perhaps some Group B skills relating to knowledge transfer to other applications. It has been discussed in Chapter 4 that visualisation as a skill in its own right pervades and reinforces most of the Group A, B and C skills.

In the problem solving process considered above, the conversion skill of being able to move smoothly when appropriate from one mathematical representation to another is at least a Group B skill. The advancement process requiring the generation of alternative strategies and accepting/rejecting them certainly involves Group B and C skills. The point of emphasis here is that **at this stage in the students' mathematical development their solution strategy is one of trying to reduce every problem to one that can be solved using Group A skills alone.** A weaker student with only Group A skills anyway is not likely to be able to achieve this reduction process and hence has to rely on his or her knowledge of a template for the given problem.

Thus any cognitive framework should emphasise the point that a mathematical knowledge base containing facts, formulae, definitions, subject knowledge, etc. is not sufficient on its own. Rather, mathematical expertise consists of an expanding knowledge base, incorporating the student's own personal set of mental images (in accordance with Dreyfus's work on mental processes and Harel and Kaput's work on conceptual entities, Chapter 2), plus a developing set of mathematical skills that interact effectively with the knowledge base. A schematic diagram illustrating this iterative development in the learning process is given in Fig. 6.02. Once again, the conversion between both discursive and visual forms of representation is shown explicitly within the constructivist teaching and learning process to emphasise its role in the development of expertise (the importance of the conversion process, which links representations, is stressed by both Dreyfus and Hitt, Chapter 2). The student's existing expertise may be limited, particularly in terms of both mental images and range of skills. The constructivist teaching and learning process, which encourages communication (Laurillard discusses the benefits of a communication model, Chapter 2), develops the higher order skills that are required in order to move successfully through the advancement phase of the problem solving flow-chart earlier. In this manner, the student's expertise is reformulated, as it not only has stronger conceptual links between subject knowledge and mental images, thus promoting relational as opposed to instrumental understanding (in accordance with Skemp's work, Chapter 2), but now has a more developed skills base so that the knowledge base can be applied successfully in problem solving.

In an iterative fashion, the newly acquired expertise becomes the foundation (i.e. new starting point) for further constructivist activity (this is analogous to Buchberger's White-Box/Black-Box principle, Chapter 2). An instructivist approach would not focus on the conversion process, and would thus not develop the higher order skills required for successful problem solving.

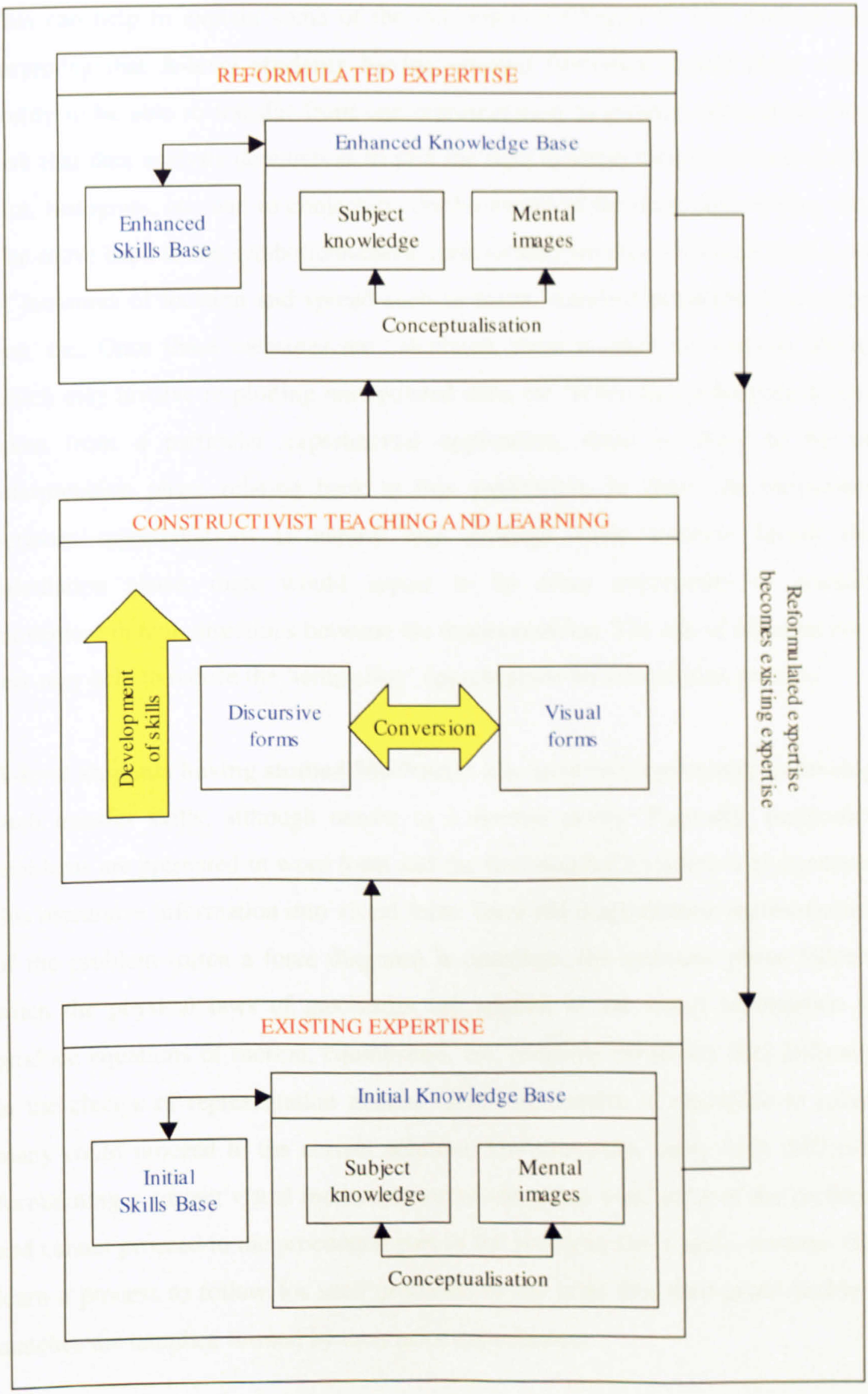

REFORMULATED EXPERTISE
Enhanced Knowledge Base
Enhanced Skills Base
Subject knowledge
Mental images
Conceptualisation
CONSTRUCTIVIST TEACHING AND LEARNING
Development of skills
Discursive forms
Conversion
Visual forms
Reformulated expertise becomes existing expertise
EXISTING EXPERTISE
Initial Knowledge Base
Initial Skills Base
Subject knowledge
Mental images
Conceptualisation

Fig. 6.02.

This can help to explain some of the outcomes of Chapter 5. It is perhaps not surprising that **A-level students having studied Statistics** should show some ability to be able to transfer from one representation to another. Almost the first task that data analysis demands is to plot the data in some form such as a scatter plot, histogram, etc. and to conjecture, on the nature of the data, any outliers, etc. The move back to the symbolic/numeric form usually involves calculation of a set of measures of location and spread such as mean, standard deviation, regression line, etc. Once these measures are calculated, there is often an analysis phase, which may involve re-plotting manipulated data, etc. When the underlying data is taken from a particular experimental application, there is likely to be an interpretation phase relating back to this application. In short, the movement between representations is natural and although some students favour the calculation phase, there would appear to be every opportunity to practice movement in both directions between the representations. The use of different data sets may help to reduce the 'templating' opportunities for the solution process.

A-level students having studied Mechanics also have had opportunity to develop such transfer skills, although maybe to a limited extent. Typically, mechanics problems are presented in word form and the first step for a student is to represent this discursive information into visual form. Once the diagrammatic representation of the problem (often a force diagram) is complete, the symbolic phase follows when the physical laws of mechanics are applied to the visual information to produce equations of motion, equilibrium, etc. Students invariably find difficulty in the change of representation phases. Given the system of equations to solve, many could proceed to the correct solution. Unfortunately, many have difficulty formulating a correct visual representation of the given word form of the problem and cannot proceed to the procedural part of the problem. Once again, students can learn a process to follow for such problems in the hope that their given problem matches the template formed by their prior experiences.

As seen in this thesis, students of pure mathematics up to A-level can often rely on the symbolic/numerical representation to solve problems and would appear to want to do so. In many examples quoted in this thesis, the visual nature of mathematics is seen as a support rather than an integrated enhancement to conceptual understanding. The movement from the symbolic to the visual, for example when plotting an equation on a Cartesian graph, is often an algorithmic process (Group A skill) that students can cope with. The movement from a visual representation to a symbolic one is often not algorithmic, requires Group B or C skills, and is found to be difficult by students.

6.5. Visualisation Skills and Constructivism

The case for the development of visualisation skills has been made above, particularly in terms of development of mathematical expertise. The teaching and learning process has been seen to influence strongly the development of this expertise also. The iterative development of this expertise, depicted in Fig. 6.02, would appear to depend very much on the linkages between the skills, the subject knowledge, the mental images, etc. at any one time, so that new skills, facts, images, etc. and the linkages between them can be easily assimilated by the learner.

The evidence from Chapters 3 and 5 suggests that an instructivist approach to teaching and learning tends to promote these items of expertise as essentially separate ideas with the links between them often lacking. A student may learn a mathematical definition but have no pictorial representation, either externally or internally as a mental image, to reinforce this definition with meaning and possible application (in Fig. 6.02, the conceptualisation link between subject knowledge and mental images would be missing, as would the conversion process and the resultant skills development). A mathematical process may be familiar to a student, but if the connections between, and generalisations of, similar processes do not exist within the student's expertise, then the knowledge base may reduce to a potentially large set of such processes which become increasingly difficult to commit to, or retrieve from, memory. To some extent, using the terminology of Buchberger, as discussed in Chapter 2, this means that the student is seeing learning as a collection of 'white boxes' and cannot necessarily consolidate knowledge into the 'black box' phase.

A constructivist approach to teaching and learning, if applied properly along the lines described in Chapter 3 and exemplified in Chapter 5, attempts to promote these linkages explicitly, primarily by self-discovery. This style of learning should train students to seek linkages between pieces of information when they believe they are missing, construct more appropriate mental images to aid understanding of concepts, conjecture results and generalisations of particular cases, and hence develop mathematical skills. In this way, for example, the white-box/black-box principle of Buchberger is possible for a student to attain in the way their learning develops.

Of course, visualisation and constructivism are not directly connected. An instructivist approach can include drill practice for students in the conversion from one representational form to another so that a student may acquire some visualisation skills. Equally, a constructivist approach may still focus on a more symbolic form of problem solution so that the learner's solution strategies are still potentially limited. However, visualisation skills and a constructivist approach to learning are complementary in that, on the basis of the evidence from Chapter 5, both add more naturally to the development of mathematical expertise.

6.6. Teaching, Learning and Assessment 'Blockage' Issues

In this section, factors that may inhibit the potential for the development of visualisation skills and/or a constructivist approach to teaching and learning described above, are considered.

6.6.1. Teaching

The teaching acts as a role model for the learning process and in particular the problem solving process. Visualisation skills will not develop in students if the teacher has devised a style of teaching that excludes the need for conversion practice. There may be many plausible reasons why this takes place, for example:

- the teacher's own mathematical training has not led to the development of visual skills and hence the teacher feels more secure using symbolic forms.

- the teacher does not see the benefit of such visual skills - the syllabus is not likely to specify this skill as a core requirement or learning objective, and his or her student pass rates may be sufficiently good.

- the teacher is under pressure to get through a prescribed syllabus and achieve acceptable pass rates - in which case the 'safest' process might be to train students to perform a number of methods capable of solving particular problems.

- the teacher wants to control the learning process, and finds such control difficult when trying to teach visualisation skills because there isn't a single instructional process that will cater for problems involving conversion.

Some of the above are also reasons cited for teachers adopting an instructivist rather than a constructivist approach to teaching and learning. However, another important factor inhibiting the use of a constructivist approach is again that of time. The evidence of the case-study in Chapter 5 suggested that the teacher has little time to investigate whether students have developed correct mental images, representational linkages, etc., especially on an individual student basis. A constructivist approach may well widen the time needed between 'more able' and 'less able' students to achieve similar skills development, even if motivation to learn is increased.

6.6.2. Learning

It has been shown that, when solving problems, learners of mathematics undergoing an instructivist teaching style:

- prefer to remain within the one representational form if possible (usually a symbolic one).

- prefer the symbolic discursive form rather than the visual form if given a choice of representations to solve a problem.

- have difficulty moving from one representational form to another and in particular more difficulty moving from the visual form to the discursive form.

The first preference is a natural one. Many problems are 'well-posed' and have a neat, closed solution and, more significantly, students expect such questions, begin to doubt their answers if they look 'untidy' (such as non-integer answers for solutions of linear equations with integer coefficients), and expect that the form of the question is indicative of the best form in which to proceed unless they have already met and practiced similar questions to the contrary.

The second preference is more strategic. As already discussed in this chapter, moving from one representation to another, particularly from the visual to the symbolic, can introduce error without necessarily being seen to advance towards a solution (and hence earning marks). Also, it is more likely that a problem posed in symbolic form will have an algorithmic 'template' for solution that the student recognises and can apply. Even if symbolic manipulation errors are made, students can still be rewarded with marks for method.

The difficulty of switching from one representation to another is more a matter of cognition. Students need sufficient practice to develop this skill of making strategic decisions as to when to utilise a certain representation. This is an important problem solving skill and should not be practiced on problems involving new (to the student) mathematical knowledge and concepts until this new material is fully consolidated into the student's cognitive structure. Otherwise, it is highly likely that a student will struggle to differentiate between lack of understanding of knowledge or lack of a capable problem solving strategy when faced with difficulties.

The other major obstacle would appear to be that of time. Increasingly in these days of modularity, learning has to be done in 'bite-size chunks', giving learners little time to reflect and assimilate learning before they are assessed on their knowledge, and also the feeling that once learning has been assessed it can then be forgotten. A task-driven approach to learning may mean that students prioritise their learning developments in a different way to those anticipated by the teacher.

6.6.3. Assessment

In recent years, many teachers, at least in the UK, have faced the prospect of their teaching performance being judged by the performance of their students in public examinations. Students are also under increasing pressure to obtain high grades, particularly for example A-level grades to aspire to select universities. Success or otherwise of government education policy has been often measured by national pass rates alone, with higher pass rates usually accompanied by cries of lowering of standards. Thus, there is plenty of support for the notion that, in schools at least, assessment drives the teaching and learning process and hence any attempt to influence teaching and learning performance must consider assessment first.

Given the statement above, that students will attempt to solve problems by reducing a given problem to a set of Group A tasks, it would seem highly desirable that assessment tasks should be set so that:

(i) such a reduction is only a relatively minor part of a larger problem and that a demonstration of Group B and/or Group C skills is needed for a complete solution.

(ii) the demonstration of conversion skills within the advancement of a problem solution is recognised and rewarded sufficiently to motivate students and teachers to reflect on the need to develop such a skill.

(iii) the reflection phase of the problem solving process described earlier, including the ability to generalise, has more significance.

Such a move would require careful judgement, particularly in public examinations. The main difficulty would be to ensure that the assessment considers not just that the mathematical topics studied are covered, but also an appropriate spread of skills ability is assessed. There must be sufficient spread of Group A, B and C skills, including visualisation skills, to ensure that gifted students can continue to score well and yet weaker students have adequate opportunity to achieve at least a pass grade.

It may well be that written, timed examinations are not the most suitable form for such an assessment and that more prominence should be given to course work to assess the Group B and C skills. A number of A-level Examining Boards have adopted this approach in their assessment of Statistics.

6.6.4. The Role of Computer Technology

Perhaps the one single influence that could change the current situation is the desire by many, particularly in government, to utilise computer technology effectively in schools. In Chapter 5, it was the use of software, designed and implemented with a particular teaching and learning function in mind, that had the most impact on students' motivation and desire to learn, and to take responsibility for their own learning.

Graphic calculators, and more generally computer algebra systems (CAS), are prime examples of a technology where both the symbolic and visual representational forms are readily available to a student. The use of software that encourages the conversion process to arise naturally in the learner's development of mathematical expertise should be encouraged. The difficulty to overcome would seem to be that a CAS can generally do many of the Group A tasks well and is seen as a threat by those students whose expertise is limited to performing such tasks themselves, usually in pen-and-paper mode. Using a CAS when attempting Group B and C tasks usually means that the user must have these skills developed, as a CAS by itself does not have much 'intelligence'. Hence we have a situation where the technology has the potential to widen the skills gap between the 'more able' and 'less able' student.

6.7. Other Opportunities for Visual Skills Development

The above has concentrated on learning, teaching and assessment given the curriculum as it currently stands. Opportunities for more practice of transfer from one mathematical representation to another within mathematics syllabuses are discussed here.

An obvious strategy is to include more aspects of geometry - a subject that has seen a decline over recent decades. However, mathematics educators are now supporting moves to reverse this trend. Geometry is clearly visual and requires a clear understanding of the pictorial information needed in a diagram and its links to mathematical knowledge. For example, given a diagram involving circles, the links to knowledge such as properties of radii, tangents, areas, etc. should be made immediately and a strategy devised as to what information will be most useful for a given problem. Geometry problems also offer opportunities for a solution without moving to the symbolic form, for example by mentally or physically rearranging parts of the given diagram to transform shapes, or when considering similarity or perspective. Of course, geometry problems still allow for symbolic/numeric manipulations and the notions of proof.

Another topic that can enrich the development of mathematical skills and expertise, by requiring adaptability in the different forms of representation, is that of mathematical modelling. If the typical 'template' for the modelling process is examined (see Fig. 6.03 below, from the Open University Course MST204 Mathematical Models and Methods), it can be seen how visualisation skills can be incorporated in all of the aspects. The real world application can start with visual as well as numeric data, and can drive a requirement that the analysis and outcomes be similarly described in a visual as well as a symbolic form. The mathematical solution part might be purely symbolic, or a mixture of symbolic and visual in a manner similar to that described above. Of course, the modelling application can only really draw upon a student's expected mathematical subject knowledge and the suitability for a given student cohort would need to be carefully thought through. There is the possibility that something of this type, in limited form, will feature in the A/S-level 'Use of Mathematics' proposed by the AQA examining board for 2003 start. It is interesting to note however that this syllabus is recommended more for those not pursuing a full A-level in mathematics.

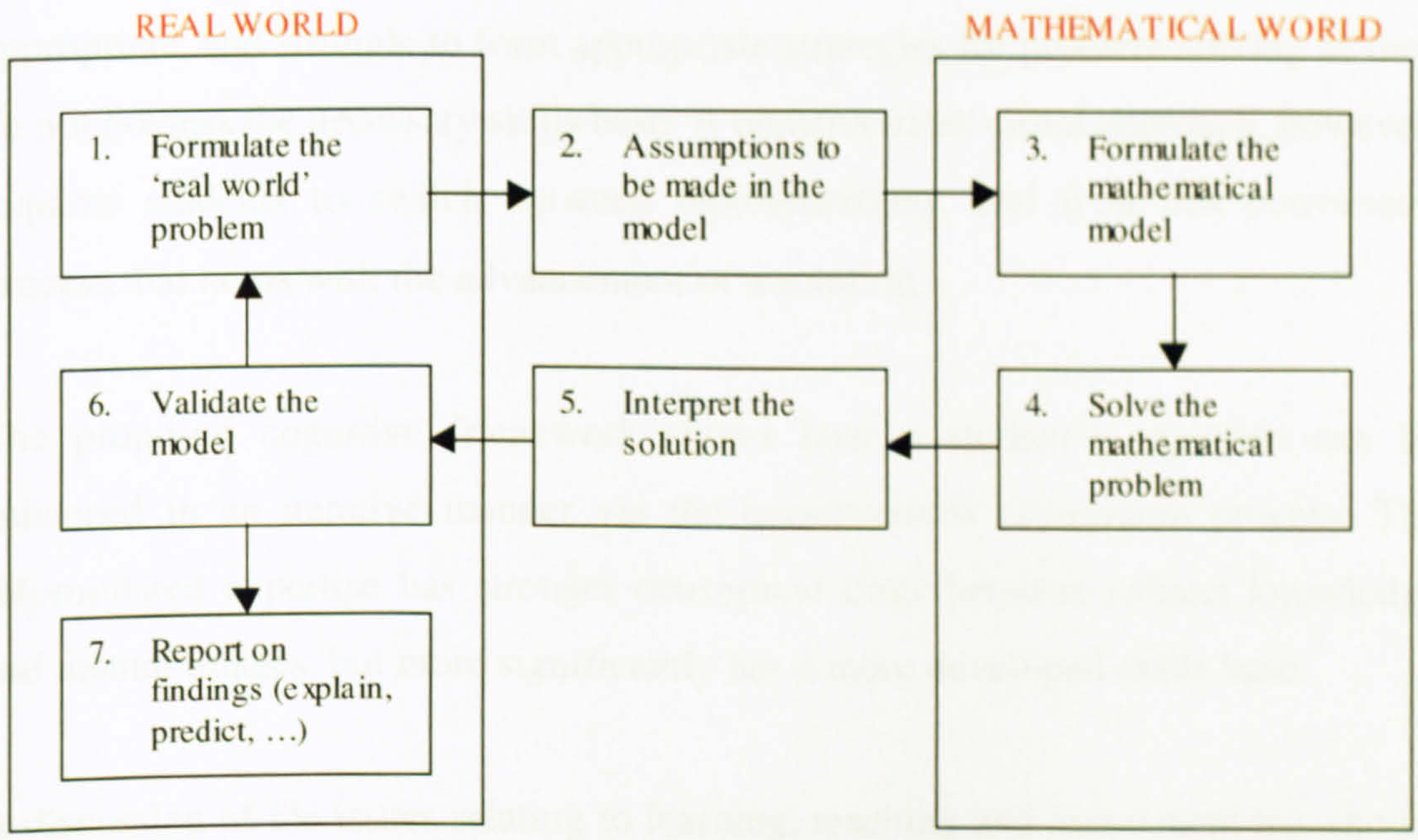


Fig. 6.03.

It is generally recognised that course work is a more appropriate form of assessment here than written, timed examinations and that aspects of assessment such as oral presentations can play a powerful role in developing students' skills.

Educators seeking to develop a wider range of mathematical skills in more students may need to reflect on the inclusion of mathematical topics that more evidently encourage and allow for the development of such learning skills, perhaps at the same time reviewing (with the aim of reducing) the breadth of topics now prevalent, particularly on A-level syllabuses. Such a reduction could be achieved by taking into account those elements of mathematical knowledge that can easily be regenerated and assimilated by students using computer technology effectively.

6.8. Summary

Having experienced instructivism, many students choose to apply algorithmic approaches to solve problems, even when a visual approach would be more appropriate, and struggle to form appropriate strategies for problem solving as they do not possess the necessary skills base. A constructivist visual approach, however, requires students to switch between representations, and it is this conversion process that helps with the advancement of a solution.

The proposed cognitive framework shows how a student's expertise can be enhanced in an iterative manner via the constructivist conversion process. The reformulated expertise has stronger conceptual links between subject knowledge and mental images, but more significantly has a more developed skills base.

A discussion of the issues relating to learning, teaching and assessment has shown that the transition to a more constructivist visual approach is not straightforward. The integration of technology into mathematics classrooms is a positive step, as it encourages the conversion process, however usage often requires the application of higher order skills that may well not be developed in students.

Alternative topics, such as geometry and mathematical modelling, can support the transfer between representations, and can encourage students to develop higher order skills, as it is not easy for students to convert such tasks into ones that require the demonstration for Group A skills only.

CHAPTER 7

Future Developments

Mathematics is the only infinite human activity. It is conceivable that humanity could eventually learn everything in physics or biology. But humanity certainly won't ever be able to find out everything in mathematics, because the subject is infinite. Numbers themselves are infinite.

PAUL ERDOS

7.1. Introduction

The foundations were laid in Chapters 1 and 2 by setting the scene regarding current thinking within mathematics education, highlighting the main areas of concern for both current and future research issues, and reviewing the effectiveness of previous work. Chapter 3 considered alternative methodologies for the learning of mathematics, and made a case for constructivism by discussing the benefits to students of adopting this approach. Chapter 4 considered the types of skills that students possess and those that mathematics educators consider more desirable, i.e. higher order skills such as the ability to conjecture and evaluate, not just lower order skills such as the ability to follow procedures. These first four chapters included the necessary background for the learning case-study described in Chapter 5, which demonstrated that a constructivist approach to learning mathematics which employs computer-based visualisation can result in students developing an enhanced skills set. These positive findings were then formalised into a generalised learning framework proposed in Chapter 6.

This final chapter considers future research that could be undertaken to extend the work of this thesis, and is split into two main sections. First, consideration is given to further developments concerning the case-study, and secondly, possible future research considerations for mathematics education in general are discussed as a direct result of the findings of this thesis.

7.2. Further Developments Concerning the Case-Study

This section considers further developments as a direct result of the case-study in Chapter 5.

Such developments include software enhancements, taking into account student feedback, and improvements concerning the execution of the case-study.

7.2.1. Software Enhancements

Ideas are suggested on how to improve the learning and teaching properties of the software, supported by comments from students, and a justification of current features is provided where necessary as a response to student feedback.

Student comment: **"Maybe it could be made so that you are able to choose your skill level and then be able to proceed to harder questions when you feel ready to."**

Presumably students feel that 'harder questions' are ones that are not recognisable (in terms of using a template for solution), and thus require the application of higher order skills. The above suggestion is a good one, however perhaps a better solution for future software would be to have a built-in level checker, i.e. diagnostic test, which only lets users proceed when they have reached mastery of that particular concept, as students may feel they are ready to continue when actually they are not. Students should only be allowed to move on when they have *demonstrated* that they have mastered the stage in question. This will require careful testing via questions that are novel to students.

Due to differences in ability, students clearly wanted different things from the software. A number of students wanted questions relating to more basic graphs to start with, but at the same time, a number wanted questions on more difficult graphs. If the software were to be used as part of students' school study, then there would need to be a wider variety of graphs in order to cater for diverse needs, and to provide additional practice. This difference in needs and desires will always exist with differing abilities. This is where the diagnostic test, as suggested above, comes to the fore, in order to start students at their appropriate levels. In this manner, there could be many more easier and harder questions (in terms of complexity of function) to cater for everybody, including perhaps a revision of, or introduction to, basic functions.

The software was designed primarily to be used in a case-study to assess the extent to which students developed visualisation skills, particularly the conversion from pictorial to symbolic representations, as well as other higher order skills. If it were to be integrated into the A-level curriculum, then some sort of diagnostic testing would certainly be appropriate in order to cater for diverse abilities by determining appropriate starting points for student explorations.

The following comments have been grouped together as they all refer to the nature of the software in the same vein (many students expressed these views):

Student comment: **"The summary sheet may have been helpful at the start, before using the software."**

"More hints should be given."

"It needs a lot more explanation about how the different numbers change the graph, etc. (instead of it being a trial and error method)."

"More explanation of how to go about writing the equations rather than just playing around with numbers, i.e. what happens when you put certain numbers in."

"More explanation of what changing the numbers actually does in a general sense whilst using the package."

"Explain and show examples of different equations before questions."

"Some information on how altering the various changing factors can affect the shape of the graph."

"Maybe a bit more explanation in the 'Hint' section."

"A quick run through the way to work out the answers."
"More instructions and help."

These comments give us some insight into how the students prefer to study, i.e. to be given the facts and instructions, and then 'turn the handle'. Many students perceive a useful 'hint' as one that leads to the answer, whereas from a constructivist perspective, the teacher perceives a useful 'hint' as one that aids the exploration process, thus leading to enhanced learning. As suggested in Chapter 6, many of the students prefer to be explicitly told the procedure and follow it to the letter, rather than explore in order to discover mathematical ideas. This reflects the way they have been taught at school, so they have come to accept it as the preferred method. This very prescriptive approach clearly would have undermined the constructivist philosophy that the case-study supported, and therefore the above suggestions regarding *more* instruction would not be incorporated into future designs. The whole idea of the case-study was to change the emphasis entirely by encouraging students to learn via exploration. It is this constructivist exploration that is so productive in the whole learning process; it is a good example of incidental learning. After this exploration stage came the reflection process, where the ideas could be brought together, reinforced, and formalised with the aid of the summary sheet.

Student comment: **"More detailed instructions on how to use the software."**

The instructions section was purposely kept fairly brief to instil an exploratory mode in the students from the outset. Those who were unsure thus have to ask, thereby initiating the interaction process. This is a positive consequence, as the learning process clearly benefits from social interaction (as discussed in Chapter 3). On this basis, no additions to the instructions would therefore be made – it would merely delay the building of a relationship between student and teacher.

Student comment: **"More explanations of what the basic graphs looked like."**

It is difficult to understand this comment. All the students had to do was plot the basic graphs to see what they looked like. However, a possible explanation is that students were used to an instructivist style, and therefore did not even know where to begin, as they had not been told what to do explicitly. They did not equate the plotting of the basic graphs with the first stage of the exploratory process. Clearly no further explanations would be offered in future software, but it would be useful to explain more fully the way in which students would be expected to explore from the outset, and that they would be supported fully in their explorations. The fact that some students had not even met the basic graphs makes the superior visualisation scores of the experimental group, compared with those of the control group, even more impressive (see Chapter 5).

Student comment: **"If the student has no idea, even with hints, show the answer so that the student can take it 'on board' for the next question."**

The event of this happening is very rare, as the students are encouraged to converse with the tutor to initiate the strategic questioning process, and will therefore hopefully arrive at the answer via this constructivist help. A few students commented that a summary after each section/question, not just at the end, in order to take the concepts 'on board' for the next section/question, would be useful. This is possibly due to their desire to use this newly acquired information as a template for the remaining exercises. Students could then merely routinely apply the appropriate rule instead of the more desirable learning process of encountering perturbations, which when resolved lead to an accommodation into reformulated conceptual structures. It is therefore not a good idea to provide summaries at such regular intervals, but instead it would be useful in future designs to provide them after sufficient constructivist activity has taken place (perhaps after each family of functions, rather than right at the end), as a means of formalising their explorations.

Student comment: **"It would have been helpful to have more numbers on the axes."**

This lack of detail was an intentional design feature, so that students could concentrate on the 'big picture', i.e. view the whole graph as a conceptual entity (see Section 2.4.3.6 of Chapter 2) instead of focusing too much on detail, for example the coordinates of certain points. The above student comment illustrates a desire for explicit information to be provided. It would be preferable for students to make decisions as to the relevance of given information. Future software would adopt the same minimalist style, and students would be encouraged to ask for detailed information that *they* think is necessary and relevant, rather than the software or the teacher providing such information.

Student comment: **"I feel that the harder examples towards the end were a little too hard."**

Students have a tendency to associate 'unfamiliar' with 'hard'. As discussed in Chapters 5 and 6, if students have not been shown explicitly how to do something, then they struggle to apply their existing knowledge to these unfamiliar scenarios. The control group would appear to struggle more in their instructivist environments. This lack of necessary skills to tackle such problems can be a source of decreased motivation. The constructivist approach (adopted by the experimental group) lends itself to this type of scenario, i.e. building on existing knowledge and being more disposed to apply newly acquired knowledge to other situations. The understanding that has been built up is therefore both conceptual and relational, and success will lead to more success. In future designs, diagnostic testing, as described earlier, would provide the students who have not had the chance to build up higher order skills (due to instructivism) with more remedial examples of an appropriate level in order to equip them with the necessary skills to tackle the 'harder' examples towards the end.

Student comment: **"A mixture of computer and traditional methods would be ideal."**

This student has 'hit the nail on the head' with the above comment. This thesis wholly supports the idea that the computer should complement traditional approaches rather than act as a replacement. Mathematical technique is important, and the computer should not be used as a black-box if a level of procedural mastery has not already been achieved. The problem with more traditional methods, normally instructivist in nature, is that they concentrate mainly on the development of Group A skills, resulting in students always having a desire to transform problems into those that require the demonstration of Group A skills (as per the discussion in Chapter 6). The power of appropriate software is that it facilitates a constructivist approach for the development of higher order skills, so that when traditional approaches deal with subject knowledge and procedural mastery, students will know when and how to apply such techniques in problem solving. Future case-studies will continue with the philosophy that a computer-based approach is by no means a replacement for traditional methods, but one which offers the opportunity to develop higher order skills which will assist in the understanding and application of techniques developed via more instructivist approaches.

Student comment: **"The package could have built-in assessment."**

It is suggested here that the learning and assessment could be in the same compact package, where instant feedback on achievements can be received. Future software could incorporate such assessment at the start and at the end of software usage, as a motivating tactic, in order to help the students feel confident that they have learnt something. It is an instant way of checking whether or not they have actually understood, i.e. if the software has done what it set out to do. Monitoring of such assessment will also ensure intervention with the student, thus helping to establish the relationship between student and teacher.

Student comment: **"Points should be awarded for each attempt and bonus points should be given for getting the question right first time. You should receive a breakdown of your scores at the end."**

The use of instant assessment in this manner, to motivate and challenge students, is very interesting. It might make them think a little harder and conjecture before making an attempt, i.e. the application of higher order skills, rather than trying anything and seeing what happens. This would be a useful addition to future software. The use of randomisation to ensure that questions are different each time a student uses the software could enable students to have another attempt and try to beat their score. This could promote healthy competition. On the other side of the coin, however, it could have a negative effect on poorer students who get very low scores. Therefore there would need to be sufficient questions in the test that assess Group A skills in order for the poorer students to pass. Additionally, if monitoring of results took place, as described above, then this intervention would provide the necessary support.

Student comment: **"The graphs' axes could have changed automatically."**

This comment, it is assumed, refers to scale. It is useful for students to experiment with scale, as the concept of scale is important. Altering the 'look' of a graph by changing the scale is clearly not the same as altering the 'look' by changing the functional notation (as per the discussion in Section 5.3.3.1 of Chapter 5). This is something the students can explore to good effect. Additionally, students could plot a graph that is outside the 'plot area', so some adjustment of scale is necessary in order to proceed. This is clearly a useful tool that provides students with the opportunity to clarify the difference between changing the scale and changing the function composition, which would certainly be incorporated into any future designs.

Student comment: **"If your graph was very wrong more hints could be given."**

In future software, something needs to be in place for those who are not progressing at all. This could be achieved by offering additional, easier examples to start with, as suggested earlier. It was hoped, however, that students would ask the tutor and strategic questioning could be put into operation. Some students are reluctant to ask for help, perhaps because they are just shy, or perhaps for fear of embarrassment. This hurdle needs to be overcome so that tutor, computer and student can work together in harmony. The tutor must stress the importance of asking for help, as this way the students will get the most out of the session. In this manner, they can take advantage of the usefulness of the software and benefit from the knowledge and experience of the tutor. In future case-studies, the tutor must explain to the students the rationale for the Socratic dialogue, stress that it is actually *useful* to make mistakes as their remediation will lead to a deeper understanding, and stress that they will be fully supported in their explorations. This will help to change the attitudes of students so that they feel entirely comfortable interacting with the tutor, which is probably more difficult to achieve in schools than in universities. The relationship between students and tutor is vitally important so that this process can run smoothly.

Student comment: **"Why not give the program the ability to retain the old graph, maybe in a different colour, to show exactly what has changed, and if the graphs got cluttered by lines, you could have a 'graph clear button'. This would make even more obvious the effect of specific function changes."**

This is an excellent idea, which would significantly improve the software. Better still, the given graph could be plotted, and then the students' attempts could be overlaid in the same window. The incorporation of the two smaller plot windows into one larger, more useful one is a better format for purposes of comparison. This is something that is possible in CAS such as DERIVE. Future designs would combine the important features of the software with the plotting capabilities of DERIVE.

The following comments refer to the nature of the hints:

Student comment: **"The hints provided are at times too leading and too specific, and perhaps a more graduated hint system, gradually leading towards the answer, would be better."**

"A more detailed 'Hint' section, which gradually gives bigger hints the more times you use it, would be better."

These comments were not common, as more students preferred even more hints and explanations than were actually given. This point could be incorporated into future software by offering very little advice at first (merely encouragement to explore), then a little more direction each time, until finally providing the current hints. This would perhaps further encourage exploration by strategically 'nudging' them little by little in the right direction in the same manner that the tutor would. Ideally, the students could explore with the smallest amount of help possible in order to succeed. Care would need to be taken, however, not to make the process too prescriptive, which is totally against the constructivist philosophy.

Student comment: **"Something that told me if I was getting closer or further away from the answer."**

This could be a positive addition to future software for motivational purposes. This should, however, be fairly obvious depending on what the graph looks like compared with the given graph. In order to determine how 'close' a student is to the correct function would require artificial intelligence, which would be an interesting area for future research.

Positive feedback included students' comments such as, "**Great package**", "**Good and informative**", and "**It was extremely good**".

As well as the ideas on how to improve the software as a result of student feedback, the following ideas have also emerged as a result of carrying out the case-study:

It would be useful if the user had the ability to obtain the co-ordinates of a point by clicking on it with the mouse. The user could then click on a part of the graph that needed further scrutiny, and zoom in to examine it more closely. The user could click on screen to determine the y-intercept, roots, etc., and then derive the equation from the key features of the graph. The user could directly modify a graph by dragging it up, down, left, right, squashing it, stretching it, or inverting it with the mouse, and observe the resultant changing elements within the symbolic form. It would also be interesting to have the ability to draw a graph, and ask the computer to provide the symbolic expression that it represents to assist in the development of conversion skills from the pictorial to the symbolic (not merely from the symbolic to the pictorial, as with currently available graph plotting software). This perhaps could be possible further into the future of graphing technology.

Special features of the graph, for example the vertex of a parabola, or y-intercept, could be highlighted in order to track transformations. This could, however, detract from students focusing on the graph as a conceptual entity.

Perhaps of most interest would be to understand how students tackle certain problems in terms of their thought processes. This could be achieved, to a limited extent, by recording button clicks (or keystrokes) to try to determine their cognitive development as they progress through a problem. Research on keystroke capture work with graphic calculators is currently being carried out in order to gain insight into student learning styles (Berry, 2002; Smith and Berry, 2002), and this could be extended to incorporate the constructivist use of visualisation, in particular the transfer from pictorial to symbolic forms. The types of questions about student learning that this could help to answer are:

- Which graphs do they plot, and in which order do they plot them?
- Do they use the hints? and if so, at what stage?
- How many attempts do they have?
- Do they attempt to come up with the correct expression first time?
- Do they try to gradually build up the expression starting with a basic graph?
- Do they learn from their mistakes?

The software would perhaps be more enticing with the addition of aesthetic multimedia enhancements, such as touchscreen capability and audio features. This could in turn improve motivation. The use of audio capability, such as sound effects or music, however, would be extremely annoying if it were used in a laboratory situation - it would only be useful for an individual using it alone. Soundbites, in the form of wave files, were taken out of the code for this reason.

As mentioned earlier, the package was designed as a vehicle for an experiment and was not intended to be a polished piece of marketable software. Minor 'glitches' could be ironed out if released for commercial use. A few students complained about the notation for the expressions being unfamiliar, for example the use of '*' for multiplication, '/' for division, 'pi' for π, etc., and it would have been slightly more user-friendly, for example, if the user could have used the 'enter' key as an alternative to the 'plot' button. They were all minor issues, however, and certainly did not detract from the usefulness of the package in any way.

7.2.2. Issues Concerning the Experimental Design

The case-study produced some positive and practical findings. However, experiments abound with potential biases, and, as with many practical experiments, there are issues associated with the experimental design. These problems are addressed specifically in this section.

There are additional factors that need to be considered concerning the analysis of the data:

1. **For the results of the experimental group, there are two 'human factor' effects, and each is confounded with the other.**

- The first is that for a class of pupils there is the 'novelty' effect of being involved in a 'non-routine' activity which thus attracts their attention more than 'routine' work might do (but this is one of the specific attractions of this type of approach). This could result in the experimental group achieving inflated visual scores in comparison with the control group.

- The second is referred to as the 'Hawthorne' effect after the name of the Western Electric Company's Chicago factory where it was first demonstrated. It is that a group who feel that they are being treated specially as participants in the evaluation of some new activity do their very best to please. The Hawthorne investigation (Mayo, 1933) was conducted over a period of five years (1927-1932), and it considered the importance of groups in affecting the behaviour of individuals at work. A series of experiments was carried out to determine the effect of illumination on production. The control groups worked under constant illumination whereas illumination in the experimental groups was varied (increased and decreased). The result was that production increased not only in the test groups, but also, at comparable rates, in the control groups; this held true not only when illumination was increased, but also when it was decreased.

It became clear that the increase in production was not caused by the changes in illumination, but apparently by the increased attention the workers received from management. This unplanned effect on the 'untreated' control group is called the 'Hawthorne' effect. In the case-study in Chapter 5, the experimental group students feel that they are being treated specially as participants in the evaluation of the software, and thus their performance could increase (leading to improved visual scores).

2. The control group is one year ahead of the experimental group.

- As a consequence of ongoing practice, the control group may be expected to perform better than the experimental group on the procedural questions. On the other hand, if visualisation of the type required for the exercises had not been practised for twelve months, mental rust may have set in and thus the control group may be expected to perform worse than the experimental group on the visual questions. Both of these factors would contribute to a decrease in the 'difference' (procedural - visual) scores for the experimental group as compared with the control group, irrespective of the mode of presentation of the visualisation material.

- It would be of interest to revisit the experimental group after a lapse of time (ideally twelve months, although this of course may not be practicable), present them with the same test, and then compare these second batch scores with those obtained by the control group twelve months previously. For further discussion, see Section 7.2.5 later, which considers retention level.

7.2.3. Experimental Techniques

In order to gain a more complete understanding of the cognitive processes involved in learning functions and graphs via the software described in Chapter 5, careful analysis of both student behaviour and the process of developing understanding are required (this is likely to vary from student to student due to knowledge structures being built up from their own personal experiential world, as per the discussions in Chapter 3). The practical study of the development of knowledge structures is sparse in the literature. It would be of great theoretical and practical interest to analyse behaviours of students as they interpret visual information and convert to symbolic form such as when studying graphs in relation to their respective algebraic forms, how students develop understanding over time, and what type of information they call upon in order to make mathematics meaningful to them. This is certainly an avenue of research that could help considerably in the design of computer-based learning materials to be used in a constructivist way.

Although some interesting feedback has been obtained, the information-seeking process could perhaps be improved with the introduction of video. This could be used to explore the effectiveness of the Socratic method of strategic questioning, to record students' questions, and tutors' answers, and to observe the resulting strategy by students. The questions could be tackled in groups, and the discussions recorded on video tape (in a non-threatening environment). This technique would be useful in helping to establish how students attempt to solve mathematics questions, and this could lead to an appreciation of the mental operations that are involved whilst tackling such questions. It would serve to confirm, or otherwise, the problem solving processes discussed in Chapter 6, which considered how students build and select solution strategies. This would remove individuality to a certain extent, but it would give a clearer picture of the students' thinking. It would provide insight into how students think during mathematical exploration.

Alternatively, students' oral responses to the use of the interactive software could be obtained. This would enable the students to informally elaborate on their mental operations, which would perhaps be more informative than a written statement.

7.2.4. Alternative Topics for Treatment

The topic of 'graphs of functions' is just one example of many potential packages that could assist in the nurturing of visualisation skills. Consideration is given here to additional and more advanced topics that could be taught in a similar fashion, together with an explanation as to how and why they were chosen.

A natural progression from the work involving two-dimensional graphs is to consider an educational package involving three-dimensional graphs. Here, the user can be shown a three-dimensional graph and, as before, asked to determine the correct mathematical expression for the function via an investigative approach. The plotting of three-dimensional graphs is a fairly straightforward routine for packages such as DERIVE and MATLAB, and even some graphics calculators. Such a study could help to answer questions such as:

- Do the outcomes of the case-study concerned with two-dimensional graphs carry over to three-dimensional work?

- Is the conversion process from the pictorial to the symbolic more difficult, or just different, in three dimensions?

Another topic for consideration is that of transformation matrices. Three-dimensional shapes can be shown to the user and the user will be able to alter a shape's orientation via the use of transformation matrices. In an exploratory manner, the user will be able to change various elements within the matrices and observe the effect of these changes on the image. Starting with the image, students could conjecture the symbolic form of the rotation matrix, reflection matrix, etc. This topic is highly 'visual', and therefore attractive computer generated visualisations are ideal for such an approach to learning.

7.2.5. Measurement of Retention Level

Concern cannot simply be given to how quickly students *grasp* a particular mathematical concept. How long they are able to retain such information is particularly relevant to mathematical development. In a study concerned with the effects of technology on student learning (Baker and Gloster, 1994), not only did college students using technology learn faster, six months after completing their studies they tested better on the subject than their peers who had been taught in traditional settings.

The case-study has illustrated that a constructivist visual approach is preferable for the acquisition of knowledge and skills. It is conjectured that a constructivist approach to learning is also preferable for increasing retention level. For example, it is the belief that a student is much more likely to remember the picture of a function, in the form of a graph, than its symbolic notation. In a similar manner to Baker and Gloster, tests need to be carried out on the knowledge and skills retention of students learning via such a constructivist approach. As mentioned earlier under issues concerning the experimental design, this would be possible by re-testing the experimental group after a period of twelve months and then comparing these scores with those obtained by the control group twelve months previously.

It is also conjectured that a constructivist visual approach is beneficial to all age ranges, however the above scenario would still only provide information specific to the chosen age range of students (16-19 years). Any results regarding the suitability of teaching approaches, in terms of the development and retention of knowledge and skills, would need to be compared with similar studies at primary, secondary and tertiary levels in order to draw generic conclusions.

7.2.6. Summary of Issues Concerning a Constructivist Approach

As a result of the case-study, the following important questions have surfaced that require further research:

- Can any **generic** conclusions be derived?
    - As mentioned above, are the outcomes limited to certain age groups? For example, is an instructivist approach necessary before a constructivist approach takes over?
    - Are the outcomes limited to particular subject domains? For example, will a constructivist approach to teaching develop better ideas of formal proof? (Alcock and Simpson, 2001).
    - Does a constructivist approach assist in the mathematical development of all students of all abilities?

- Do traditional **assessment** methods favour an instructivist approach and hence limit constructivist activities?
    - Which methods of assessment effectively document genuine learning?
    - Should technology be allowed to be used in examinations, when appropriate, to measure abilities in conceptual understanding?

- How are psychological and motivational factors taken into account when using a constructivist approach?
    - Is learning via a constructivist approach more 'fun' and does it lead to increased motivation for *all* students?

Consideration also needs to be given to the effective switching between knowledge/subject domains. Once skills have been built up, they need to be transferable between domains. It is conjectured here that the software described in

Chapter 5 facilitates domain-general problem solving, but further research on the application of students' skills in other mathematical topics is required to substantiate this claim.

7.3. More General Future Mathematics Education Research

Building on the issues discussed in Chapter 6, consideration is given to further research into learning styles of students, mental imagery, the integration of innovative approaches into the school curriculum in terms of teaching, learning and assessment, and the future of mathematics education in relation to developing computer technologies.

7.3.1. Holistic and Serialistic Learning Styles

One of the outcomes of the case-study in Chapter 5 was that the control group and the experimental group demonstrated a different skills set after experiencing different teaching approaches, and hence different learning styles. Chapter 6 highlighted conflicts between the types of skills that students require to solve problems, and those that they actually possess following instructivist methods. Moreover, Chapter 6 described how students, as a result of their learning style, try to convert problems that require the application of Group C skills into ones in which Group A skills can be employed via a template.

Constructivism, including the use of computers in learning mathematics, has been discussed, together with the associated benefits, but so far different student learning styles and the effect that these might have on the teaching approach adopted have not been fully considered.

In order to optimise learning effectiveness, students' preferred learning styles should be taken into account. For example, in Higher Education there could be implications for teaching mathematics to mathematics students compared to 'service' students, e.g. engineers, sports scientists, etc. Students who act holistically, having experienced a constructivist approach, tend to adopt a global approach to learning, concentrating first on building a broad conceptual overview into which detail can subsequently be fitted (Ford, 1995). Typically they address several aspects of the learning matter at the same time, and make use of a rich variety of stimuli. Ford explains that serialists, having experienced an instructivist approach, tend to use a predominantly local learning approach, concentrating on one thing at a time. They tend to relate new concepts to previously learnt ones using simple logical links, building up their understanding on a relatively narrow front. They proceed on the basis of thoroughly mastering one component of the subject matter before proceeding to the next, in a very linear fashion. Relative to the holistic approach, the broad conceptual overview of the subject matter thus emerges later in the learning process, if at all.

These learning styles are, however, relatively difficult to measure in an individual, which compounds the complexity of the choice of approach. The test questions used in the case-study were useful for assessing students' skills, but were rather limited in terms of determining students' preferred learning styles as they had multiple-choice answers. Future research should seek responses that illustrate how students actually tackle the questions, in terms of solution strategy and application of a range of skills.

Educators must strive to develop educational materials that are appropriate for all types of learners, not just those well suited to the constructivist approach. Consideration must be given to which teaching and learning strategies are appropriate in the light of individual learning style preferences. There is the argument, therefore, that we should perhaps teach holists in a constructivist manner, and serialists in an instructivist manner. It is more likely, however, that learning styles will adapt to the chosen teaching style. In this manner, students would unfortunately be forced to learn in a particular way, regardless of their preferred learning style. It is important, therefore, that a range of teaching and learning materials should be made available to cater for all learning styles.

Students prior to a case-study could fill out a learning styles questionnaire, and on the basis of responses they could be split into two groups, holists and serialists. It could then be seen whether or not constructivist and instructivist teaching approaches do in fact mould students' learning styles accordingly. Additionally, comparisons could be made between results from serialists experiencing an instructivist approach and holists experiencing a constructivist approach to see if a certain approach is more appropriate for a certain favoured learning style.

7.3.2. Mental Imagery

The interlay between pictorial and symbolic forms, i.e. conversion, and the problems that students have in switching between these representations during problem solving, i.e. advancement, have been discussed in Chapter 6. The constructivist use of visualisation directly influences the mental images created by learners, and in turn conceptualisation. Knowing more about the nature of students' mental images would help educators design software that uses visualisation as a means of enhancing them.

There seems very little information in the literature on how to actually evaluate a person's set of mental images, how they are used, whether they are 'correct', etc. Mental images are different from person to person, and are clearly extremely difficult to assess due to their personalised nature. Such fundamental differences in image making could have important implications in terms of teaching and learning, yet attempts to measure this complex factor in terms of educational significance have been relatively unsuccessful (Thompson, 1990), and there have been no significant additions to the literature since this report. Increased awareness of the issue of mental imagery could lead to more flexible teaching methods which take greater account of a student's individual style of thought and its interaction with the nature of the learning task. This appears a rather daunting avenue to explore, but one which would be of significant interest in terms of designing educational materials based on visualisation.

Mental imagery is implicitly linked to visualisation ability. It would be interesting to study the process of how visualisation ability is utilised to form 'correct' mental images, and to see how a learner progresses from a novice state to an expert state in terms of their mental images. As indicated in Chapter 2, there has been some work reported in the literature (Presmeg and Bergsten, 1995) considering students' preferences for visual or non-visual methods, but a number of questions remain unanswered, such as:

- Which students prefer visual techniques, if any?
- Can students visualise, but just choose not to?
- If so, why do they choose not to?
- Or can they just not visualise?

It would be of value to see whether or not a student's 'artistic' ability is influential in favouring a visual approach to mathematics. A controlled trial could be set up in which students are segregated on the basis of whether or not they have good qualifications in Art or related subjects. Results in visual exercises, together with how the students chose to tackle the problems, would illustrate whether or not there is a significant link. Visualisation ability could also simply be linked to personality, which would clearly make any relationship more difficult to assess.

In Section 3.3 of Chapter 3, problems with students having different rates of knowledge construction were considered from a teaching perspective. It is conjectured here that visualisation can play an important role in providing remedial action for the poorer students to enhance their powers of mental imagery, which in turn will speed up the cognitive structuring process.

7.3.3. Integration into the School Curriculum

There are difficulties involved with integrating a more constructivist, visual approach to mathematics into the school curriculum, as discussed in Chapter 6. Curricular implications of such an approach in terms of teaching, learning and assessment are discussed further.

7.3.3.1. Teaching and Learning Issues

It was reported a decade ago that the wide availability of useful mathematical software, together with changes in approaches to teaching and learning, has important curricular implications at A-level (Philipp et al., 1993). This is still the case today, as little has been done in terms of updating the A-level curriculum to encompass change in technology. Technology and more constructivist approaches need to play a more prominent role and shape the future of mathematics teaching. They reduce the pre-requisite of extensively developed manipulative skills, so that students can spend more time actually solving problems and developing more desirable higher order skills.

The introduction of computers into the mathematics classroom changes the distribution of responsibilities for teaching and learning (Squires and McDougall, 1996). Learning environments need to be encouraged in which students assume more responsibility. Teachers need to become less concerned with control and more involved in facilitating student-centred learning. Cooperation between peers is often a feature of classrooms in which software is being used, generating more discussion. With such enhanced discussion, students learn from each other. The need to be sensitive to peer group learning may lead to changes in the role of the teacher, requiring emphasis on reacting to student initiatives rather than planning and overseeing a pre-determined teaching plan. The specific effects of student-teacher interactions have not been measured in this thesis, and little is known about their real impact. It would be interesting to explore the impact of such interactions. This could be achieved by offering students different levels of interaction in a controlled trial, and observing the effect on student learning.

The professional development of teachers in their use of technology is obviously important as the teachers must feel competent with the technology before using it in the classroom. This development should also include some form of peer support and review, promoting reflection amongst both teachers and students (Brown, 1994b). Successful applications of the use of technology in teaching and learning may serve to inspire teachers to commit themselves to the change process. However, there are possible barriers, such as lack of institutional support and commitment of resources, even if teacher attitudes are favourable.

Research needs to be more proleptic (Kaput, 1993), i.e. anticipate the future. Research being planned in the early part of this century will be conducted and reported midway through the first decade, and then will have its impact on curriculum design in the following two or three years. The actual classroom impact will not occur until towards 2010. The total time elapsed is about 8 to 10 years. By this time, technology and attitudes towards best teaching practices may well have moved on. Educators therefore need to try to anticipate what the future holds in terms of technological capability so that in 8 to 10 years time the technology will be at the right stage of development. More proleptic research, however, could well require expensive technology, institutional support, and imagination, and the researchers will need to resist pressures to provide quick 'payback'.

This situation may well change, however, as technology is rapidly becoming easier to integrate. Vast numbers of people find computers part of their daily lives, because as computers become more powerful, they become easier to use. Perhaps in the future, technological capability will not be a key issue for researchers, as it will be an accepted norm. Instead, the major issue will be how to exploit this technological capability for the purpose of enhancing both teaching and learning in the classroom.

7.3.3.2. Assessment Issues

One of the main obstacles blocking increased utilisation of software that supports visualisation is the degree to which assessment incorporates the use of technology (Philipp et al., 1993). Obviously if a visual approach is used in teaching, then assessment will need to respond accordingly. Computer-based visualisation cannot be fully employed in the classroom if the assessment does not allow appropriate software.

The employment of technology clearly has important implications for mathematics curricula and how they are taught. Given the availability of CAS and powerful handheld technology, the key issue now is the art of asking the 'right questions'. Comprehension tasks (Houston, 1996) are a positive step, as they require students to explain, to justify, and to produce a mathematical argument, rather than merely carry out procedures. Any assessment that incorporates the use of technology must be appropriate for all ranges of ability.

Assessment must therefore move with the times. Traditional questions which ask students to sketch rational functions, for example, are now trivialised by the available technology, and are no longer a valid exercise (Hope, 1994). Curriculum topics must reflect the technology available, otherwise they will merely become exercises in outdated techniques.

Finally, consideration is given to the role of assessment in determining the nature of students' cognitive structures. In Section 3.2 of Chapter 3, a discussion was provided which explained that the way students build their cognitive structures is essentially subjective. A teacher therefore cannot possibly know the make-up of individuals' cognitive structures, which is a serious problem in the teaching of mathematics due to its hierarchical nature. A possible solution to this is the introduction of formative assessment (rather than summative), with the aim of gaining insight into students' existing cognitive structures. On the basis of this information, *individual* pre-requisites could be determined for a given topic, thus moving towards more individualised teaching and learning. Note that the summative assessment at the end of the case-study in Chapter 5 was used as the best way to measure an individual's cognitive structure at any one time.

7.3.4. Application of Outcomes to Developing Computer Technologies

It is difficult to estimate the extent of the effect of developing technologies on mathematics education, as there is little general theory in the literature concerning the effects of innovative technology on improved learning. Difficulties in speculation on the effects of technology, and research employing technology, is compounded by the speed with which it advances. Educators face the problem of attempting to design something that is constantly changing. This section considers various advances in technology and their possible impact on mathematics education.

An exciting progression could be the introduction of virtual reality. As well as being widely used in the field of computer games, virtual reality has serious applications that far exceed standard visualisation techniques. It is a medium for direct experience, as for example in a flight simulator. Virtual reality also offers significant potential as a mathematical tool, and for many it enhances the left-right brain connection so vital to understanding (Cochrane, 1996). Inside a virtual world, a user could 'manually' change the shape of a function's graph, for example, by being immersed in two-dimensional or three-dimensional images. This may mean that students could gain direct understanding of functions, for example, by viewing and manipulating them 'hands-on'. The activities that virtual reality can support are discussed in terms of the issues outlined in Chapter 6:

- The direct, dynamic interaction could assist in the conversion process between visual and algebraic representations.

- As a result of enhanced skill in switching between representations, students may be less inclined to convert problems into those that merely require Group A skills.

- Students would be 'physically' handling the conversion process by being immersed in the environment, which could help them to see the linkages more clearly.

New ways of teaching mathematics with the help of multimedia tools need to be explored, as children now come from a culture of hi-tech computer games. Putting mathematics into that same framework could have a hugely positive effect on the speed of the learning process. Future technology will bring with it an increasingly visual, interactive and virtual world in which radical changes in the teaching and learning of mathematics (and many other subjects) can be explored.

Finally, there is the consideration of mathematics via the Internet. Working groups have discussed the benefits of using the Internet for teaching mathematics (Edwards et al., 2000; Butler et al., 2000), and practical examples of web-based systems for enhancing mathematics teaching and learning are beginning to emerge (Mavrikis and Macciocia, 2002; Oevel et al., 2002). An enormous amount of information is available on the Internet, it is easy to use, and is accessible from just about anywhere. The Internet could therefore be used as a resource for disseminating software, such as that described in Chapter 5, in order to satisfy the growing trend for distance learning. Although this would allow for self-pacing, the problem with using the software as a distance learning tool is that the outcomes of the case-study are not entirely applicable due to the limited opportunity for peer or teacher interaction. Without the facility for integrating software use with the Socratic method of strategic questioning (Chapter 3), the software alone would not fully support a constructivist approach. There are clearly implications here for the design of web-based materials for 'e-learning' of this nature. Preliminary attempts at a constructivist approach to software design on the Internet, which support and encourage dialogue, have been reported (Gabbard, 2000; Scott, 2001). From a constructivist perspective, the Internet could be used in a classroom situation for the exploration of visual information, of which there is a colossal amount. Students are often less inhibited in an electronic environment, and are more likely to interact in a group situation, thus developing desirable key skills (Houston, 2001). The use of the Internet would allow for the sharing of information and the promotion of collaboration. There appears to be little, if any, research reported on collaborative working across such a medium in the mathematics classroom. Such activity would encourage peer interaction and debate.

7.4. Summary

This final chapter has considered future developments in the form of enhancements to both the software and the case-study, taking into account student feedback, and possible future research considerations for mathematics education in general.

Integrating the constructivist use of computer-based visualisation into school curricula, to aid the development of conceptual and relational understanding, will serve to bridge the gap between what is expected at A-level and what is expected at degree level mathematics. As technology continues to advance, educators need to continually re-assess what is important in the curriculum.

Students are more likely to be successful in mathematics if they have knowledge in different contexts, for example symbolic and graphical. It can provide different ways of thinking around problems, and can offer alternative solution strategies depending on the viewpoint. In this manner, students can check their answers via alternative approaches, for example the checking of an algebraic solution by means of a graph. Powerful environments link together different representational forms, and allow the user to switch between representations. This makes the concepts easier to handle, and aids in the acquisition of a more flexible and holistic understanding. However, a number of questions still need to be answered:

- Should all topics be taught this way?
- Is there an ideal 'mix' of symbols and pictures in the learning process?
- Does it vary depending on the individual?
- Should certain topics be introduced visually or symbolically?
- For a visual start, at what stage should symbols be introduced (if at all) and vice versa?
- Is either treatment unsuitable for certain topics?
- Are symbols better for more accuracy, more dimensions, generic cases?

By adopting the computer-based constructivist approach to learning described in this thesis, students will have a greater understanding of the underlying concepts, will be able to apply them to other problem solving situations at a later date, and will be able to demonstrate the ability to transfer knowledge across different subject domains. Constructivism shifts the focus of attention from the product of mathematical activity, i.e. the solution, to the cognitive processes of that activity, i.e. how we arrive at the solution. Visually compelling graphical software can enrich these cognitive processes.

Visual software can provide an environment in which students can develop relational understanding, and can develop a more versatile way of thinking about mathematics, without having a detrimental effect on their symbolic manipulation skills. Moreover, these symbolic manipulation skills can be developed in more meaningful contexts.

The computer can be used to carry out routine processes (act as a black box) while the student concentrates at a higher conceptual level, and via constructivist activities, the student can process a number of pieces of information as a single conceptual entity.

Students need to be engaged in activities that will result in the creation of meaningful mental constructions that are necessary for conceptual and relational understanding. Powerful and appropriate images help students to create quality mental images that can be re-presented and utilised in mathematical thinking. These mental images need to be 'correct' and meaningful. The computer can assist greatly in this image-making process.

The thesis has produced some positive and practical findings, however this chapter has illustrated that there is still much work to be done.

It is vitally important for technological tools to be seen and treated as exactly that - tools, and not as divine bearers of knowledge. Comments such as, 'Well that's what the computer told me', are not uncommon. Students have never thought that blackboards have been able to do mathematics, but many tend to think that computers can.

The nature of the virtual world that we are increasingly forced to live in often means that the power of interacting with peers and teachers, and all the educational benefits that ensue, can be lost. Educators need to pause for breath and take a step back - it is easy to get carried away with advances in technology. Technology must not be used for the sake of it, but instead there must be sound educational reasons for employing appropriate technology in the classroom. It is vital that effective, traditional practices are not replaced, and that the use of technology is seen as an integrated resource. Technology must only be used if it enhances the learning prospects of students. It is not desirable to move towards a 'virtual university', losing in the process all the rich sources of interaction so important for students' academic and social development.

REFERENCES

Abbott, S. (2001). "Rank and File: Vision and Visualisations, The 2001 Presidential Address", *The Mathematical Gazette*, Vol. 85, No. 504, pp 386-402.

Abboud, M. (2002). "Introducing Experiments into a First Course in Calculus", *Proceedings of the 2nd International Conference on the Teaching of Mathematics (at the undergraduate level)*, University of Crete, Greece, July, CD-ROM.

Ahlander, B. (2002). "Let the Students Explore Algebra with CAS, TI89", *Proceedings of the 2nd International Conference on the Teaching of Mathematics (at the undergraduate level)*, University of Crete, Greece, July, CD-ROM.

Ahmad, J., Lissaman, R., Malabar, I., Price, L., Smith, F., Stewart, B. (2000). "Attracting Students to Mathematics", *Proceedings of the 26th Undergraduate Mathematics Teaching Conference (UMTC)*, Sheffield Hallam University, September, pp 54-59.

Alcock, L.J., Simpson, A.P. (2001). "The Warwick Analysis Project: Practice and Theory", *Teaching and Learning Mathematics at University Level: An ICMI Study*, ed. Holton, D., Dordrecht, The Netherlands: Kluwer Academic, pp 99-111.

Al-Jumeily, D. (2002). "The Potential of Multimedia in Teaching and Learning of Mathematics", *Proceedings of the 2nd International Conference on the Teaching of Mathematics (at the undergraduate level)*, University of Crete, Greece, July, CD-ROM.

Amrhein, B., Gloor, O., Maeder, R.E. (1997). "Visualizations for Mathematics Courses Based on a Computer Algebra System", *Journal of Symbolic Computation*, Academic Press Ltd., Vol.23, pp 447-452.

Anderson, J.A. (1996). "Experiments to Investigate Instrumental and Relational Understanding Among Undergraduates", *International Journal of Mathematics Education in Science and Technology*, Taylor & Francis Ltd., Vol. 27, No. 6, pp 813-819.

Ardahan, H., Ersoy, Y. (2002). "Issues on Integrating CAS in Teaching Mathematics: Teaching Concepts in Calculus with Derive", *Proceedings of the 2nd International Conference on the Teaching of Mathematics (at the undergraduate level)*, University of Crete, Greece, July, CD-ROM.

Assessment and Qualifications Alliance (AQA) <http://www.aqa.org.uk/> (accessed 30th November 2003).

AUTOGRAPH
<http://www.autograph-math.com/> (accessed 30th November 2003).

Bailey, C., Chambers, J. (1996). "Interactive Learning and Technology in the US Science and Mathematics Reform Movement", *British Journal of Educational Technology*, Vol. 27, No. 2, pp 123-133.

Baker, W.J., Gloster II, A.S. (1994). "Moving Towards the Virtual University: a Vision of Technology in Higher Education", *Cause/Effect*, Summer, pp 4-11.

Baldin, Y. (2002). "On some Important Aspects in Preparing Teachers to Teach Mathematics with Technology", *Proceedings of the 2nd International Conference on the Teaching of Mathematics (at the undergraduate level)*, University of Crete, Greece, July, CD-ROM.

Ball, G., Stephenson, B., Smith, G., Wood, L., Coupland, M., Crawford, K. (1998). "Creating a Diversity of Mathematical Experiences for Tertiary Students", *International Journal of Mathematics Education in Science and Technology*, Vol. 29, No. 6, pp 827-841.

Barwise, J., Etchemendy, J. (1991). "Visual Information and Valid Reasoning", *Visualization in Teaching and Learning Mathematics*, ed. Zimmermann, W., Cunningham, S., Vol.19, pp 9-24.

Ben-Chaim, D., Lappan, G., Houang, R.T. (1989). "The Role of Visualization in the Middle School Mathematics Curriculum", *Focus on Learning Problems in Mathematics*, Winter Edition, Vol. 11, No. 1, pp 49-60.

Berry, J. (2002). "Do You Know What Your Students are doing on their Graphic Calculators?", *Proceedings of the International Teachers Teaching with Technology Conference*, Calgary, Canada, CD-ROM.

Berry, J., Graham, T., Watkins, T. (1987). "Teaching Calculus with Derive", *Innovations in Mathematics Teaching*, SEDA Paper, pp 29-34.

Bishop, A.J. (1989). "Review of Research on Visualization in Mathematics Education", *Focus on Learning Problems in Mathematics*, Winter Edition, Vol. 11, No. 1, pp 7-16.

Bishop, P. (1997). "How Computers Can Help Teachers and Learners", *MathSkills Newsletter*, March, pp 6-7.

Bloch, I. (2002). "Contents and Nature of Preparation of Mathematics Teachers", *Proceedings of the 2nd International Conference on the Teaching of Mathematics (at the undergraduate level)*, University of Crete, Greece, July, CD-ROM.

Bloom, B.S., Engelhart, M.D., Furst, E.J., Hill, W.H., Krathwohl, D.R. (1956). "Taxonomy of Educational Objectives: Cognitive Domain", New York: McKay.

Blyth, B., Vouzas, P., Saunders, J. (2002). "Animations to Illustrate Ill-Conditioning and an Introduction to Matrices using Maple", *Proceedings of the 2nd International Conference on the Teaching of Mathematics (at the undergraduate level)*, University of Crete, Greece, July, CD-ROM.

Boaler, J. (1997). "Experiencing School Mathematics", *Buckingham, Open University Press.*

Booth, D., Butler, M., Emery, D., Horan, R., Middleton, W., Pountney, D. (2001). "Issues for Web-Delivered Assessment", *Proceedings of the 27th Undergraduate Mathematics Teaching Conference (UMTC)*, University of Birmingham, September, pp 57-63.

Bowles, R., Calay, P., Flores, R., Formosa, J., Goda, D., Malabar, I., Norrington, P., O'Rourke, F., Scott, P., Simpson, A., Street, D. (2002). "Deep and Surface Learning in Undergraduate Mathematics", *Proceedings of the 28th Annual Undergraduate Mathematics Teaching Conference (UMTC)*, University of Birmingham, September, pp 58-68.

Brown, A. (1994a). "Undergraduate Mechanical Engineering Students' Experience of Peer Support in Computer-Based Workshops", *Association of Learning Technology Conference on Enabling Active Learning*, University of Hull, September.

Brown, A. (1994b). "Processes to Support the use of Information Technology to Enhance Learning", *Computers Education*, Pergamon Press Ltd., Vol. 22, No.1/2, pp 145-153.

Buchberger, B. (1989). "Should Students Learn Integration Rules?", *Technical Report RISC-LINZ, Series No. 89-07.0*, March.

Butler, M.K., Kay, A., Booth, D.J., Gordon, N.A., Middleton, W., Stone, J.A. (2000). "Use of the Internet in Teaching Mathematics (Group B)", *Proceedings of the 26th Undergraduate Mathematics Teaching Conference (UMTC)*, Sheffield Hallam University, September, pp 72-86.

Butterfield, A., Molyneux-Hodgson, S., Sutherland, R. (2000). "Learning Conversions in Science: The Case of Vocational Students in the UK", *ALT-J*, Vol. 8, No. 3, pp 89-104.

CABRI-GEOMETRE
<http://www-cabri.imag.fr/> (accessed 30th November 2003).

CALMAT
<http://www.calmat.gcal.ac.uk/> (accessed 30th November 2003).

Candlish, R. (1997). "Reach for the Sky", *graphiti, The Newsletter of the Technology User Group*, Edition 2: Spring, p 5.

Cha, S., Kent, P. (2002). "x^2 or 2x?: Experiments on Geometric Algebra with Dynamic Geometry", *Proceedings of the Day Conference of the British Society for Research into Learning Mathematics (BSRLM)*, University of Bristol, May, Vol. 22, No. 2, pp 1-6.

Chae, S.D., Tall, D.O. (2001). "Construction of Conceptual Knowledge: the Case of Computer-Aided Exploration of Period Doubling", *Research in Mathematics Education, Papers of the British Society for Research into Learning Mathematics (BSRLM)*, ed. Morgan, C., Jones, K., Vol. 3, pp 199-209.

Challis, N., Gretton, H. (2000). "Visualization: a Powerful Aid for Conceptual Development", *Proceedings of the 4th International DERIVE-TI92/89 Conference*, Liverpool, July, CD-ROM.

Challis, N., Gretton, H., Pitt, D., Robinson, M. (2002). "Activity Mathematics", *Proceedings of the 2nd International Conference on the Teaching of Mathematics (at the undergraduate level)*, University of Crete, Greece, July, CD-ROM.

Christel, M.G. (1994). "The Role of Visual Fidelity in Computer-Based Instruction", *Human-Computer Interaction*, Lawrence Erlbaum Associates, Inc., Vol. 9, pp 183-223.

Clements, D.H., Battista, M.T. (1994). "Computer Environments for Learning Geometry", *Journal of Educational Computing Research*, Baywood Publishing Co. Inc., Vol. 10(2), pp 173-197.

Cochrane, P. (1996). "Virtual Mathematics", *The Mathematical Gazette*, July, pp 267-278.

Connor, J. (2002). "Using Technology and Co-operative Groups to Develop a 'Deep Understanding' of Secondary School Geometry", *Proceedings of the 2nd International Conference on the Teaching of Mathematics (at the undergraduate level)*, University of Crete, Greece, July, CD-ROM.

Connors, M.A., Snook, K.G. (2001). "A Technology Tale: Integrating Hand-held CAS into a Mathematics Curriculum", *Teaching Mathematics and its Applications*, Vol. 20, No. 4, pp 171-190.

Corfield, D. (2001). "The Importance of Mathematical Conceptualisation", *Studies in History and Philosophy of Science Part A*, Elsevier Science, Vol. 32, No. 3, pp 507-533.

Corporate Author School Mathematics Project. (1996). "Maths for Work: Higher Assignments Copymasters", Cambridge University Press.

Cretchley, P. (1998). "Enhancing Achievement using Technology? Feedback is Effective", *Proceedings of the International Conference on the Teaching of Mathematics,* University of the Aegean, Samos, Greece, July, pp 83-85.

CTI Centre for Mathematics and Statistics (1995). "Mathematics and Statistics Guide to Software for Teaching", Third Edition, ed. Bishop, P.

Cunningham, S. (1991). "The Visualization Environment for Mathematics Education", *Visualization in Teaching and Learning Mathematics*, ed. Zimmermann, W., Cunningham, S., Vol.19, pp 67-76.

Dalgarno, B. (2001). "Interpretations of Constructivism and Consequences for Computer Assisted Learning", *British Journal of Educational Technology*, Blackwell Publishers Ltd., Oxford, UK, and Boston, USA, Vol. 32, No. 2, pp 183-194.

Daskalogianni, K., Simpson, A. (2002). " 'Cooling-Off': the Phenomenon of a Problematic Transition from School to University", *Proceedings of the 2nd International Conference on the Teaching of Mathematics (at the undergraduate level)*, University of Crete, Greece, July, CD-ROM.

Davis, A. (2001). "Teaching for Understanding in Primary Mathematics", *Evaluation and Research in Education*, Multilingual Matters, Vol.15, No. 3, pp 136-142.

de Alwis, T. (2002). "Computer Algebra Systems in a Multivariable Calculus Course and Center of Gravity Problems", *Proceedings of the 2nd International Conference on the Teaching of Mathematics (at the undergraduate level)*, University of Crete, Greece, July, CD-ROM.

Demana, F., Schoen , H.L., Waits, B. (1993). "Graphing in the K12 Curriculum: the Impact of the Graphing Calculator", *Integrating Research on the Graphical Representation of Functions*, ed. Romberg, T.A., Fennema, E., Carpenter, T.P., Lawrence Erlbaum Associates, Chapter 2, pp 11-39.

DERIVE
<http://www.derive.com/> (accessed 30th November 2003).

Dhillon, A.S. (1997). "An Interactive System for Learning Rotational Dynamics", *Journal of Computer Assisted Learning*, Vol.13, pp 59-67.

Dörfler, W. (1991). "Meaning: Image Schemata and Protocols", *Proceedings of PME XV*, Assisi, Italy, Vol. 1, pp 17-32.

Dreyfus, T. (1991). "Advanced Mathematical Thinking Processes", *Advanced Mathematical Thinking*, ed. Tall, D.O., Kluwer, Chapter 2, pp 25-41.

Dreyfus, T. (1995). "Imagery for Diagrams", *Exploiting Mental Imagery with Computers in Mathematics Education*, NATO ASI Series F, Vol. 138, pp 3-19.

Drijvers, P. (1995). "White-Box/Black-Box Revisited", *The International DERIVE Journal*, Vol. 2, No. 1, pp 3-14.

Duatepe, A., Ersoy, Y. (2002). "Effects of using Advanced Calculator (TI-92/Cabri) on Learning Transformational Geometry", *Proceedings of the 2nd International Conference on the Teaching of Mathematics (at the undergraduate level)*, University of Crete, Greece, July, CD-ROM.

Dubinsky, E. (1991). "Reflective Abstraction in Advanced Mathematical Thinking", *Advanced Mathematical Thinking*, ed. Tall, D.O., Kluwer, Chapter 7, pp 95-126.

Dugdale, S. (1993). "Functions and Graphs – Perspectives on Student Thinking", *Integrating Research on the Graphical Representation of Functions*, ed. Romberg, T.A., Fennema, E., Carpenter, T.P., Lawrence Erlbaum Associates, Chapter 5, pp 101-129.

Duval, R. (1995). "The Cognitive Analysis of Problems of Comprehension in the Learning of Mathematics", *Proceeding of PME 25*, Utrecht, The Netherlands, pp 1-15.

Duval, R. (1999). "Representation, Vision and Visualization: Cognitive Functions in Mathematical Thinking. Basic Issues for Learning", *Proceedings of the 21st Meeting of the American Group of the PME*, Mexico, pp 3-26.

Edwards, P. (1995). "A Visual Approach to Understanding Mathematics", *Maths and Stats Newsletter*, August, pp 2-7.

Edwards, P. (1996). "Implementing Diagnostic Testing for non-Specialist Mathematics Courses", *Open Learning Foundation Handbooks*.

Edwards, P. (1997). "Mathematics Diagnostic Testing in OLF Member Universities", *MathSkills Newsletter*, March, pp 4-5.

Edwards, P., Harley, P., Hermans, D., Parkhurst, S., Pountney, D. (2000). "Use of the Internet in Teaching Mathematics (Group A)", *Proceedings of the 26th Undergraduate Mathematics Teaching Conference (UMTC)*, Sheffield Hallam University, September, pp 60-71.

Eisenberg, T., Dreyfus, T. (1991). "On the Reluctance to Visualize in Mathematics", *Visualization in Teaching and Learning Mathematics*, pp 25-37.

Elliott, S. (1998). "Visualization and Using Technology in A-Level Mathematics", *Proceedings of the Day Conference of the British Society for Research into Learning Mathematics (BSRLM)*, University of Birmingham, June, pp 45-50.

Elliott, S., Hudson, B., O'Reilly, D. (2000). "Visualisation and the Influence of Technology in 'A' Level Mathematics: a Classroom Investigation", *Research in Mathematics Education, Papers of the British Society for Research into Learning Mathematics (BSRLM)*, ed. Rowland, T., Morgan, C., Vol. 2, pp 151-168.

Ernest, P. (1994). "Social Constructivism and the Psychology of Mathematics Education", *Constructing Mathematical Knowledge: Epistemology and Mathematics Education*, ed. Ernest, P., The Falmer Press, London.

Escher, M.C.
<http://www.mcescher.com/> (accessed 30th November 2003).

Etchells, T., Monaghan, J. (1994). "Handheld Technology: Assessment Issues", *Acta Didactica Univ. Comenianae Mathematics*, Issue 3.

Ferrini-Mundi, J. (1987). "Spatial Training for Calculus Students: Sex Differences in Achievement and in Visualization Ability", *Journal for Research in Mathematics Education*, Vol. 18, pp 126-140.

Ford, N. (1995). "Levels and Types of Mediation in Instructional Systems: an Individual Differences Approach", *International Journal of Human-Computer Studies*, Academic Press Ltd., Vol. 43, pp 241-259.

Forster, P.A., Mueller, U. (2002). "What Effect does the Introduction of Graphics Calculators have on the Performance of Boys and Girls in Assessment in Tertiary Entrance Calculus?", *International Journal of Mathematics Education in Science and Technology*, Vol. 33, No. 6, pp 801-818.

Fosnot, C.T. (1996a). "Constructivism: a Psychological Theory of Learning", *Constructivism: Theory, Perspectives and Practice*, ed. Fosnot, C.T., Teachers College Press, Chapter 2, pp 8-33.

Fosnot, C.T. (1996b). "Teachers Construct Constructivism: the Center for Constructivist Teaching/Teacher Preparation Project", *Constructivism: Theory, Perspectives and Practice*, ed. Fosnot, C.T., Teachers College Press, Chapter 13, pp 205-216.

Francis, B. (1996). "Effective Investigation", *graphiti, The Newsletter of the Technology User Group*, Edition 1: Autumn, pp 13-15.

French, D. (1991). "Varying the Approach to A-Level Mathematics", *The Mathematical Gazette*, March, pp 62-68.

French, D. (1998). "School Algebra and the Challenge of the TI 92", *The Mathematical Gazette*, March, pp 62-71.

Fujita, H., Iitaka, S., Uetake, T., Yokochi, K. (1996). "Mathematics Education at Risk", *The Mathematical Gazette*, July, pp 352-355.

Gabbard, R.B. (2000). "Constructivism, Hypermedia and the World Wide Web", *CyberPsychology & Behavior*, Mary Ann Liebert, Inc., Vol. 3, No. 1, pp 103-110.

Gage, M., Pizer, A., Roth, V. (2002). "Webwork – Generating, Delivering and Checking Math Homework via the Internet", *Proceedings of the 2nd International Conference on the Teaching of Mathematics (at the undergraduate level)*, University of Crete, Greece, July, CD-ROM.

Galbraith, P.L., Haines, C.R. (1995). "Students' Mathematical Characteristics: Some Development Skills for Undergraduate Programmes", *The Institute of Mathematics and its Applications*, Vol. 31, pp 175-179.

Galbraith, P.L., Haines, C.R. (2000a). "Conceptual (Mis)understandings of Beginning Undergraduates", *International Journal of Mathematics Education in Science and Technology*, Vol. 31, No. 5, pp 651-678.

Galbraith, P.L., Haines, C.R. (2000b). "Prioritising Mathematical Skills for an Educated Workforce", *Cultural Diversity in Mathematics (Education): CIEAEM 51*, ed. Ahmed, A., Kramer, J.M., Williams, H., Chichester: Horwood Publishing Ltd., pp 103-112.

Garcia, A., Garcia, F., Hoya, S., Rodriguez, G., de la Villa, A. (2002). "Differential Calculus of Several Variables with Mathematica or Maple", *Proceedings of the 2nd International Conference on the Teaching of Mathematics (at the undergraduate level)*, University of Crete, Greece, July, CD-ROM.

Gardiner, J., Hudson, B., Povey, H. (2000). "Aspects of Construction and Proof in a Hand-Held Dynamic Geometry Environment", *Research in Mathematics Education, Papers of the British Society for Research into Learning Mathematics (BSRLM)*, ed. Rowland, T., Morgan, C., Vol. 2, pp 125-136.

Gill, P. (1998). "Mathematical Support for Engineering and Science Students", *Proceedings of the Day Conference of the British Society for Research into Learning Mathematics (BSRLM)*, University of Birmingham, June, pp 73-78.

Gil-Pérez, D., Guisasola, J., Moreno, A., Cachapuz, A., Pessoa De Carvalho, A.M., Martinez Torregrosa, J., Salinas, J., Valdés, P., Gonzáles, E., Gené Duch, A., Dumas-Carré, A., Tricárico, H., Gallego, R. (2002). "Defending Constructivism in Science Education", *Science and Education*, Kluwer Academic Publishers, Dordrecht, The Netherlands, Vol. 11, No. 6, pp 557-571.

Giraldo, V., Carvalho, L.M., Tall, D.O. (2002). "Theoretical-Computational Conflicts and the Concept Image of Derivative", *Proceedings of the Day Conference of the British Society for Research into Learning Mathematics (BSRLM)*, Nottingham University, November, Vol. 22, No. 3, pp 37-42.

Goldenberg, E.P. (1991). "The Difference Between Graphing Software and Educational Graphing Software", *Visualization in Teaching and Learning Mathematics*, ed. Zimmermann, W., Cunningham, S., Vol.19, pp 77-86.

Goldenberg, E.P. (1995). "Ruminations About Dynamic Imagery (and a Strong Plea for Research)", *Exploiting Mental Imagery with Computers in Mathematics Education*, NATO ASI Series F, Vol. 138, pp 202-224.

Graham, E., Rowlands, S. (2000). "Using Computer Software in the Teaching of Mechanics", *International Journal of Mathematics Education in Science and Technology*, Vol. 31, No. 4, pp 479-493.

Graham, T. (1996). "Practical Activities", *graphiti, The Newsletter of the Technology User Group*, Edition 1: Autumn, p 5.

Gray, E.M., Tall, D.O. (1994). "Duality, Ambiguity and Flexibility: a 'Proceptual' View of Simple Arithmetic", *Journal for Research in Mathematics Education*, Vol. 25, No. 2, pp 116-140.

Gray, E.M., Tall, D.O. (2001). "Relationships Between Embodied Objects and Symbolic Procepts: an Explanatory Theory of Success and Failure in Mathematics", *Proceedings of the 25th Conference of the International Group for the Psychology of Mathematics Education*, Utrecht, The Netherlands, Vol. 3, pp 65-72.

Gretton, H., Challis, N. (1996). "Mathematical Fun", *graphiti, The Newsletter of the Technology User Group*, Edition 1: Autumn, pp 10-12.

Gretton, H., Challis, N. (1999). "Bungee Jumping - A Mathematical Perspective using a CBL™/CBR™", *TI-TIME*, Issue 2, pp 1-4.

Habre, S. (2001). "Visualization Enhanced by Technology in the Learning of Multivariate Calculus", *The International Journal of Computer Algebra in Mathematics Education*, Vol. 8, No. 2, pp 115-129.

Hannah, J. (1998). "Student Use of the Graphics Calculator: Tool or Crutch?", *Proceedings of the International Conference on the Teaching of Mathematics*, University of the Aegean, Samos, Greece, July, pp 134-136.

Harel, G. (1994). "On Teacher Education Programmes in Mathematics", *International Journal of Mathematics Education in Science and Technology*, Vol. 25, No. 1, pp 113-119.

Harel, G., Kaput, J. (1991). "The Role of Conceptual Entities and their Symbols in Building Advanced Mathematical Concepts", *Advanced Mathematical Thinking*, ed. Tall, D.O., Kluwer, Chapter 6, pp 82-93.

Herring, M. (1995). "Solutions of $x = \lambda \cos x$, a Student Investigation", *International DERIVE Journal*, Vol. 2, No. 1, pp 15-23.

Hershkowitz, R., Arcavi, A., Bruckheimer, M. (2001). "Reflections on the Status and Nature of Visual Reasoning - the Case of the Matches", *International Journal of Mathematics Education in Science and Technology*, Vol. 32, No. 2, pp 255-265.

Hitt, F. (1998a). "The Role of the Semiotic Representations in the Learning of Mathematics", *Proceedings of the Day Conference of the British Society for Research into Learning Mathematics (BSRLM)*, University of Leeds, November, pp 23-28.

Hitt, F. (1998b). "Difficulties in the Articulation of Different Representations Linked to the Concept of Function", *Journal of Mathematical Behaviour*, Ablex Publishing Corp., Vol. 17, No. 1, pp 123-134.

Holt-Reynolds, D. (2000). "What does the Teacher Do? The Many Faces of Constructivism", *Teaching and Teacher Education*, Elsevier Science, Vol. 16, No. 1, pp 21-32.

Honebein, P.C., Duffy, T.M., Fishman, B.J. (1993). "Constructivism and the Design of Learning Environments: Context and Authentic Activities for Learning", *Designing Environments for Constructive Learning*, Springer-Verlag, Berlin, pp 87-108.

Hooper, A. (2002). "Diagnostic Testing and Follow-Up for First-Year Mathematics Students", *Proceedings of the 2nd International Conference on the Teaching of Mathematics (at the undergraduate level)*, University of Crete, Greece, July, CD-ROM.

Hope, L. (1994). "Mission Possible: Computers in Chess and A-Level Mathematics", *The Mathematical Gazette*, March, pp 11-17.

Houston, S.K. (1996). "Assessing A-level Mathematics", *The Mathematical Gazette*, November, pp 538-541.

Houston, S.K. (1997). "The Role of Mathematical Modelling in Undergraduate Courses", *Bulletin of the Irish Mathematical Society*, 1997, Vol. 38, pp 30-38.

Houston, S.K. (1998). "Innovation in University Mathematics Teaching", *MathSkills Newsletter*, February, pp 4-6.

Houston, S.K. (2001). "Appropriate Assessment of Key/Core Skills", *paper presented at the MSOR LTSN Workshop "Matching Assessment to Learning in Mathematics and Statistics"*, Birmingham, May, and recorded in *MSOR Connections*, Vol. 1, No. 3, August, pp 29-31.

Houston, S.K., Blum, W., Huntley, I., Neill, N.T. (1997). "Teaching and Learning Mathematical Modelling: Innovation, Investigation and Applications", Albion Publications, Chichester.

Hubbard, R. (1997). "Designing Questions that Develop Understanding", *International Journal of Mathematics Education in Science and Technology*, Taylor & Francis Ltd., Vol. 28, No. 6, pp 793-801.

Hughes-Hallett, D. (1991). "Visualization and Calculus Reform", *Visualization in Teaching and Learning Mathematics*, ed. Zimmermann, W., Cunningham, S., Vol.19, pp 121-126.

Ihejieto, D.O., Emenalo, S.I. (1996). "Parametric Analysis of Performance Trends for Mathematics Degree Students vis-à-vis Directional Bearing on Entry Qualification", *International Journal of Mathematics Education in Science and Technology*, Taylor & Francis Ltd., Vol. 27, No. 2, pp 163-175.

Jackson, A. (1997a). "The Math Wars: California Battles It Out over Mathematics Education Reform (Part 1)", *Notices of the AMS*, June/July, Vol. 44, No. 6, pp 695-702.

Jackson, A. (1997b). "The Math Wars: California Battles It Out over Mathematics Education Reform (Part 2)", *Notices of the AMS*, August, Vol. 44, No. 7, pp 817-823.

Jones, K., Bills, C. (1998). "Visualisation, Imagery and the Development of Geometrical Reasoning", *Proceedings of the Day Conference of the British Society for Research into Learning Mathematics (BSRLM)*, University of Birmingham, June, pp 123-128.

Kahn, P. (1998). "A Distinctive Skills Profile for Undergraduate Mathematics", *MathSkills Newsletter*, February, pp 7-8.

Kaput, J.J. (1993). "The Urgent Need for Proleptic Research in the Representation of Quantitative Relationships", *Integrating Research on the Graphical Representation of Functions*, ed. Romberg, T.A., Fennema, E., Carpenter, T.P., Lawrence Erlbaum Associates, Chapter 10, pp 279-311.

Kautschitsch, H. (2002). "Dynamic Geometry Software Not Only for Simple Dragging", *Proceedings of the 2nd International Conference on the Teaching of Mathematics (at the undergraduate level)*, University of Crete, Greece, July, CD-ROM.

Kawski, M. (2002). "Interactive Visualization in Complex Analysis", *Proceedings of the 2nd International Conference on the Teaching of Mathematics (at the undergraduate level)*, University of Crete, Greece, July, CD-ROM.

Kendal, M., Stacey, K. (1999). "Varieties of Teacher Privileging for Teaching Calculus with Computer Algebra Systems", *The International Journal of Computer Algebra in Mathematics Education*, Vol. 6, No. 4, pp 233-247.

Kidron, I., Zehavi, N. (2002). "The Role of Animation in Teaching the Limit Concept", *The International Journal of Computer Algebra in Mathematics Education*, Vol. 9, No. 3.

Kieran, C. (1993). "Functions, Graphing and Technology: Integrating Research on Learning and Instruction", *Integrating Research on the Graphical Representation of Functions*, ed. Romberg, T.A., Fennema, E., Carpenter, T.P., Lawrence Erlbaum Associates, Chapter 8, pp 189-237.

Kopp, E., Higgins, D. (1997). "What do Employers Want?", *MathSkills Newsletter*, March, pp 13-14.

Kozma, R.B. (1991). "Learning with Media", *Review of Educational Research*, Vol. 61, No. 2, pp 179-211.

Kutzler, B. (1996). "Improving Maths Teaching with DERIVE", *Improving Mathematics Teaching with DERIVE*, Chartwell-Bratt, pp 1-6.

Kutzler, B. (1999). "The Algebraic Calculator as a Pedagogical Tool for Teaching Mathematics", *Improving Mathematics Teaching with the TI-92*, June.

Laurillard, D. (1990). "Computers and the Emancipation of Students: Giving Control to the Learner", *Computers and Learning*, ed. Boyd-Barrett, O., Scanlon, E., Addison-Wesley Publishing Company, Paper 1.5, pp 64-80.

Learning and Teaching Support Network (MSOR), Institute of Mathematics and its Applications (IMA), London Mathematical Society (LMS), Engineering Council. (2000). "Measuring the Mathematics Problem".

Leinbach, C., Aspetsberger, K., Barzel, B., Fuchs, K., Furukawa, A., Heugl, H., Mann, G., Rothery, A., Sato, T., Schweiger, F. (1997). "The Curriculum in a CAS Environment", *The State of Computer Algebra in Mathematics Education*, ed. Berry, J. et al., pp 21-31.

Leinbach, C., Pountney, D., Etchells, T. (2002). "Appropriate use of a CAS in the Teaching and Learning of Mathematics", *International Journal of Mathematics Education in Science and Technology*, Taylor & Francis Ltd., Vol.33, No.1, pp 1-14.

Lemerise, T. (1993). "Piaget, Vygotsky and Logo", *The Computing Teacher*, April, pp 24-28.

Lepper, M.R., Henderlong, J. (2000). "Turning 'Play' into 'Work' and 'Work' into 'Play': 25 Years of Research on Intrinsic Versus Extrinsic Motivation", *Intrinsic and Extrinsic Motivation: the Search for Optimal Motivation and Performance*, Academic Press.

Lepper, M.R., Woolverton, M., Mumme, D.L., Gurtner, J.L. (1993). "Motivational Techniques of Expert Human Tutors: Lessons for the Design of Computer-Based Tutors", *Computers as Cognitive Tools*, pp 75-105.

Lerman, S. (1994). "Articulating Theories of Mathematics Learning", *Constructing Mathematical Knowledge: Epistemology and Mathematical Education*, ed. Ernest, P., The Falmer Press, London.

Lerman, S. (1996a). "Some Problems in Research on Mathematics Teaching and Learning from a Socio-Cultural Approach", *Proceedings of the Day Conference of the British Society for Research into Learning Mathematics (BSRLM)*, Institute of Education, London, November, pp 33-36.

Lerman, S. (1996b). "Intersubjectivity in Mathematics Learning: a Challenge to the Radical Constructivist Paradigm", *Journal for Research into Mathematics Education*, Vol. 27, No. 2, pp 133-150.

Levin, T. (1986). "Formative Evaluation in the Context of Computerised Instruction", *Studies in Educational Evaluation*, Vol. 12, pp 121-129.

Liao, T. (1993). "Designing Microcomputer Courseware: Using Computers as a Tool for Learning Engineering Concepts", *Advanced Educational Technology in Technology Education*, NATO ASI Series F, Vol. 109, pp 199-226.

Llinares, S. (2000). "Secondary School Mathematics Teacher's Professional Knowledge: a Case from the Teaching of the Concept of Function", *Teachers and Teaching: Theory and Practice*, Carfax Publishing, part of the Taylor and Francis Group, Vol. 6, No. 1, pp 41-62.

London Mathematical Society (LMS), Institute of Mathematics and its Applications (IMA), Royal Statistical Society (RSS). (1995). "Tackling the Mathematics Problem".

Mackie, D.M. (2002). "Using Computer Algebra to Encourage a Deep Learning Approach to Calculus", *Proceedings of the 2nd International Conference on the Teaching of Mathematics (at the undergraduate level)*, University of Crete, Greece, July, CD-ROM.

Maeder, R.E., Amrhein, B., Gloor, O. (1995). "Illustrated Mathematics: Visualization of Mathematical Objects", *Mathematica in Education and Research*, Vol. 4, No. 3, pp 5-11.

Malabar, I., Pountney, D.C. (2000). "How do Traditional Examination Questions Fare in the Presence of a Computer Algebra System (CAS)?", *International Journal of Computer Algebra in Mathematics Education*, Vol. 7, No. 4, pp 241-254.

Malabar, I., Pountney, D.C. (2002). "Using Technology to Integrate Constructivism and Visualisation in Mathematics Education", *Proceedings of the 2nd International Conference on the Teaching of Mathematics*, University of Crete, Greece, July, CD-ROM.

Malabar, I., Pountney, D.C., Townend, M.S. (1998). "Combining Visual and Symbolic Skills in the Teaching and Learning of Mathematics", *Proceedings of the 3rd International DERIVE and TI-92 Conference*, Gettysburg, Pennsylvania, USA, July, CD-ROM.

Malone, T.W. (1980). "What Makes Things Fun to Learn? A Study of Intrinsically Motivating Computer Games", *Cognitive and Instructional Sciences Series*, Tech. Rep. CIS-7 (SSL-80-11).

Malone, T.W., Lepper, M.R. (1987). "Making Learning Fun: a Taxonomy of Intrinsic Motivations for Learning", *Aptitude, Learning and Instruction: Conative and Affective Process Analyses*, pp 223-253.

Mandelbrot set
<http://www.mbfractals.com/> (accessed 30th November 2003).

MAPLE
<http://www.maplesoft.com/> (accessed 30th November 2003).

Marin, N., Benarroch, A., Gomez, E.J. (2000). "What is the Relationship Between Social Constructivism and Piagetian Construcivism? An Analysis of the Characteristics of the Ideas within both Theories", *International Journal of Science Education*, Taylor and Francis Ltd., Vol. 22, No. 3, pp 225-238.

Mariotti, M.A. (1995). "Images and Concepts in Geometrical Reasoning", *Exploiting Mental Imagery with Computers in Mathematics Education*, NATO ASI Series F, Vol. 138, pp 97-116.

Marshall, A.D., Hurley, S., McIntosh-Smith, S.N., Martin, R.R., Stephens, N.M. (1994). "Novel Uses of Computers for Teaching Computing", *Axis*, Vol. 1, No. 2, pp 30-41.

Mason, J. (1984). "Learning and Doing Mathematics", *The Open University Press, Milton Keynes.*

Mason, J., Heal, B. (1995). "Mathematical Screen Metaphors", *Exploiting Mental Imagery with Computers in Mathematics Education*, NATO ASI Series F, Vol. 138, pp 291-308.

MATHEMATICA
<http://www.wolfram.com/> (accessed 30th November 2003).

Mathews, D.M. (1994). "Precalculus Investigations Using DERIVE", Harper Collins College Publishers.

Mathskills Discipline Network.
<http://www.hull.ac.uk/mathskills/> (accessed 30th November 2003).

MATHWISE
<http://www.bham.ac.uk/mathwise/> (accessed 30th November 2003).

Mavrikis, M., Macciocia, A. (2002). "Developing WALLIS: a Web-Based System to Enhance Mathematics Teaching", *Proceedings of the 2nd International Conference on the Teaching of Mathematics (at the undergraduate level)*, University of Crete, Greece, July, CD-ROM.

Mayo, E. (1933). "The Human Problems of an Industrial Civilisation", Macmillan.

McDill, J.M., Rash, A. (2002). "Interactive Visualization Software for Calculus", *Proceedings of the 2nd International Conference on the Teaching of Mathematics (at the undergraduate level)*, University of Crete, Greece, July, CD-ROM.

Middleton, W. (2001). "The Lights are Going Out All Over the UK", *Proceedings of the 27th Undergraduate Mathematics Teaching Conference (UMTC)*, University of Birmingham, September, pp 12-13.

Milheim, W.D. (1993). "How to use Animation in Computer Assisted Learning", *British Journal of Educational Technology*, Vol. 24, No. 3, pp 171-178.

Miller, T., Abramson, J. (2002). "Virtually Face-to-Face Mathematics Instruction", *Proceedings of the 2nd International Conference on the Teaching of Mathematics (at the undergraduate level)*, University of Crete, Greece, July, CD-ROM.

MINITAB
<http://www.minitab.com/> (accessed 30th November 2003).

Mogetta, C. (2000). "Cabri as a Cognitive Tool for the Evolution of a Sense of Proving", *Research in Mathematics Education, Papers of the British Society for Research into Learning Mathematics (BSRLM)*, ed. Rowland, T., Morgan, C., Vol. 2, pp 107-124.

Molyneux-Hodgson, S., Butterfield, A., Sutherland, R. (1998). "Mathematical Competencies of GNVQ Science Students: the Role of Computers", *Final report to the Leverhulme Trust, University of Bristol.*

Molyneux-Hodgson, S., Sutherland, R. (1996). "Mathematics in the Practice of Vocational Science", *Proceedings of the Day Conference of the British Society for Research into Learning Mathematics (BSRLM)*, Institute of Education, London, November, pp 37-42.

Molyneux-Hodgson, S., Sutherland, R. (2002). "Mathematics for Post-16 Vocational Courses", *Aspects of Secondary Mathematics*, ed. Haggarty, L., Routledge, London, pp 52-70.

Monaghan, J., Johnson, P., Berry, J., Maull, W. (1998). "Routine Questions and A-Level Mathematics Grades", *Proceedings of the Day Conference of the British Society for Research into Learning Mathematics (BSRLM)*, University of Leeds, November, pp 11-16.

Moreno, L.E., Sacristán, A.I. (1995). "On Visual and Symbolic Representations", *Exploiting Mental Imagery with Computers in Mathematics Education*, NATO ASI Series F, Vol. 138, pp 178-189.

Moritz, R.E. (1914). "Memorabilia Mathematica: the Philomath's Quotation Book: 1140 Anecdotes, Aphorisms and Passages by Famous Mathematicians, Scientists and Writers". Mathematical Association of America: Washington D.C.

Moschkovich, J., Schoenfeld, A.H., Arcavi, A. (1993). "Aspects of Understanding: on Multiple Perspectives and Representations of Linear Relations and Connections Among Them", *Integrating Research on the Graphical Representation of Functions*, ed. Romberg, T.A., Fennema, E., Carpenter, T.P., Lawrence Erlbaum Associates, Chapter 4, pp 69-100.

Murphy, T., White, J., Stone, A., Beauchamp, A. (2002). "Using Web-Based Interactive Graphics to Enhance Understanding of Parametric Equations: Lessons Learned", *Proceedings of the 2nd International Conference on the Teaching of Mathematics (at the undergraduate level)*, University of Crete, Greece, July, CD-ROM.

Myers, K.A. (1997). "Function Fundamentals", *Mathematical Activities with DERIVE*, ed. Graham, E., Berry, J.S., Watkins, A.J.P., Chartwell-Bratt, pp 1-10.

Nagy, P., Traub, R., MacRury, K., Roslyn, K. (1991). "High School Calculus: Comparing the Content of Assignments and Tests", *Journal for Research in Mathematics Education*, Vol. 22, No. 1, pp 69-75.

National Council for Vocational Qualifications (NCVQ) <http://www.qca.org.uk/> (accessed 30th November 2003).

National Research Council. (1990). "Renewing U.S. Mathematics", National Academy Press, Washington, D.C.

Nelson, R.B. (2000). "Proofs Without Words II: More Exercises in Visual Thinking", *Mathematical Association of America.*

Noguera, N. (2001). "A Description of Tenth Grade Algebra Students' Attitudes and Cognitive Development when Learning Algebra using CAS", *The International Journal of Computer Algebra in Mathematics Education*, Vol. 8, No. 4.

Noguera, N. (2002). "Integrating Mathematical Knowledge and Pedagogy in Teacher Preparation Programs", *Proceedings of the 2nd International Conference on the Teaching of Mathematics (at the undergraduate level)*, University of Crete, Greece, July, CD-ROM.

Nyman, M.A., Berry, J. (2002). "Developing Transferable Skills in Undergraduate Mathematics Students Through Mathematical Modelling", *Teaching Mathematics and its Applications*, Oxford University Press, Vol. 21, No. 1, pp 29-45.

Nyondo, A.C. (1993). "Mathematics Courses with a PC", *International Journal of Mathematics Education in Science and Technology*, Vol. 24, No. 4, pp 569-574.

Oevel, G., Unger, L., Mertsching, B. (2002). "Web-Based Teaching and Learning with Math-Kit", *Proceedings of the 2nd International Conference on the Teaching of Mathematics (at the undergraduate level)*, University of Crete, Greece, July, CD-ROM.

Olivero, F. (2001). "Conjecturing in Open Geometric Situations using Dynamic Geometry: an Exploratory Classroom Experiment", *Research in Mathematics Education, Papers of the British Society for Research into Learning Mathematics (BSRLM)*, ed. Morgan, C., Jones, K., Vol. 3, pp 229-246.

O'Reilly, D., Pratt, D., Winbourne, P. (1997). "Constructive and Instructive Representation", *Journal of Information Technology for Teacher Education*, Vol. 6, No. 1, pp 73-92.

Ormell, C. (1997). "Mathematics and the School Crisis", *MathSkills Newsletter*, March, pp 16-18.

Panoutsopoulos, A., Potari, D. (1995). "Applying 'Short' Computer Programs to Elementary Mathematics for Children", *International Journal of Mathematics Education in Science and Technology*, Vol. 26, No. 1, pp 11-28.

Pappas, J., Koleza, E., Rizos, J., Skordoulis, C. (2002). "Using Interactive Digital Video and Motion Analysis to Bridge Abstract Mathematical Notions with Concrete Everyday Experiences", *Proceedings of the 2nd International Conference on the Teaching of Mathematics (at the undergraduate level)*, University of Crete, Greece, July, CD-ROM.

Pea, R.D. (1987). "Cognitive Technologies for Mathematics Education", *Cognitive Science and Mathematics Education*, pp 89-122.

Pear, J.J., Crone-Todd, D.E. (2002). "A Social Constructivist Approach to Computer-Mediated Instruction", *Computers and Education*, Elsevier Science, Vol. 38, No. 1, pp 221-231.

Pesonen, M. (2002). "Interactive Animated Problem Sets in Learning Functions and Binary Operations", *Proceedings of the 2nd International Conference on the Teaching of Mathematics (at the undergraduate level)*, University of Crete, Greece, July, CD-ROM.

Philipp, R.A., Martin, W.O., Richgels, G.W. (1993). "Curricular Implications of Graphical Representations of Functions", *Integrating Research on the Graphical Representation of Functions*, ed. Romberg, T.A., Fennema, E., Carpenter, T.P., Lawrence Erlbaum Associates, Chapter 9, pp 239-277.

Phillips, R.J., Gillespie, J., Pead, D. (1995). "On Designing Screen Images to Generate Mental Images", *Exploiting Mental Imagery with Computers in Mathematics Education*, NATO ASI Series F, Vol. 138, pp 225-232.

Phillips, R.J., Pead, D. (1994). "Multimedia Resources in the Mathematics Classroom", *Journal of Computer Assisted Learning*, Vol. 10, No. 4, pp 216-228.

Poblete, A., Diaz, V. (2002). "Educational Reforms and the Competencies of the Mathematics Teacher", *Proceedings of the 2nd International Conference on the Teaching of Mathematics (at the undergraduate level)*, University of Crete, Greece, July, CD-ROM.

Pope, S. (2002). "Using Small Software on Graphic Calculators", *Proceedings of the Day Conference of the British Society for Research into Learning Mathematics (BSRLM)*, University of Durham, February, Vol. 22, No. 1, pp 36-40.

Porkess, R. (1996). "Perceptions of Change", *The Mathematical Gazette*, July, pp 356-359.

Porkess, R. (2002). "Supply and Demand: the School-University Interface in Mathematics", *Proceedings of the 28th Undergraduate Mathematics Teaching Conference (UMTC)*, University of Birmingham, September, pp 10-31.

Presmeg, N.C. (1986). "Visualisation and Mathematical Giftedness", *Educational Studies in Mathematics*, Vol. 17, pp 297-311.

Presmeg, N.C., Bergsten, C. (1995). "Preference for Visual Methods: an International Study", *Proceedings of PME*, Vol. 19, No. 3, pp 58-65.

Rickel, J.W. (1989). "Intelligent Computer-Aided Instruction: a Survey Organized Around System Components", *I.E.E.E. Transactions on Systems, Man and Cybernetics*, Vol. 19, No. 1, pp 40-57.

Romberg, T.A., Carpenter, T.P., Fennema, E. (1993). "Toward a Common Research Perspective", *Integrating Research on the Graphical Representation of Functions*, ed. Romberg, T.A., Fennema, E., Carpenter, T.P., Lawrence Erlbaum Associates, Chapter 1, pp 1-7.

Rowlands, S., Graham, T., Berry, J. (1996). "The Vygotskian Perspective and the Radical versus the Social Constructivism Debate", *Proceedings of the Day Conference of the British Society for Research into Learning Mathematics (BSRLM)*, Institute of Education, London, November, pp 49-56.

Rowlands, S., Graham, T., Berry, J. (1997). "The Socratic Method of Strategic Questioning to Facilitate the Construction of the Target-Concept within the Students Zone of Proximal Development", *Proceedings of the Day Conference of the British Society for Research into Learning Mathematics (BSRLM)*, University of Bristol, November, pp 50-55.

Ruthven, K. (1989). "An Exploratory Approach to Advanced Mathematics", *Educational Studies in Mathematics*, Kluwer Academic Publishers, Vol. 20, No. 4, pp 449-467.

Ruthven, K. (1990). "The Influence of Graphic Calculator Use on Translation from Graphic to Symbolic Forms", *Educational Studies in Mathematics*, Kluwer Academic Publishers, Vol. 21, No. 5, pp 431-450.

Ruthven, K. (1991). "Advanced Calculators and Advanced-Level Mathematics", *The Mathematical Gazette*, March, pp 48-54.

Ruthven, K. (1997). "Computer Algebra Systems (CAS) in Advanced-Level Mathematics", *Report to SCAA*, University of Cambridge, School of Education/School Curriculum and Assessment Authority, March.

Sahin, T.Y. (2003). "Student Teachers' Perceptions of Instructional Technology: Developing Materials Based on a Constructivist Approach", *British Journal of Educational Technology*, Blackwell Publishing Ltd., Oxford, UK, Boston, USA, Vol. 34, No. 1, pp 67-74.

Sam, L.C., Ernest, P. (2000). "A Survey of Public Images of Mathematics", *Research in Mathematics Education, Papers of the British Society for Research into Learning Mathematics (BSRLM)*, ed. Rowland, T., Morgan, C., Vol. 2, pp 193-206.

Scardamalia, M., Bereiter, C. (1991). "Higher Levels of Agency for Children in Knowledge Building: a Challenge for the Design of New Knowledge Media", *The Journal of the Learning Sciences*, 1(1), pp 37-68.

Schank, R.C. (1994). "Active Learning Through Multimedia", *I.E.E.E. Multimedia*, Spring, pp 69-78.

Schifter, D. (1996). "A Constructivist Perspective on Teaching and Learning Mathematics", *Constructivism: Theory, Perspectives and Practice*, ed. Fosnot, C.T., Teachers College Press, Chapter 5, pp 73-91.

Scott, B. (2001). "Conversation Theory: a Constructivist, Dialogical Approach to Educational Technology", *Cybernetics & Human Knowing*, Imprint Academic, Exeter, Vol. 8, No. 4, pp 25-46.

Sein, M.K., Olfman, L., Bostrom, R.P., Davis, S.A. (1993). "Visualization Ability as a Predictor of User Learning Success", *International Journal of Man-Machine Studies*, Academic Press Ltd., Vol. 39, pp 599-620.

Sharp, J. (1997). "What's my Line?", *graphiti. The Newsletter of the Technology User Group*, Edition 2: Spring, p 12.

Shire, A.J. (1992). "Applicability of Perceptual Psychology to Computational Image Understanding", *University of Southampton Technical Report Series*, Department of Electronics and Computer Science, CSTR 92-18.

Short, L. (1998). "Teaching with the TI92: the Napier Pilot Project", *MathSkills Newsletter*, February, pp 12-14.

Skemp, R.R. (1976). "Relational Understanding and Instrumental Understanding", *Mathematics Teaching*, Vol. 77, pp 20-26.

Smith, A., Berry, J. (2002). "Observing Student Working Styles when using Graphic Calculators", *Technology in Mathematics Teaching*, ed. Borovcnik, M., Kautschitsch, H., *Proceedings of ICTMT5*, Klagenfurt, Austria, 2001.

Smith, E. (1998). "Social Constructivism, Individual Constructivism and the Role of Computers in Mathematics Education", *Journal of Mathematical Behaviour*, Elsevier Science, Vol. 17, No. 4, pp 411-425.

Smith, G., Wood, L., Coupland, M., Stephenson, B., Crawford, K., Ball, G. (1996). "Constructing Mathematical Examinations to Assess a Range of Knowledge and Skills", *International Journal of Mathematics Education in Science and Technology*, Vol. 27, No. 1, pp 65-77.

Soloway, E. (1991). "How the Nintendo Generation Learns", *Communications of the ACM*, Vol. 34, No. 9.

SPSS
<http://www.spss.com/> (accessed 30th November 2003).

Squires, D., McDougall, A. (1996). "Software Evaluation: a Situated Approach", *Journal of Computer Assisted Learning*, Vol. 12, pp 146-161.

Stehlikova, N. (2002). "Geometrical Transformations - Constructivist Analytical Approach", *Proceedings of the 2nd International Conference on the Teaching of Mathematics (at the undergraduate level)*, University of Crete, Greece, July, CD-ROM.

Stepien, W., Gallagher, S.A. (1993). "Problem-based Learning: As Authentic as it Gets", *Educational Leadership*, Vol. 50, No. 7, pp 25-28.

Stevenson, I. (2000). "Mechanics, Modelling and Pedagogy: Reflections on the Design of Contexts for New Technologies", *Research in Mathematics Education, Papers of the British Society for Research into Learning Mathematics (BSRLM)*, ed. Rowland, T., Morgan, C., Vol. 2, pp 137-150.

Stewart, I., Cohen, J. (1997). "Figments of Reality", Cambridge University Press, Chapter 3, pp 63-76.

Steyn, T., Maree, J.G. (2002). "A Profile of First Year Students' Learning Preferences and Study Orientation in Mathematics", *Proceedings of the 2nd International Conference on the Teaching of Mathematics (at the undergraduate level)*, University of Crete, Greece, July, CD-ROM.

Sutherland, R. (1995). "Mediating Mathematical Action", *Exploiting Mental Imagery with Computers in Mathematics Education*, NATO ASI Series F, Vol. 138, pp 71-81.

Tall, D.O. (1986a). "Using the Computer to Represent Calculus Concepts", *Lecture given at the I Vème Ecole d'Etè de Didactique des Mathematiques*, Orleans, July, pp 1-27.

Tall, D.O. (1986b). "Building and Testing a Cognitive Approach to the Calculus using Interactive Computer Graphics", PhD. Thesis, University of Warwick.

Tall, D.O. (1987). "Graphical Packages for Mathematics Teaching and Learning", *Informatics and the Teaching of Mathematics*, ed. Johnson, D.C., Lovis, F., North Holland, pp 39-47.

Tall, D.O. (1991). "Intuition and Rigour: the Role of Visualization in the Calculus", *Visualization in Teaching and Learning Mathematics*, ed. Zimmermann, W., Cunningham, S., Vol. 19, pp 105-119.

Tall, D.O. (1992). "Students' Difficulties in Calculus", *Plenary Presentation in Working Group 3, ICME-7*, Québec, Canada, August, pp 13-28.

Tall, D.O. (1993). "Interrelationships Between Mind and Computer: Processes, Images, Symbols", *Advanced Technologies in the Teaching of Mathematics and Science*, ed. Ferguson, D.L., Springer-Verlag, pp 385-413.

Tall, D.O. (1995). "Visual Organisers for Formal Mathematics", *Exploiting Mental Imagery with Computers in Mathematics Education*, NATO ASI Series F, Vol. 138, pp 52-70.

Tall, D.O. (2000a). "Technology and Versatile Thinking in Mathematical Development", *Proceedings of TIME 2000*, Auckland , New Zealand, ed. Thomas, M. O.J., pp 33-50.

Tall, D.O. (2000b). "Biological Brain, Mathematical Mind and Computational Computers (how the Computer can Support Mathematical Thinking and Learning)", *Proceedings of the Fifth Asian Technology Conference in Mathematics*, Chiang Mai, Thailand, ed. Wei-Chi Yang, Sung-Chi Chu, Jen-Chung Chuan, ATCM Inc., Blackwood V.A., pp 3-20.

Tall, D.O. (2000c). "Cognitive Development in Advanced Mathematics Using Technology", *Mathematics Education Research Journal*, Vol. 12, No. 3, pp 196-218.

Tall, D.O., Razali, M.R. (1993). "Diagnosing Students' Difficulties in Learning Mathematics", *International Journal of Mathematics Education in Science and Technology*, Vol. 24, No. 2, pp 209-222.

Tall, D.O., Thomas, M. (1991). "Encouraging Versatile Thinking in Algebra Using the Computer", *Educational Studies in Mathematics*, Kluwer Academic Publishers, Vol. 22, No. 2, pp 125-147.

TEXAS INSTRUMENTS
<http://www.ti.com/> (accessed 30th November 2003).

Thompson, S.V. (1990). "Visual Imagery: a Discussion", *Educational Psychology*, Vol. 10, No. 2, pp 141-167.

TI-89, TI-92, etc.
<http://www.ti.com/> (accessed 30th November 2003).

TOOLBOOK
<http://www.asymetrix.com/> (accessed 30th November 2003).

Townend, M.S. (1994). "Mathematical Skills Maintenance with DERIVE", *SEDA Publication Paper 87*, December, pp 35-40.

Vinner, S. (1989). "The Avoidance of Visual Considerations in Calculus Students", *Focus on Learning Problems in Mathematics*, Spring Edition, Vol. 11, No. 2, pp 149-156.

Vinner, S. (1991). "The Role of Definitions in the Teaching and Learning of Mathematics", *Advanced Mathematical Thinking*, ed. Tall, D.O., Kluwer, Chapter 5, pp 65-81.

von Glasersfeld, E. (1984). "An Introduction to Radical Constructivism", *The Invented Reality*, ed. Watzlawick, P., Norton & Co., New York, pp 17-40.

von Glasersfeld, E. (1990). "An Exposition of Constructivism: Why Some Like it Radical", *Constructivist Views on the Teaching and Learning of Mathematics, Journal for Research into Mathematics Education Monograph No. 4*, ed. Davis, R.B., Maher, C.A., Noddings, N., pp 19-29.

von Glasersfeld, E. (1992). "A Radical Constructivist View of Basic Mathematical Concepts", *Paper Presented at Topic Group 16, Seventh International Congress on Mathematical Education,* Quếbec.

von Glasersfeld, E. (1995). "Radical Constructivism: a Way of Knowing and Learning", The Falmer Press, London.

von Glasersfeld, E. (1996). "Introduction: Aspects of Constructivism", *Constructivism: Theory, Perspectives and Practice*, ed. Fosnot, C.T., Teachers College Press, Chapter 1, pp 3-7.

Voyage 200
<http://www.ti.com/> (accessed 30th November 2003).

Vygotsky, L. (1962). "Thought and Language", (First published 1934), Massachusetts Institute of Technology Press, Cambridge, Massachusetts.

Waldock, J., Gretton, H., Challis, N., Thomlinson, M. (2002). "Using the Web to Enhance Student Learning", *Proceedings of the 2nd International Conference on the Teaching of Mathematics (at the undergraduate level)*, University of Crete, Greece, July, CD-ROM.

Watson, J. (2001). "Social Constructivism in the Classroom", *Support for Learning*, Blackwell Publishers Ltd., Oxford, UK, Boston, USA, Vol. 16, No. 3, pp 140-147.

Wheatley, G.H. (1991). "Enhancing Mathematics Learning Through Imagery", *Arithmetic Teacher*, Vol. 39, No. 1, pp 34-36.

Wheatley, G.H., Brown, D. (1994). "The Construction and Re-presentation of Images in Mathematical Activity", *Proceedings of PME*, Vol. 18, Pt. 1, p 8.

Williams, S.R. (1993). "Some Common Themes and Uncommon Directions", *Integrating Research on the Graphical Representation of Functions*, ed. Romberg, T.A., Fennema, E., Carpenter, T.P., Lawrence Erlbaum Associates, Chapter 11, pp 313-337.

Woods, D. (2002). "Netmath Paving the Way on the Web", *Proceedings of the 2nd International Conference on the Teaching of Mathematics (at the undergraduate level)*, University of Crete, Greece, July, CD-ROM.

Woods, D.R. (1994). "Problem-Based Learning: How to Gain the Most from PBL", Hamilton, Ontario, Canada.

Yerushalmy, M., Schwartz, J.L. (1993). "Seizing the Opportunity to Make Algebra Mathematically and Pedagogically Interesting", *Integrating Research on the Graphical Representation of Functions*, ed. Romberg, T.A., Fennema, E., Carpenter, T.P., Lawrence Erlbaum Associates, Chapter 3, pp 41-67.

Zeuli, J. (1986). "The Use of the Zone of Proximal Development in Everyday and School Contexts: a Vygotskian Critique", *The Institute for Research on Teaching (Occasional Paper No. 100)*, Michigan State University, Michigan.

Zimmermann, W. (1991). "Visual Thinking in Calculus", *Visualization in Teaching and Learning Mathematics*, ed. Zimmermann, W., Cunningham, S., Vol. 19, pp 127-137.

Zimmermann, W., Cunningham, S. (1991). "What is Mathematical Visualization?", *Visualization in Teaching and Learning Mathematics*, ed. Zimmermann, W., Cunningham, S., Vol. 19, pp 1-7.

Zorn, P. (2002). "Algebra, Computer Algebra and Mathematical Thinking", *Proceedings of the 2nd International Conference on the Teaching of Mathematics (at the undergraduate level)*, University of Crete, Greece, July, CD-ROM.

APPENDICES

Appendix A - Software Screens

Appendix B - Summary Sheet

Appendix C - Feedback Questionnaire

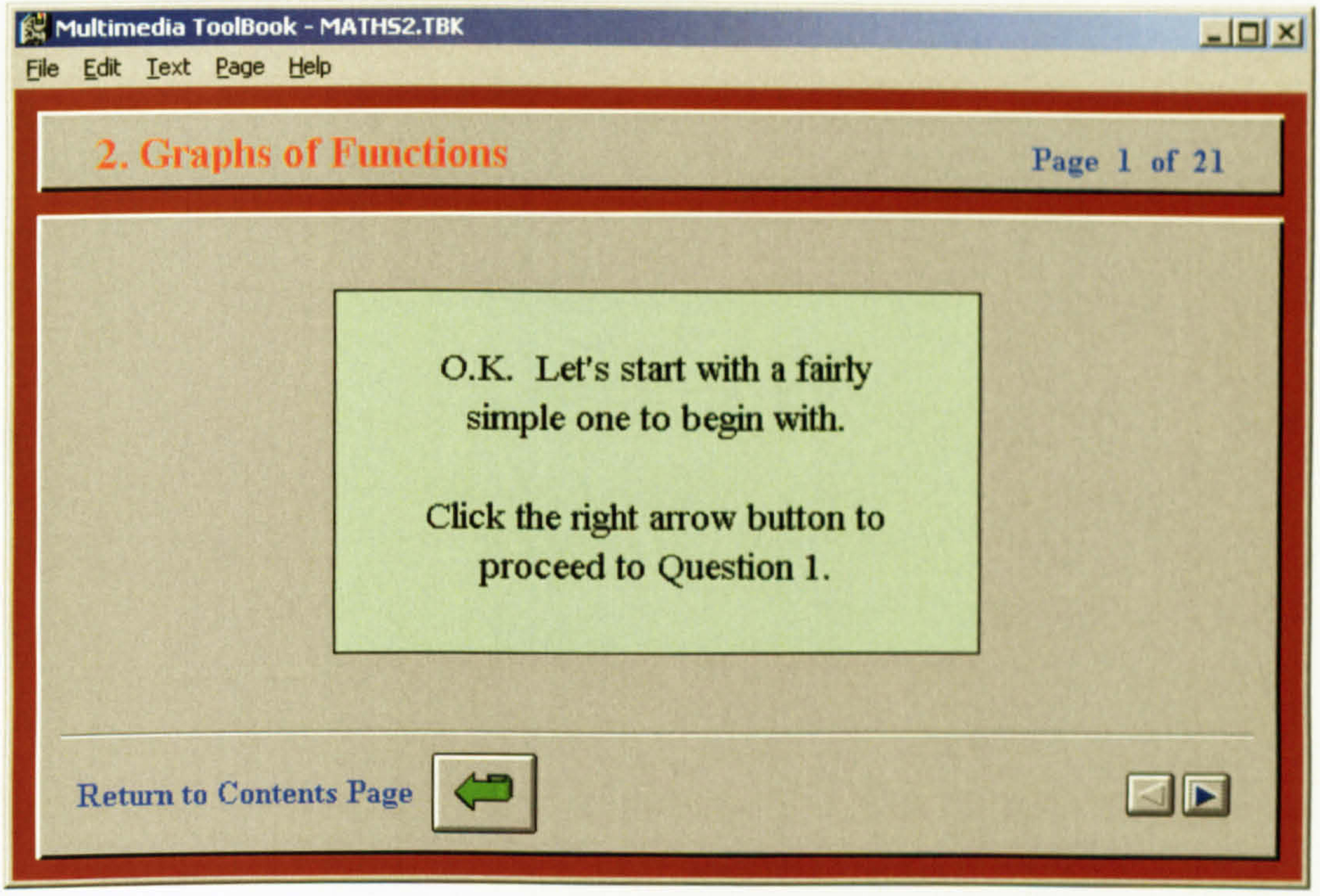
Appendix D - Analysis of Test Questions

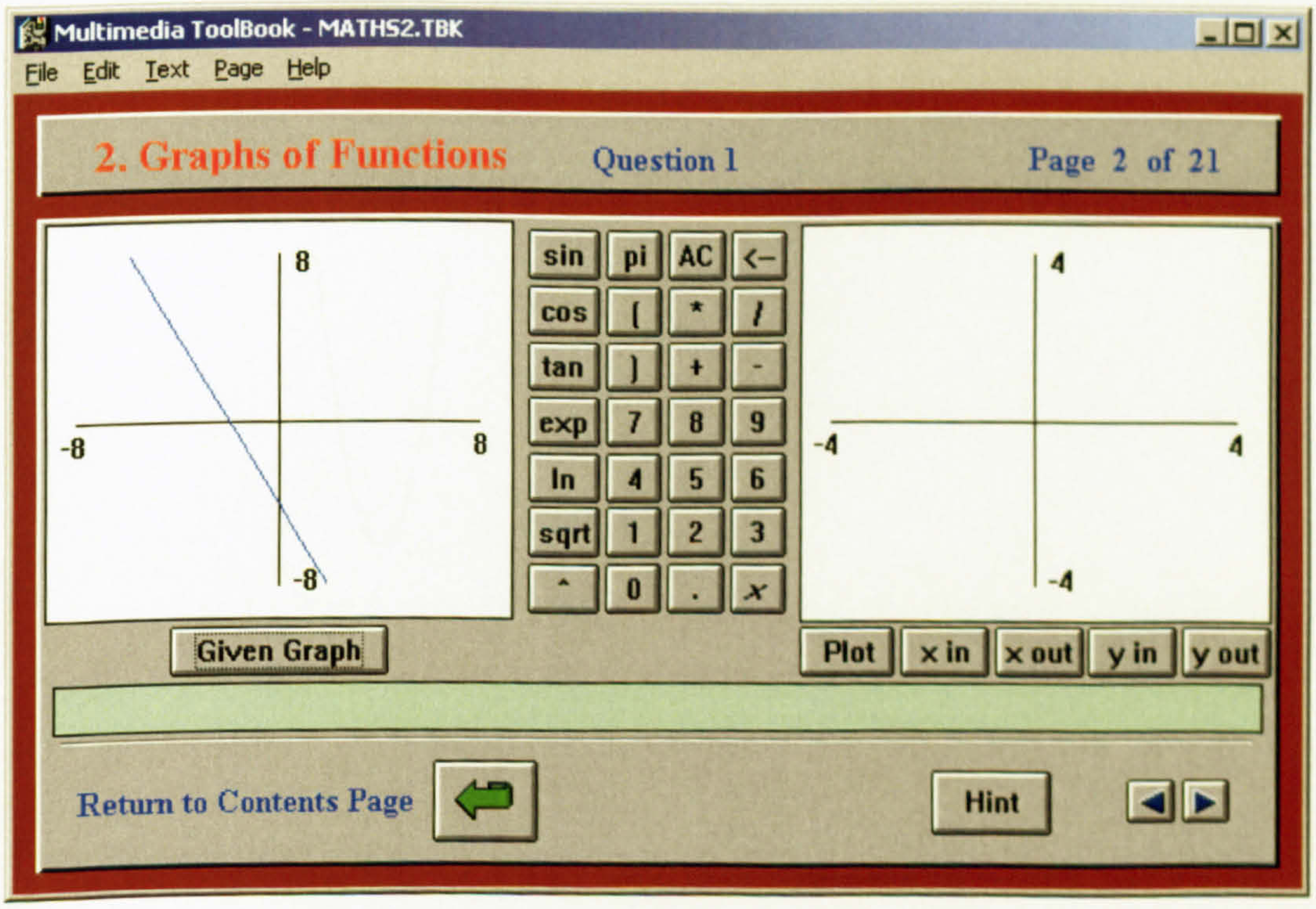
Appendix E - Experimental Data

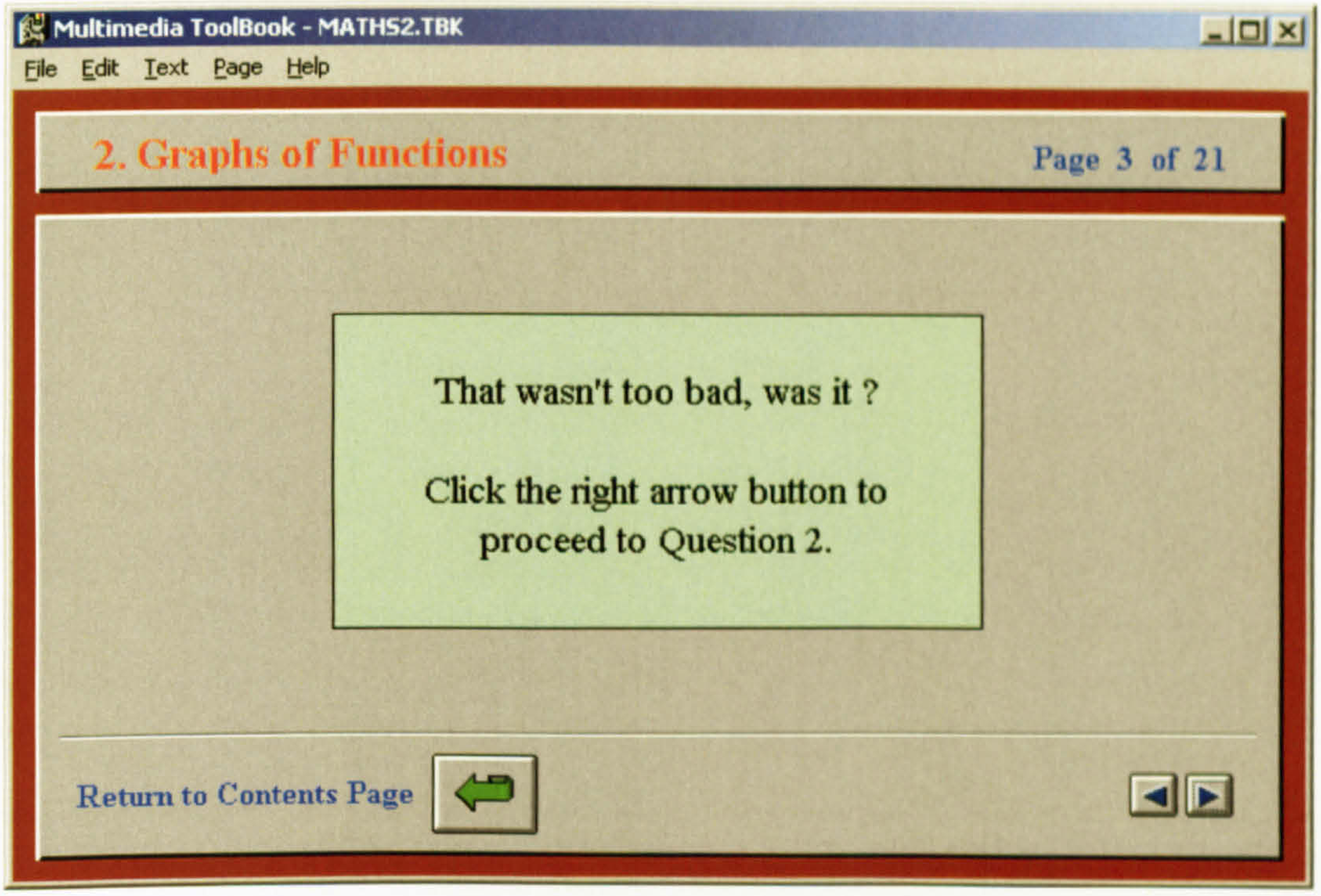
Appendix F - Publications by the Author

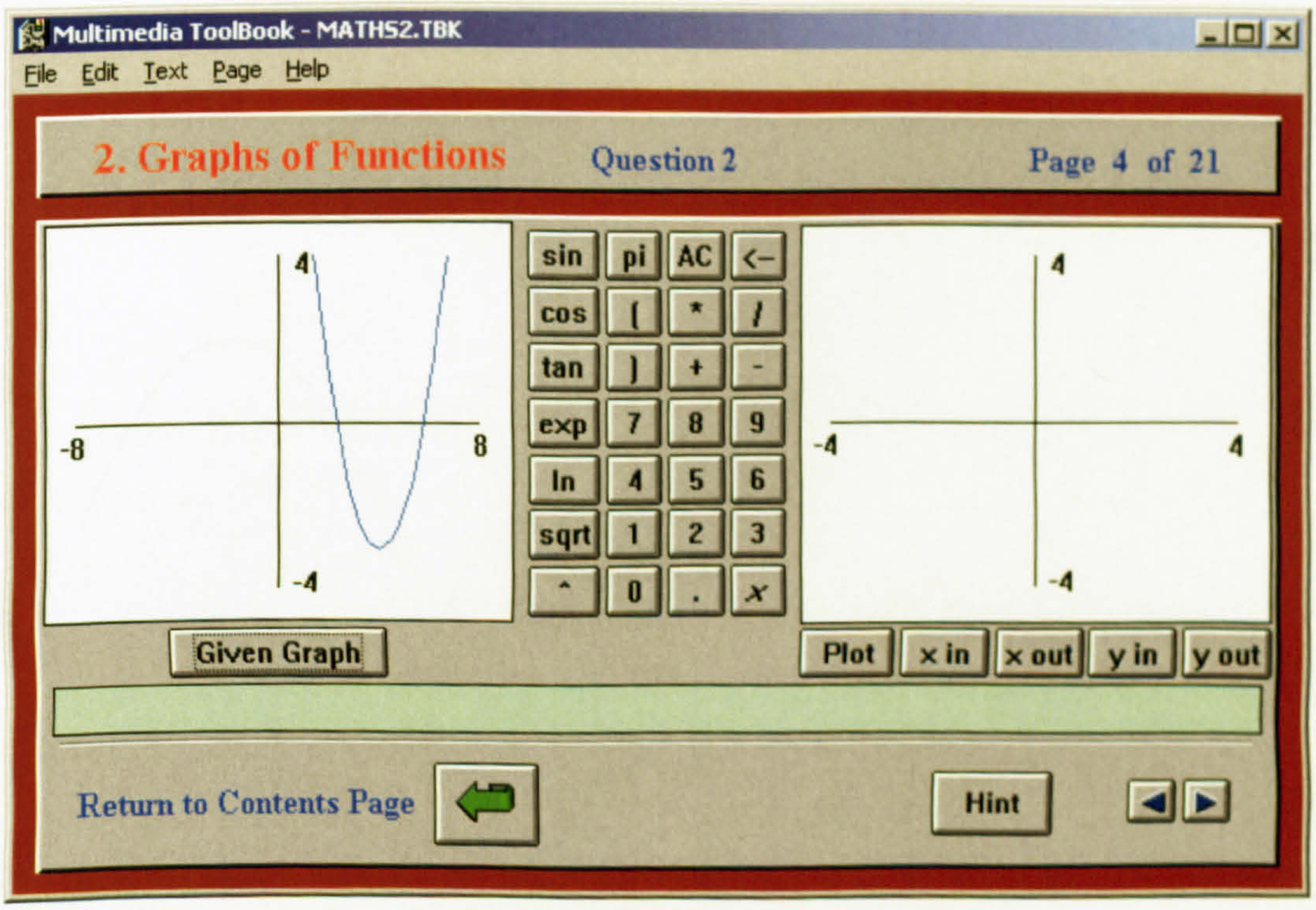
Appendix A

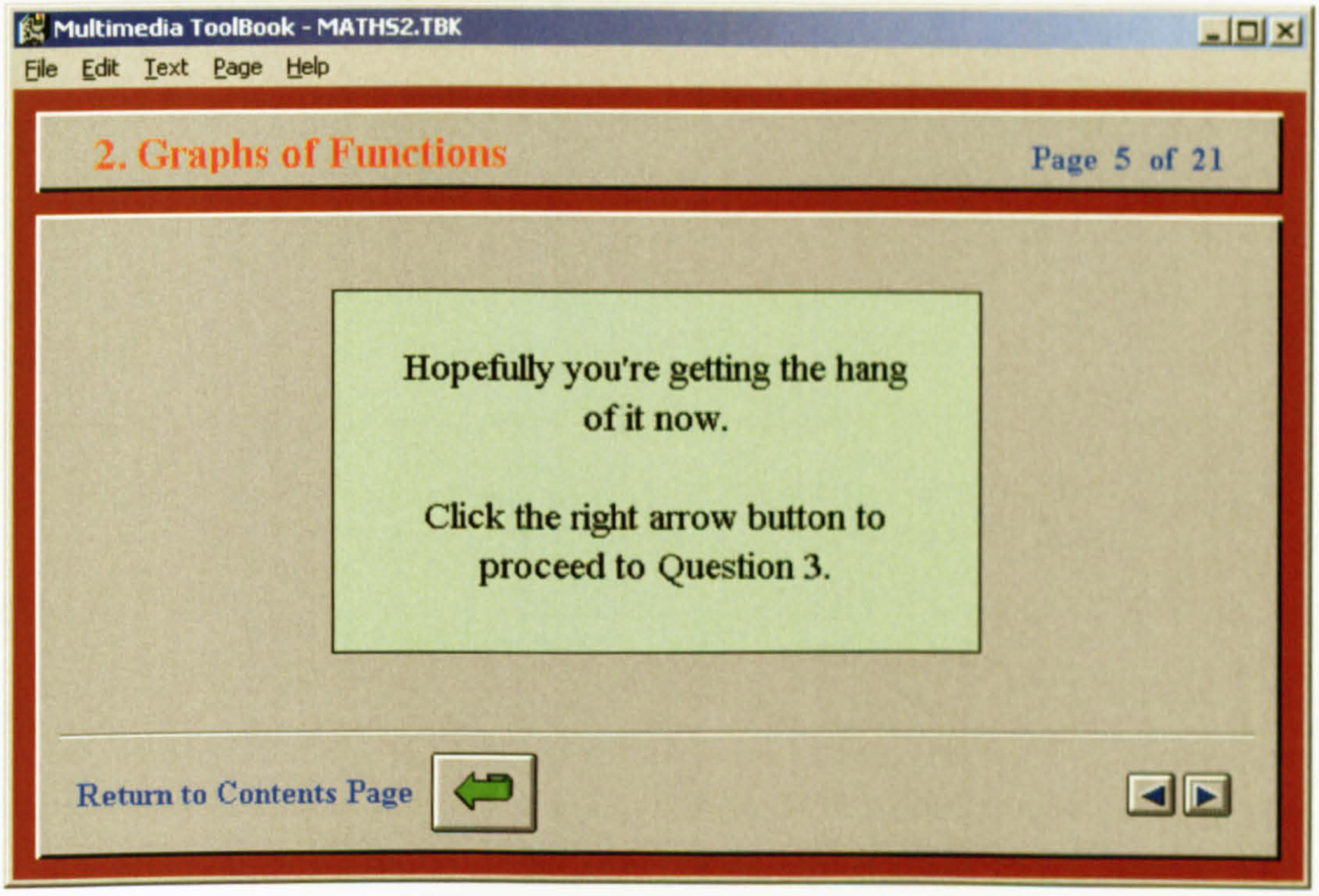
Software Screens

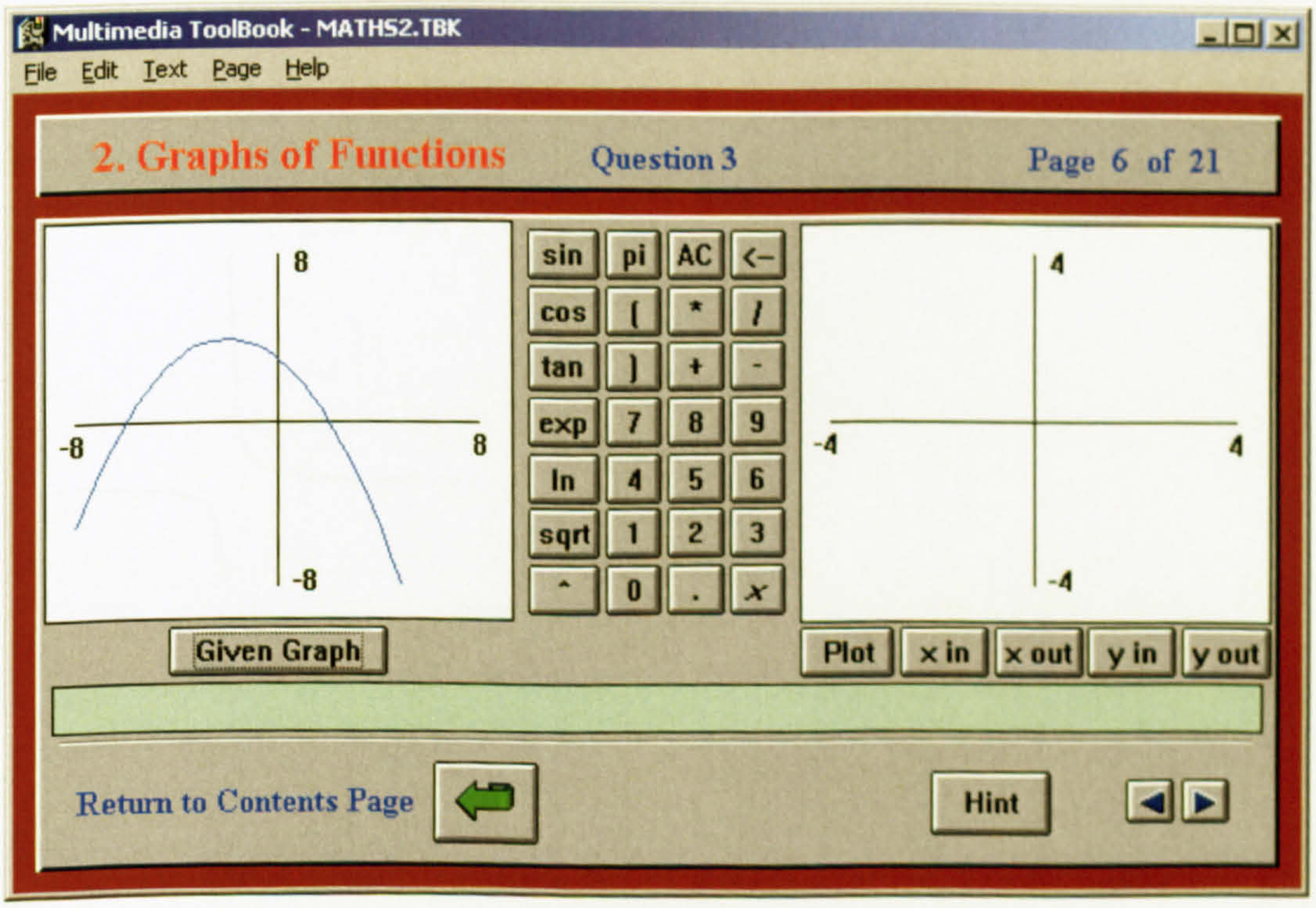

Multimedia ToolBook - MATHS2.TBK
File Edit Text Page Help
2. Graphs of Functions
Page 1 of 21
O.K. Let's start with a fairly simple one to begin with.
Click the right arrow button to proceed to Question 1.
Return to Contents Page

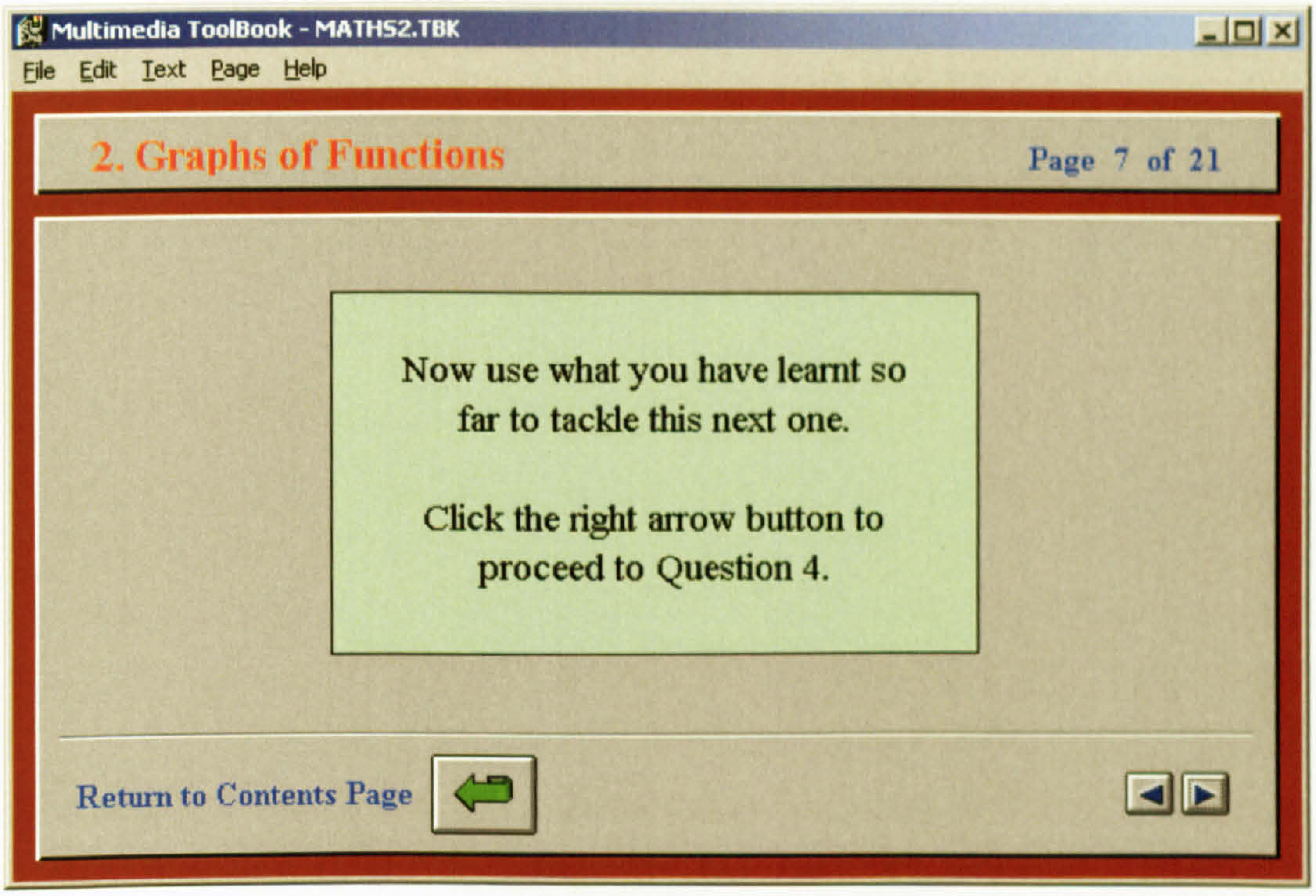

Multimedia ToolBook - MATHS2.TBK
File Edit Text Page Help
2. Graphs of Functions
Question 1
Page 2 of 21
8
-8
8
-8
Given Graph
sin pi AC <-
cos (* /
tan) + -
exp 7 8 9
ln 4 5 6
sqrt 1 2 3
^ 0 . x
4
-4
4
-4
Plot x in x out y in y out
Return to Contents Page
Hint

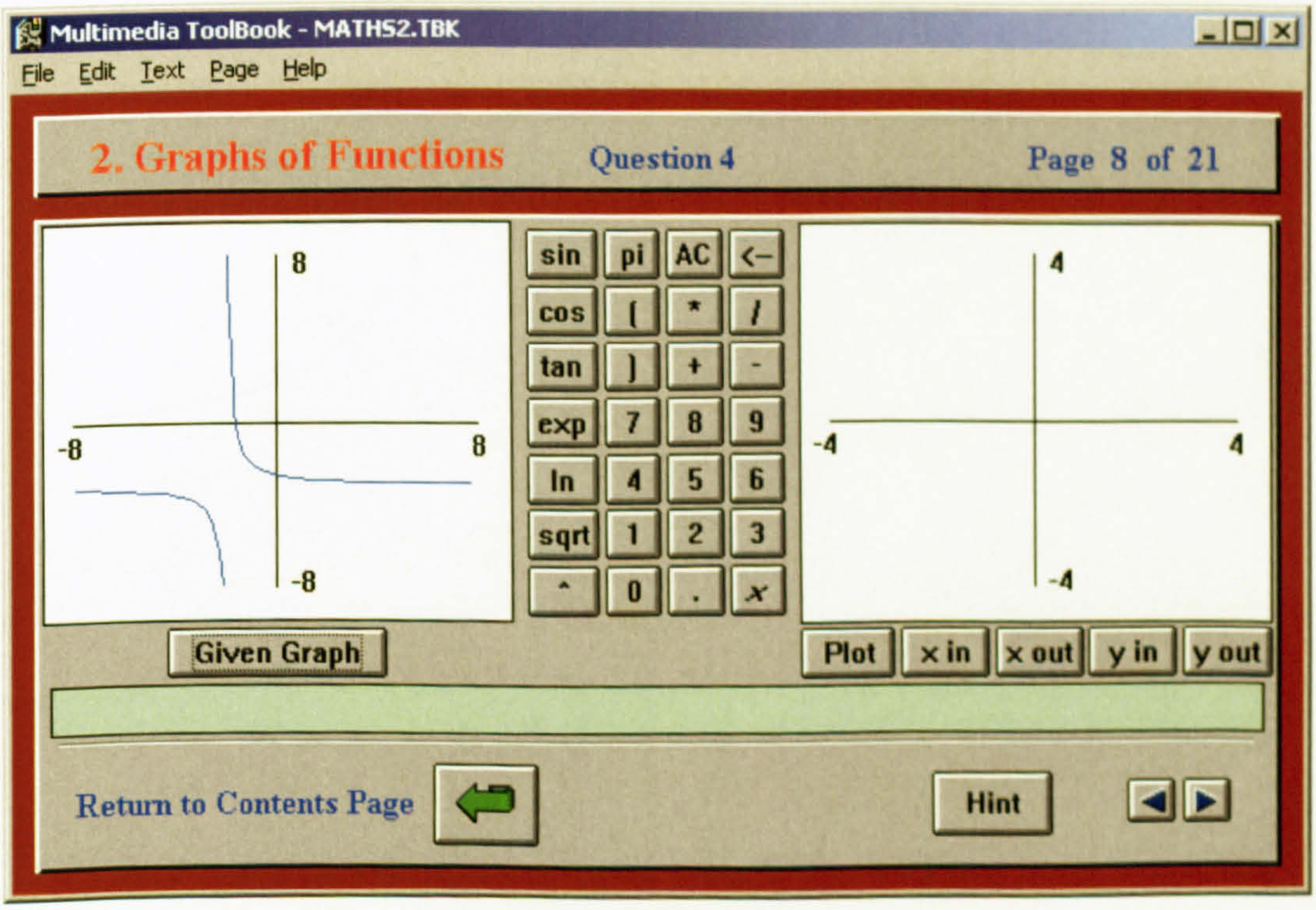

Multimedia ToolBook - MATHS2.TBK
File Edit Text Page Help
2. Graphs of Functions
Page 3 of 21
That wasn't too bad, was it ?
Click the right arrow button to proceed to Question 2.
Return to Contents Page

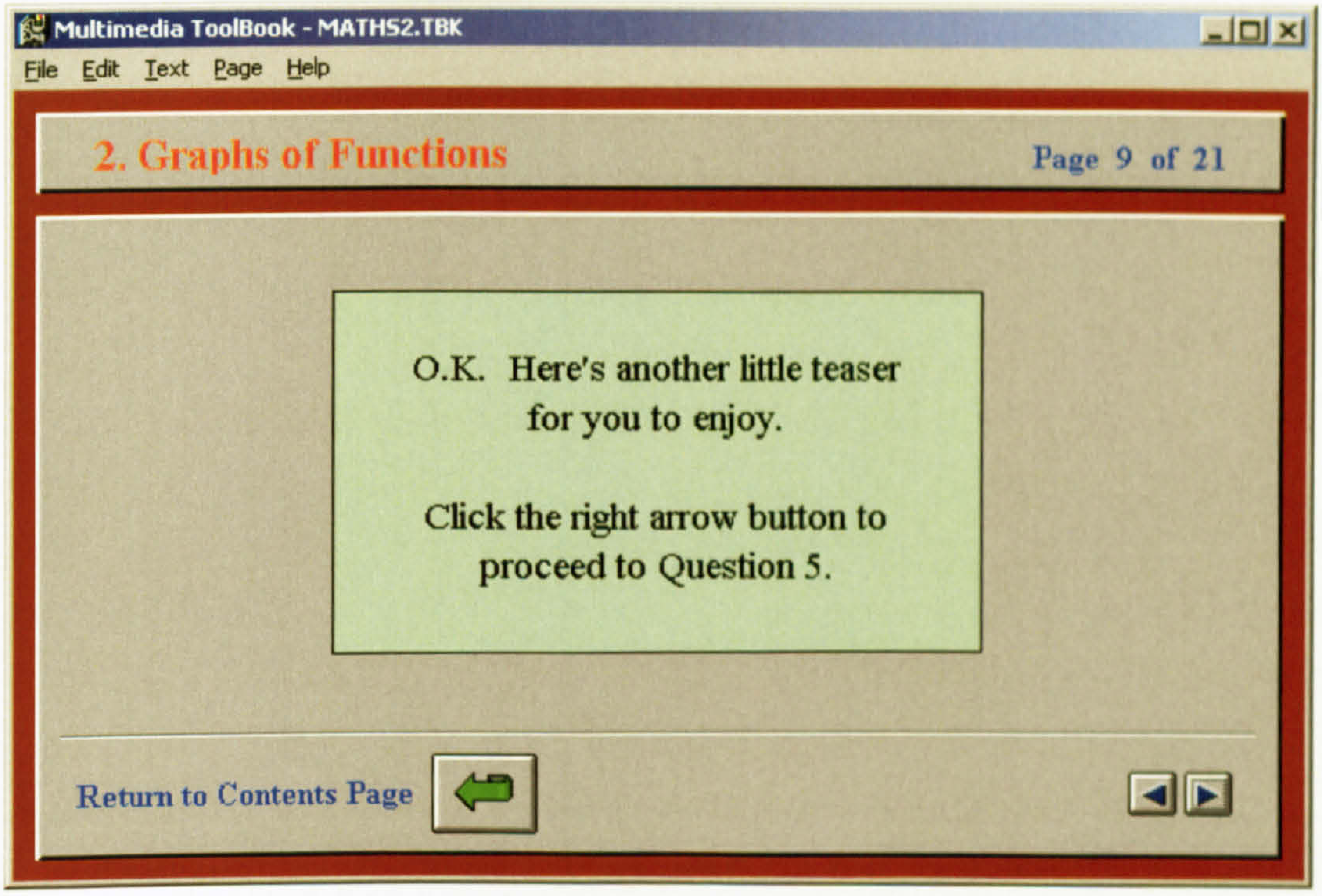

Multimedia ToolBook - MATHS2.TBK
File Edit Text Page Help
2. Graphs of Functions
Question 2
Page 4 of 21
4
-8
8
-4
sin pi AC <-
cos (* /
tan) + -
exp 7 8 9
ln 4 5 6
sqrt 1 2 3
^ 0 . x
4
-4
4
-4
Given Graph
Plot x in x out y in y out
Return to Contents Page
Hint


Multimedia ToolBook - MATHS2.TBK
File Edit Text Page Help
2. Graphs of Functions
Page 5 of 21
Hopefully you're getting the hang of it now.
Click the right arrow button to proceed to Question 3.
Return to Contents Page

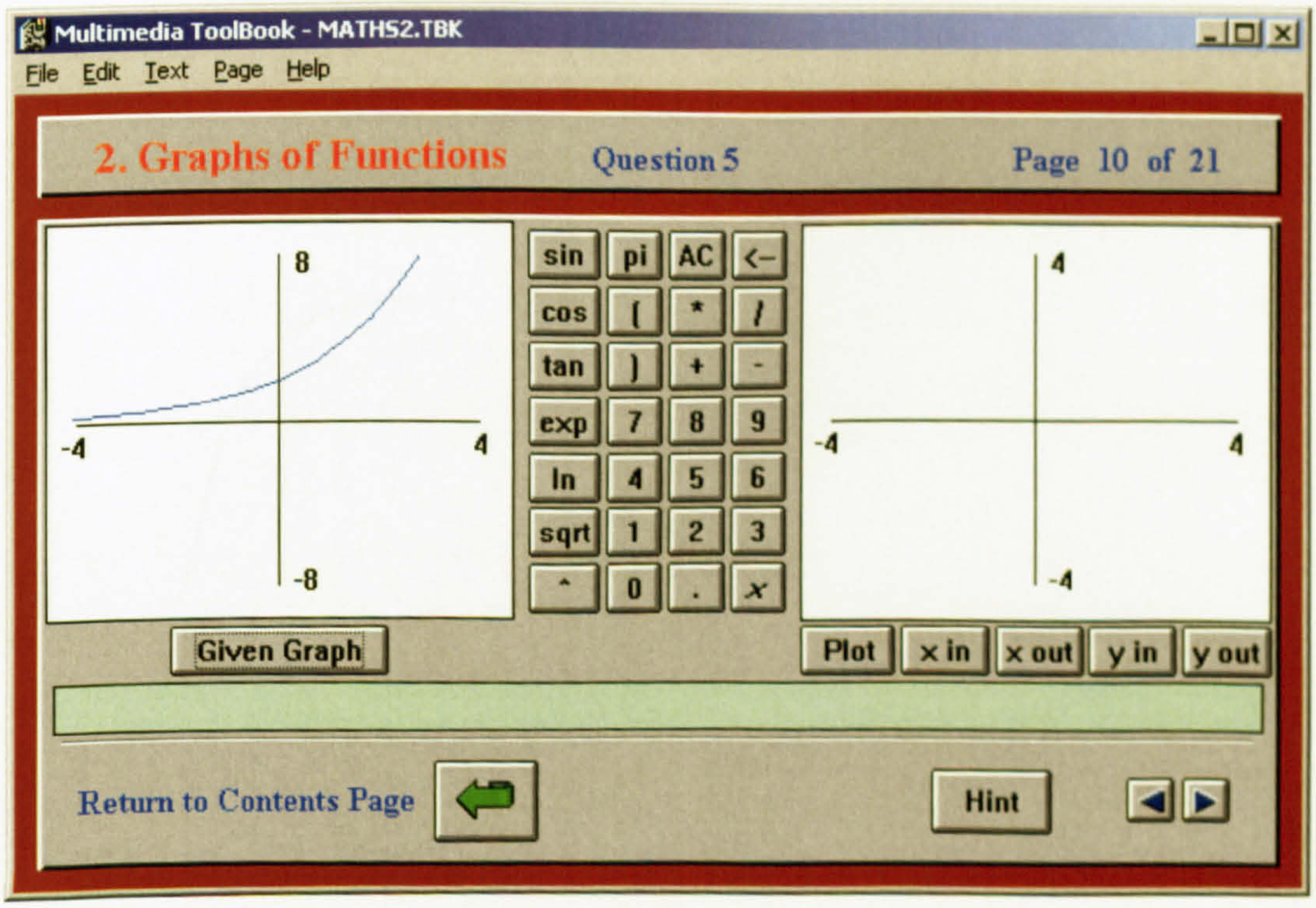

Multimedia ToolBook - MATHS2.TBK
File Edit Text Page Help
2. Graphs of Functions
Question 3
Page 6 of 21
8
-8
8
-8
sin pi AC <-
cos (* /
tan) + -
exp 7 8 9
ln 4 5 6
sqrt 1 2 3
^ 0 . x
4
-4
4
-4
Given Graph
Plot
x in
x out
y in
y out
Return to Contents Page
Hint


Multimedia ToolBook - MATHS2.TBK
File Edit Text Page Help
2. Graphs of Functions
Page 7 of 21
Now use what you have learnt so far to tackle this next one.
Click the right arrow button to proceed to Question 4.
Return to Contents Page

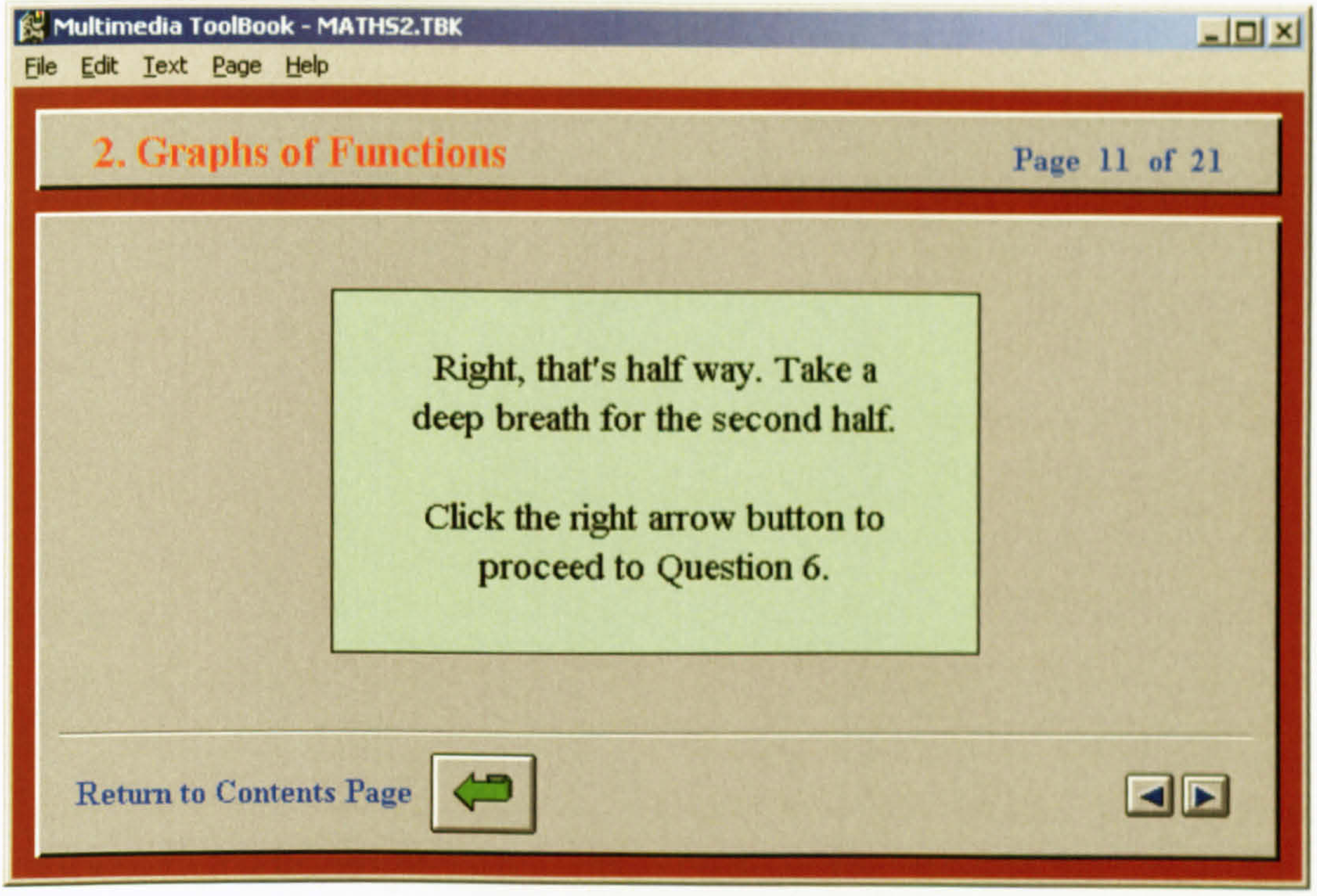

Multimedia ToolBook - MATHS2.TBK
File Edit Text Page Help
2. Graphs of Functions
Question 4
Page 8 of 21
sin pi AC <-
cos (* /
tan) + -
exp 7 8 9
ln 4 5 6
sqrt 1 2 3
^ 0 . x
Given Graph
Plot x in x out y in y out
Return to Contents Page
Hint

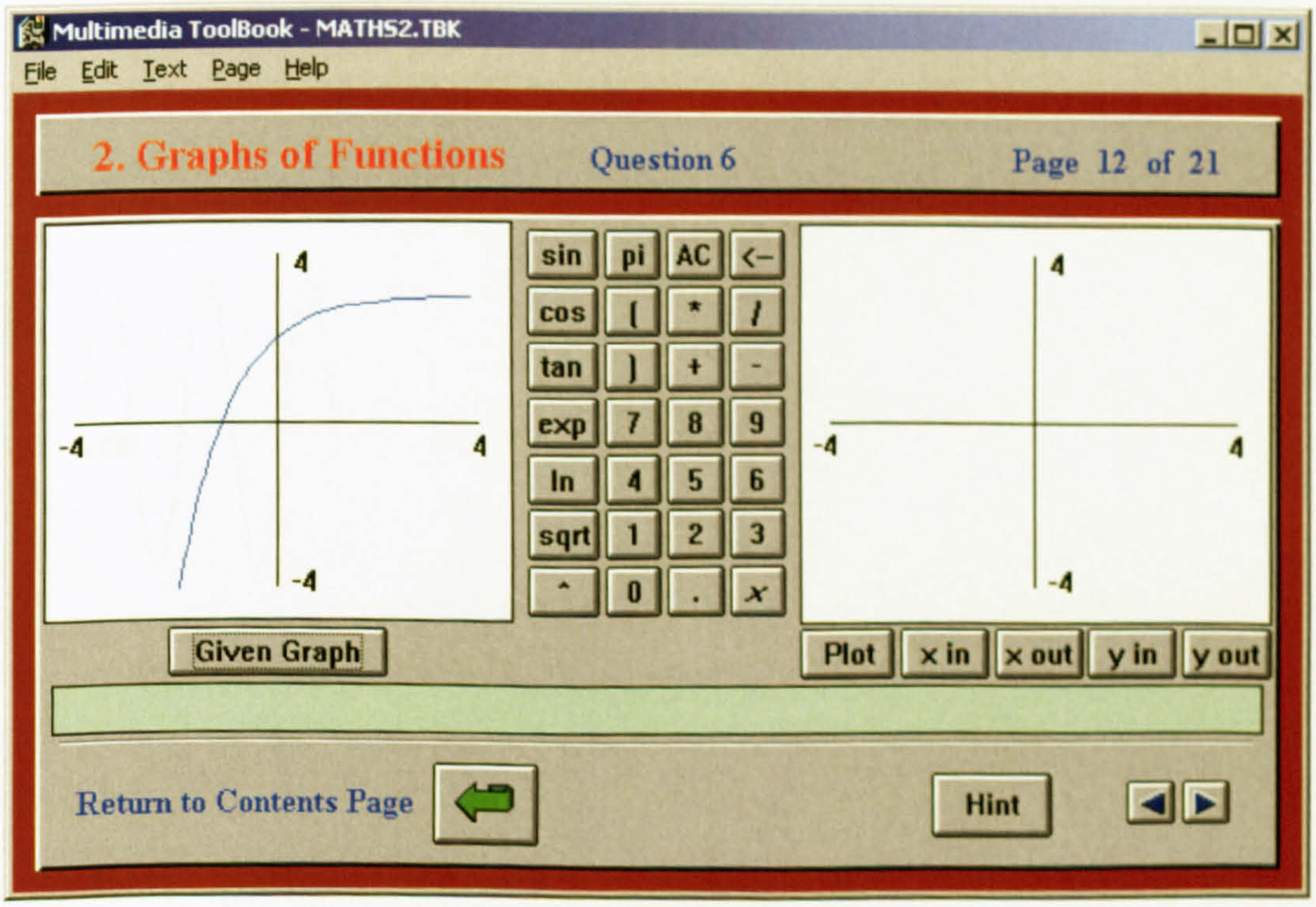


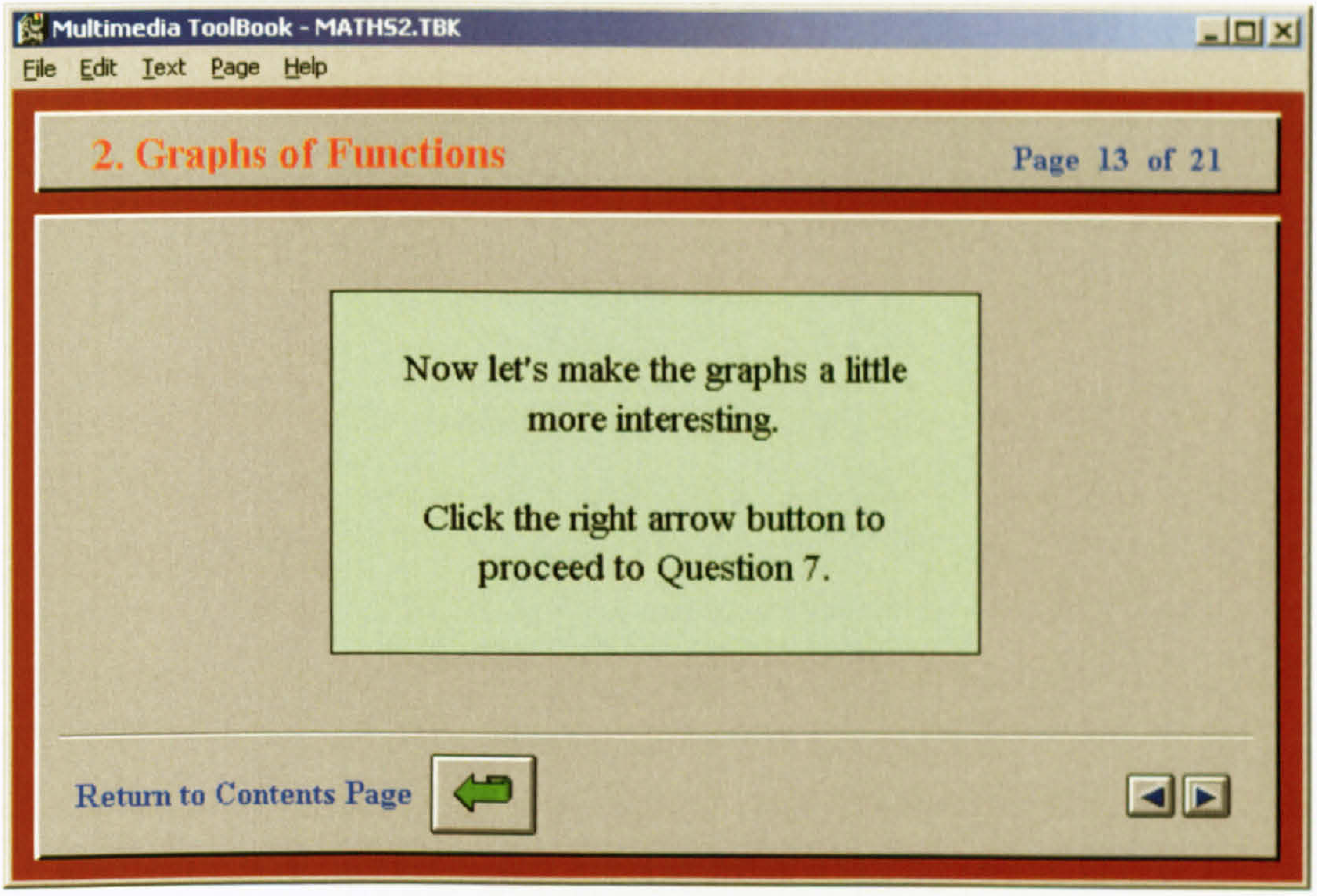
Multimedia ToolBook - MATHS2.TBK
File Edit Text Page Help
2. Graphs of Functions
Page 9 of 21
O.K. Here's another little teaser for you to enjoy.
Click the right arrow button to proceed to Question 5.
Return to Contents Page



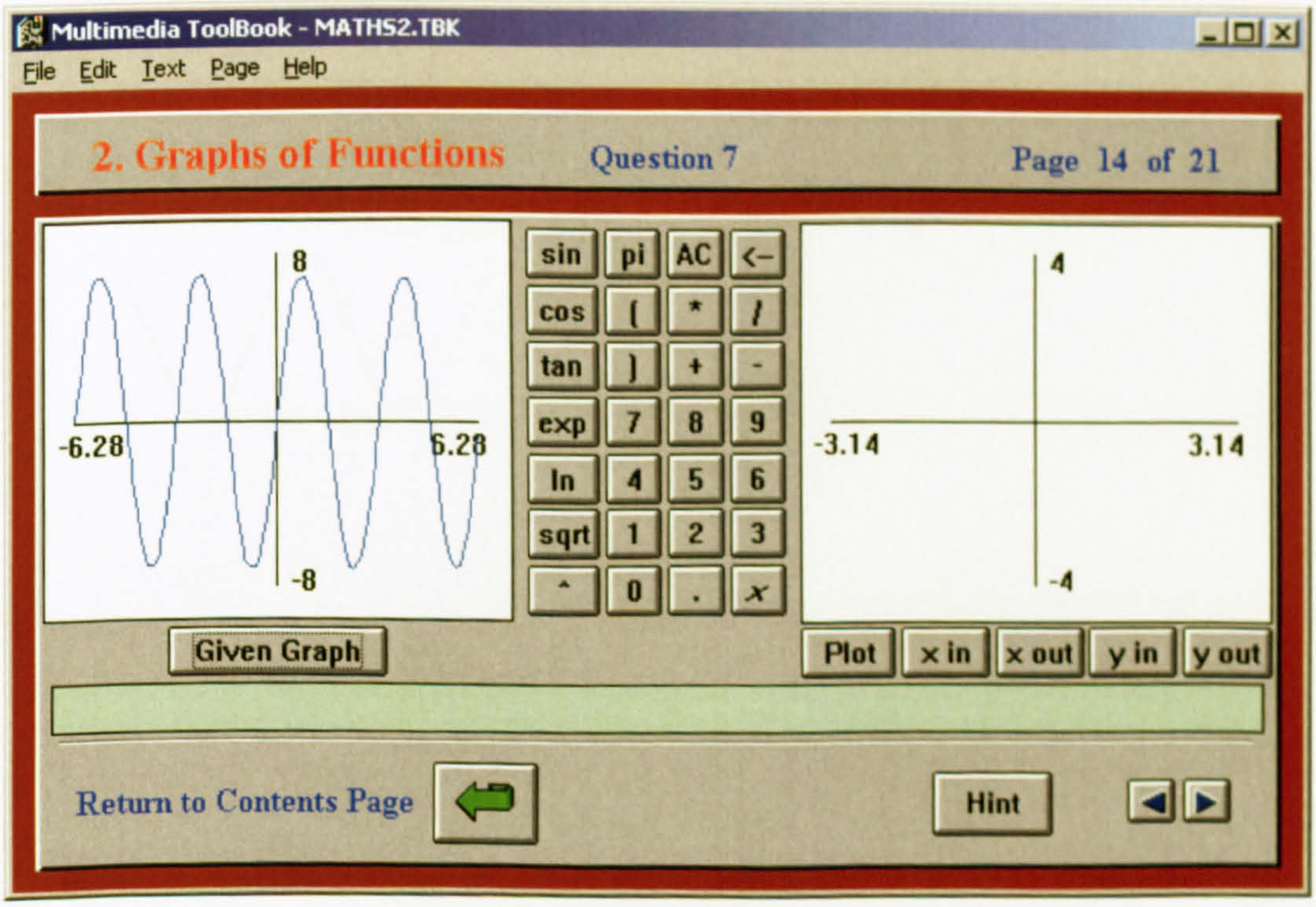
Multimedia ToolBook - MATHS2.TBK
File Edit Text Page Help
2. Graphs of Functions
Question 5
Page 10 of 21
8
-4
4
-8
sin pi AC <-
cos (* /
tan) + -
exp 7 8 9
ln 4 5 6
sqrt 1 2 3
^ 0 . x
4
-4
4
-4
Given Graph
Plot
x in
x out
y in
y out
Return to Contents Page
Hint


Multimedia ToolBook - MATHS2.TBK
File Edit Text Page Help
2. Graphs of Functions
Page 11 of 21
Right, that's half way. Take a deep breath for the second half.
Click the right arrow button to proceed to Question 6.
Return to Contents Page

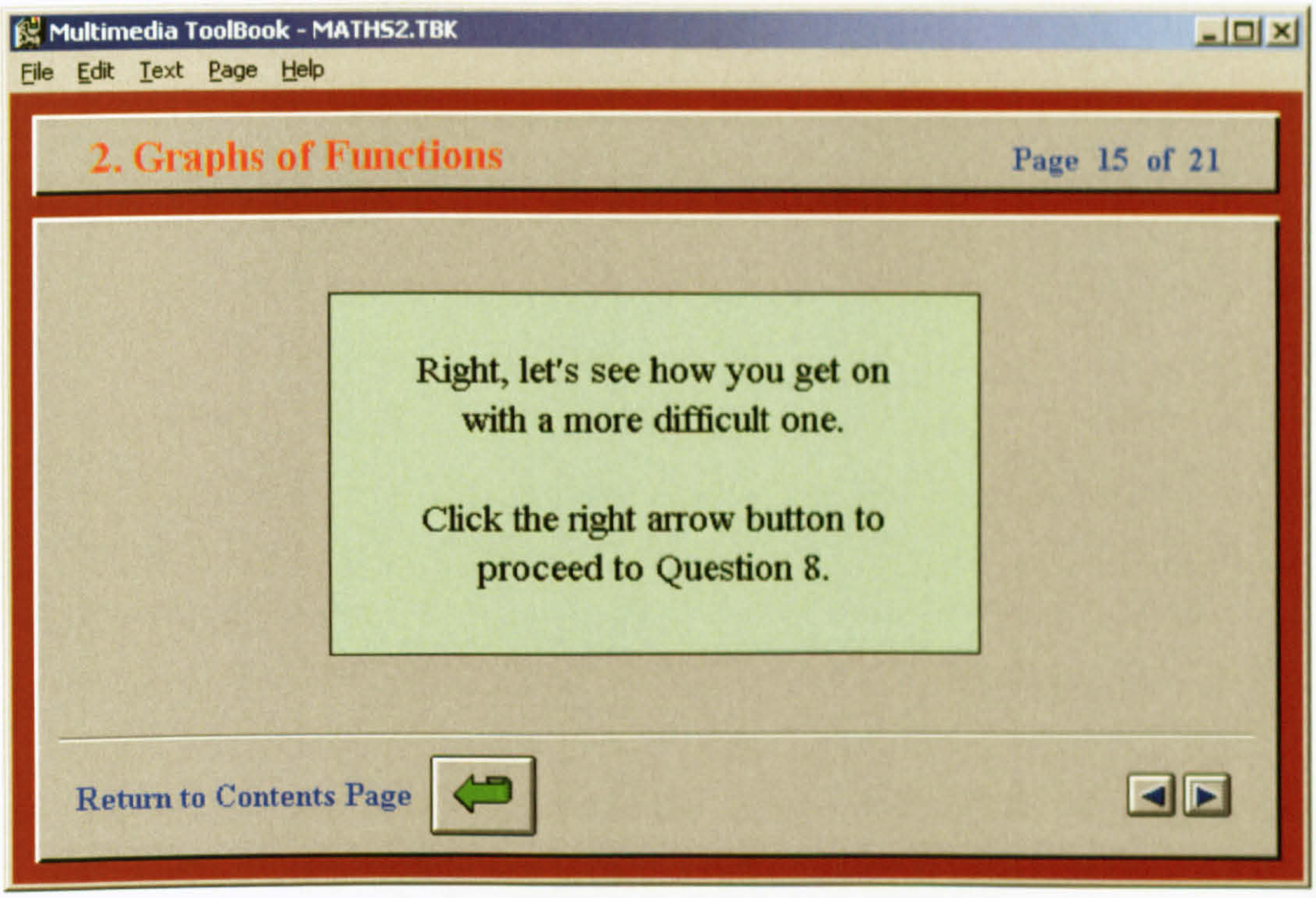

Multimedia ToolBook - MATHS2.TBK
File Edit Text Page Help
2. Graphs of Functions
Question 6
Page 12 of 21
4
-4
4
-4
sin pi AC <-
cos (* /
tan) + -
exp 7 8 9
ln 4 5 6
sqrt 1 2 3
^ 0 . x
4
-4
4
-4
Given Graph
Plot
x in
x out
y in
y out
Return to Contents Page
Hint

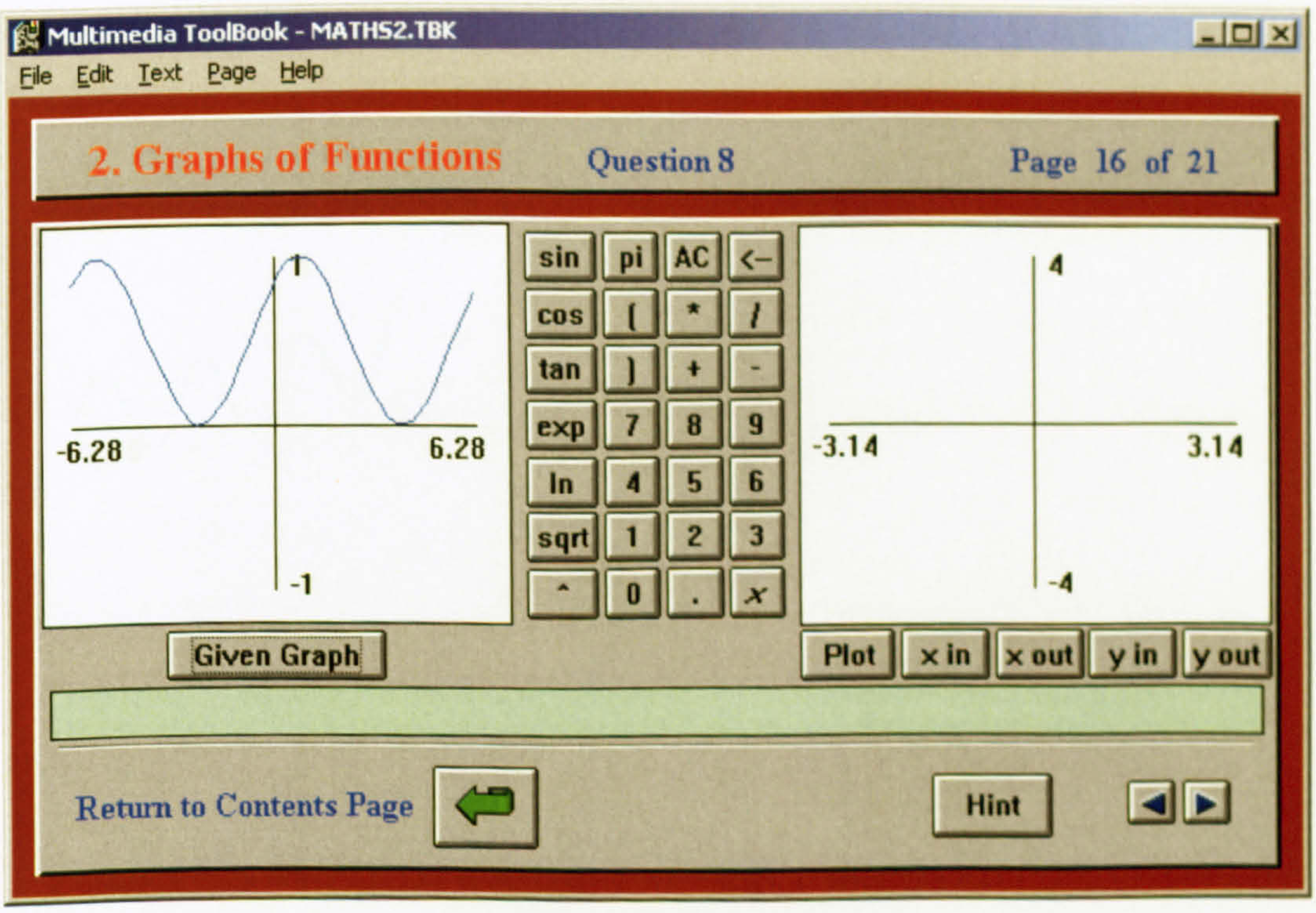


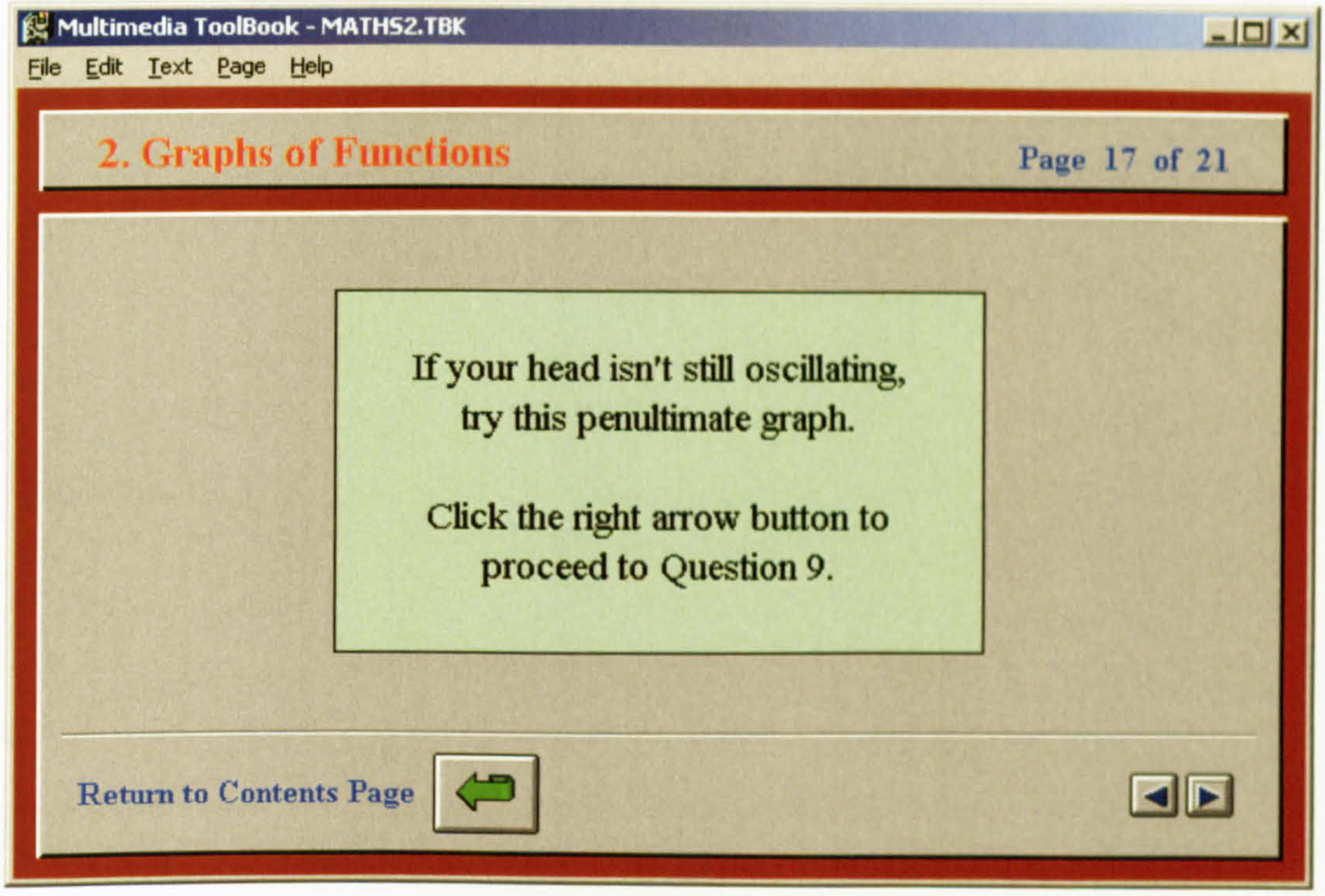
Multimedia ToolBook - MATHS2.TBK
File Edit Text Page Help
2. Graphs of Functions
Page 13 of 21
Now let's make the graphs a little more interesting.
Click the right arrow button to proceed to Question 7.
Return to Contents Page



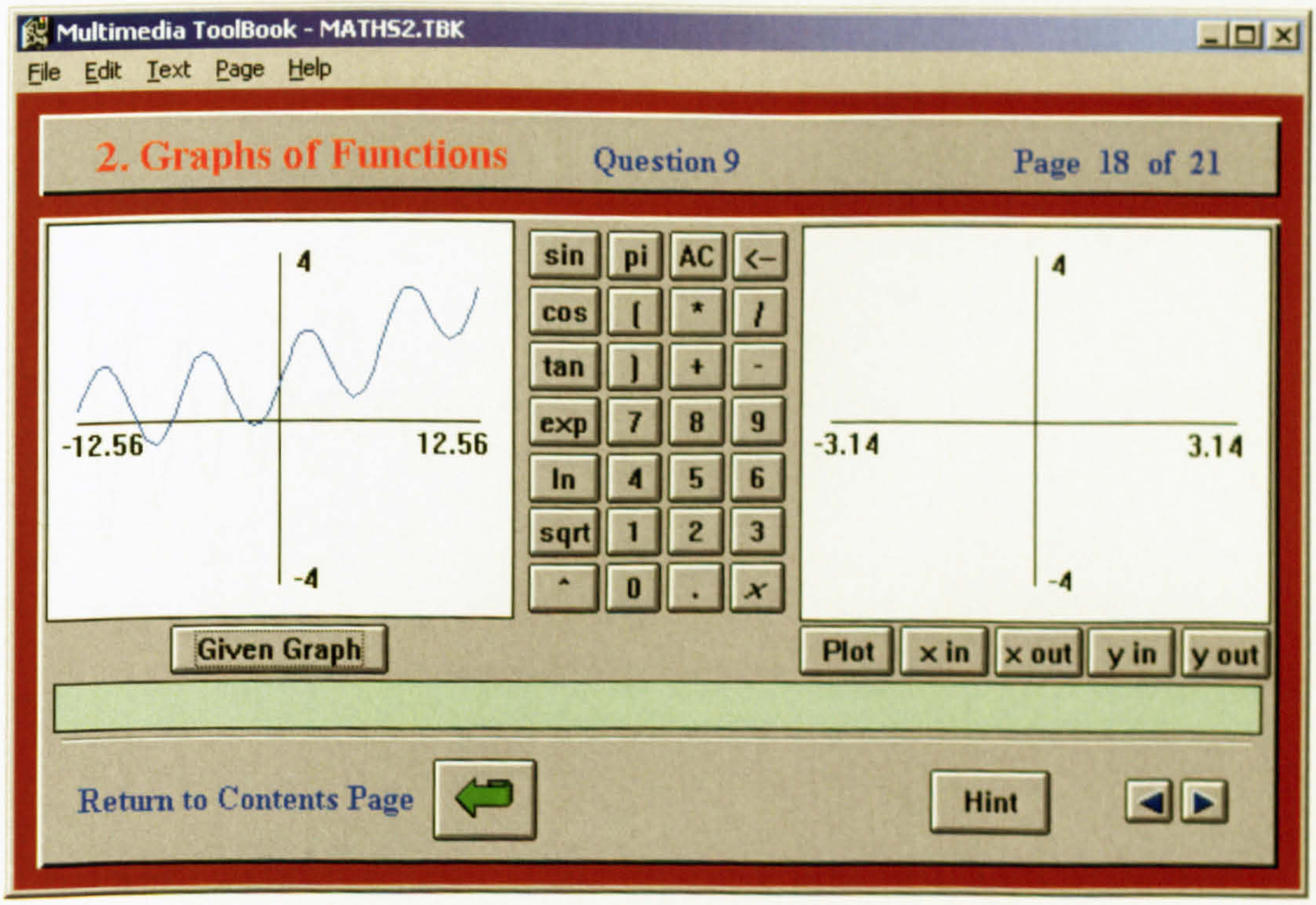
Multimedia ToolBook - MATHS2.TBK
File Edit Text Page Help
2. Graphs of Functions
Question 7
Page 14 of 21
8
-6.28
6.28
-8
sin pi AC <-
cos (* /
tan) + -
exp 7 8 9
ln 4 5 6
sqrt 1 2 3
^ 0 . x
4
-3.14
3.14
-4
Given Graph
Plot
x in
x out
y in
y out
Return to Contents Page
Hint


Multimedia ToolBook - MATHS2.TBK
File Edit Text Page Help
2. Graphs of Functions
Page 15 of 21
Right, let's see how you get on with a more difficult one.
Click the right arrow button to proceed to Question 8.
Return to Contents Page

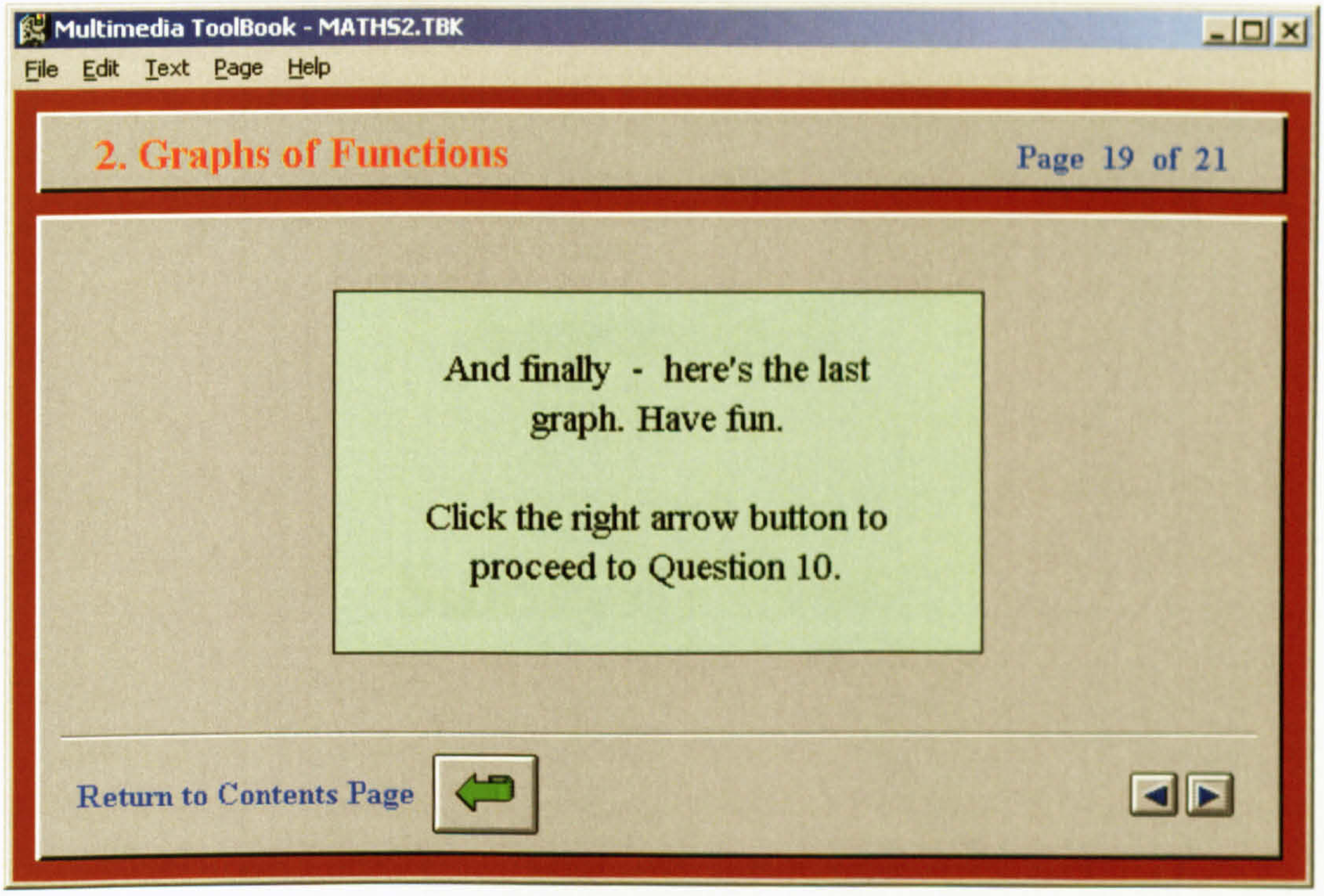

Multimedia ToolBook - MATHS2.TBK
File Edit Text Page Help
2. Graphs of Functions
Question 8
Page 16 of 21
1
-6.28
6.28
-1
Given Graph
sin pi AC <-
cos (* /
tan) + -
exp 7 8 9
ln 4 5 6
sqrt 1 2 3
^ 0 . x
4
-3.14
3.14
-4
Plot x in x out y in y out
Return to Contents Page
Hint

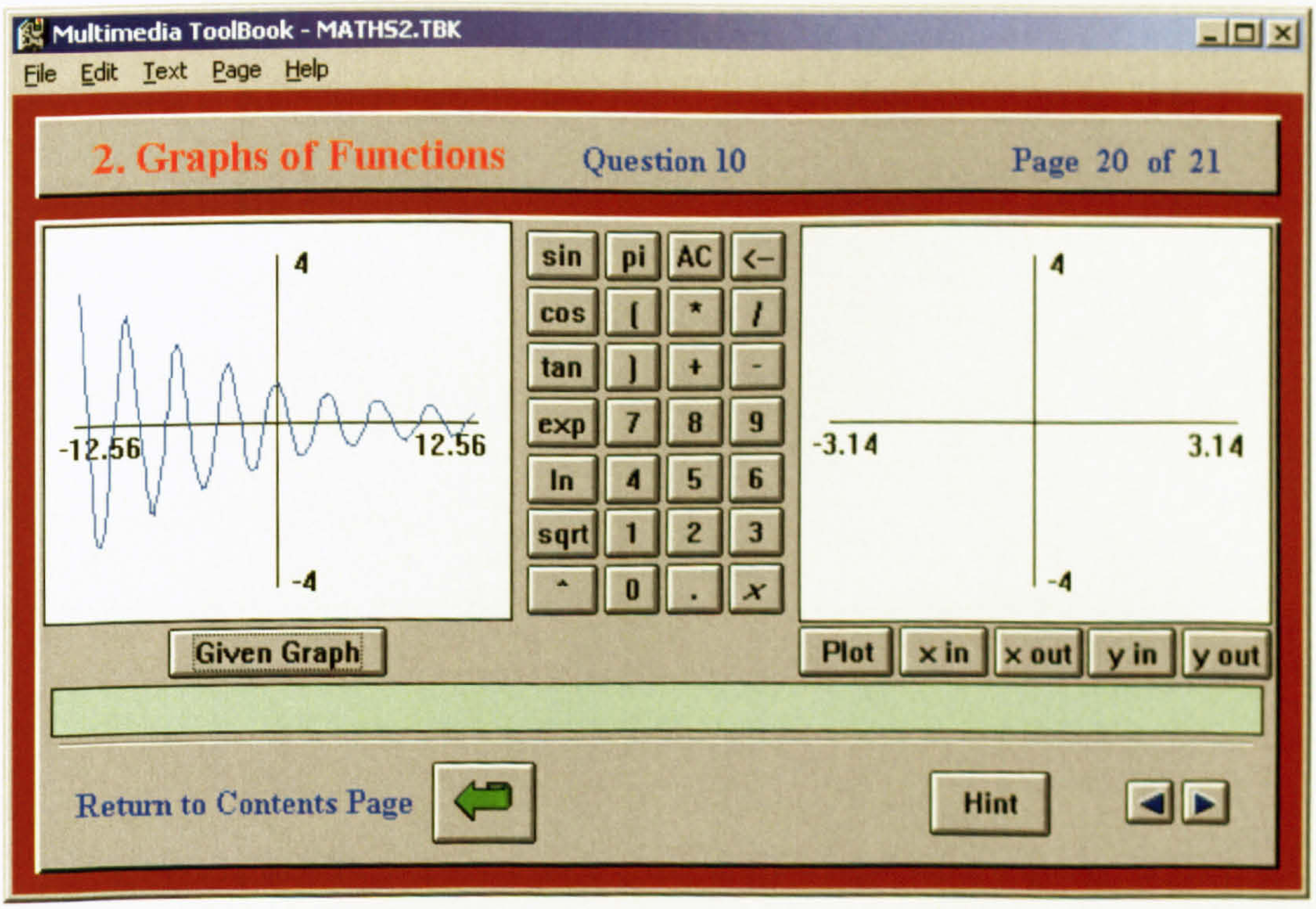


Multimedia ToolBook - MATHS2.TBK
File Edit Text Page Help
2. Graphs of Functions
Page 17 of 21
If your head isn't still oscillating, try this penultimate graph.
Click the right arrow button to proceed to Question 9.
Return to Contents Page



Multimedia ToolBook - MATHS2.TBK
File Edit Text Page Help
2. Graphs of Functions
Question 9
Page 18 of 21
4
-12.56
12.56
-4
sin pi AC <–
cos (* /
tan) + -
exp 7 8 9
ln 4 5 6
sqrt 1 2 3
^ 0 . x
4
-3.14
3.14
-4
Given Graph
Plot
x in
x out
y in
y out
Return to Contents Page
Hint


Multimedia ToolBook - MATHS2.TBK
File Edit Text Page Help
2. Graphs of Functions
Page 19 of 21
And finally - here's the last graph. Have fun.
Click the right arrow button to proceed to Question 10.
Return to Contents Page


Multimedia ToolBook - MATHS2.TBK
File Edit Text Page Help
2. Graphs of Functions
Question 10
Page 20 of 21
4
-12.56
12.56
-4
Given Graph
sin pi AC <-
cos (* /
tan) + -
exp 7 8 9
ln 4 5 6
sqrt 1 2 3
^ 0 . x
4
-3.14
3.14
-4
Plot
x in
x out
y in
y out
Return to Contents Page
Hint

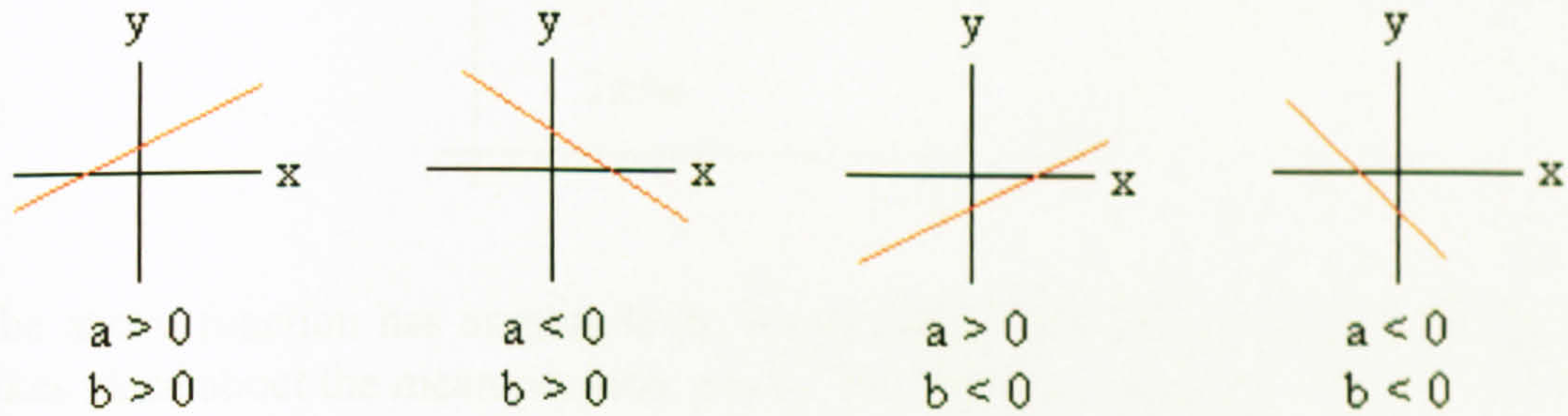
Appendix B

Summary Sheet

Summary Sheet - Some Useful Hints on the Shape and Position of Graphs

1. Linear Functions

The equation of a straight line may be expressed as $y = \mathbf{a}x + \mathbf{b}$ where **a** represents the slope of the line and **b** represents the intercept (the 'cut' on the y-axis corresponding to $x = 0$). To summarise, we have the following:



2. Quadratic Functions

Let us consider the function $y = \mathbf{a}(x + \mathbf{b})^2 + \mathbf{c}$, where **a** describes how steep or shallow the graph is, **b** describes movement in the horizontal direction, and **c** describes movement in the vertical direction.

If **a** is negative, the parabola (this is the name given to the shape of a quadratic curve) will be reflected in the x-axis, i.e. upside down. If **a** increases in magnitude the slope of the graph becomes steeper, and if **a** decreases in magnitude the slope of the graph becomes shallower.

If **b** is positive the vertex of the graph will be **b** units to the left of the origin, and if **b** is negative the vertex of the graph will be **b** units to the right of the origin.

If **c** is positive the vertex of the graph will be **c** units above the x-axis, and if **c** is negative the vertex of the graph will be **c** units below the x-axis.

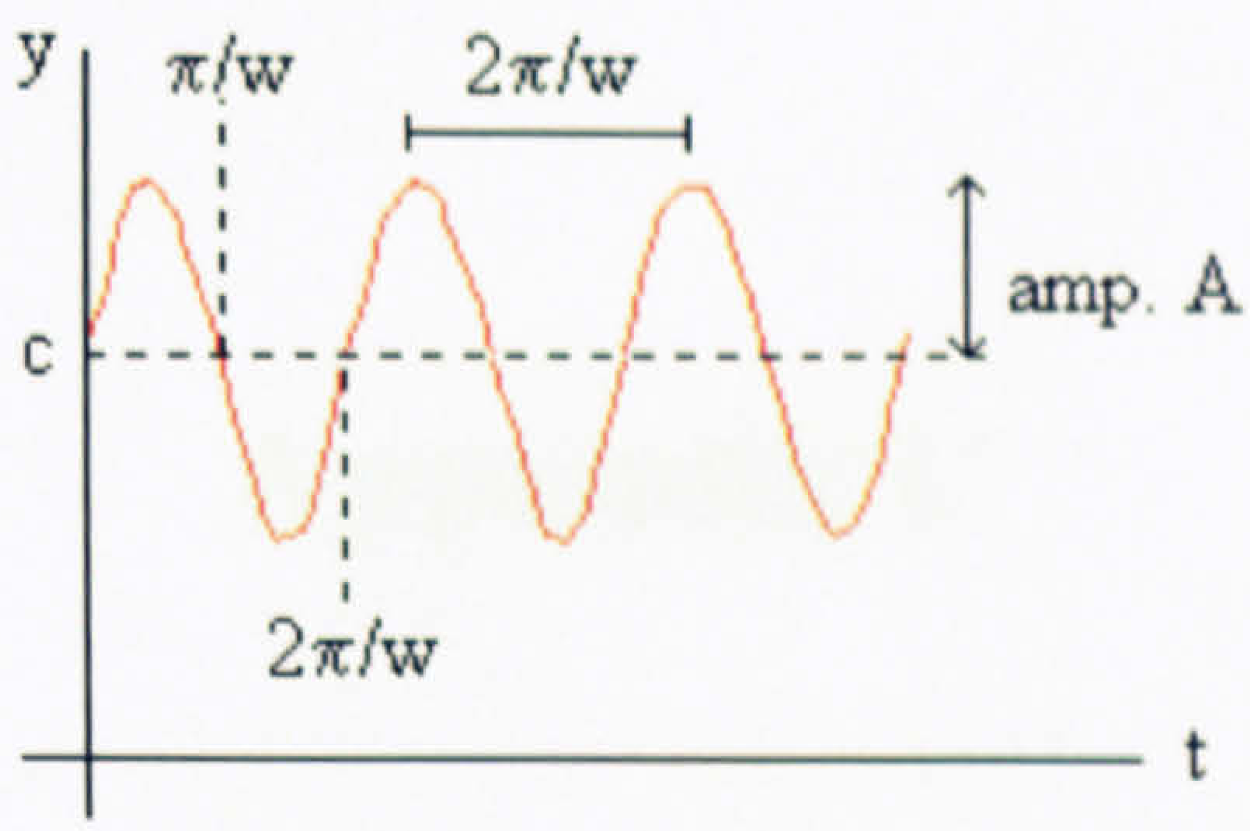
3. Exponential Functions

Let us consider the function $y = \mathbf{a}e^{\mathbf{b}x}$ [or $y = \mathbf{a}\exp(\mathbf{b}x)$] where **a** determines the intercept of the graph and **b** affects the gradient of the curve. The following show **a** fixed at the value 1, while we investigate the effect of varying **b**:

If we make **a** negative, the graph becomes a reflection in the x-axis.

4. Trigonometric Functions

Let us consider the function $y = \mathbf{A}\sin(\mathbf{w}t) + \mathbf{c}$ in the following diagram:



The above function has amplitude **A**, wavelength $2\pi/\mathbf{w}$, and frequency **w**, which takes place about the mean position $y = \mathbf{c}$. The function $\mathbf{A}\sin(\mathbf{w}t + \phi) + \mathbf{c}$ has the same properties as the above, but also has a phase difference of $\phi/\mathbf{w}$.

The same properties apply for the function $\mathbf{A}\cos(\mathbf{w}t + \phi) + \mathbf{c}$.

5. Functions in General

Certain general properties of functions and graphs are as follows:

Consider some function $f(x)$.

If we alter the function to $f(x + \mathbf{a})$, then its graph will shift horizontally. If **a** is positive the graph will shift **a** units to the left, and if **a** is negative the graph will shift **a** units to the right.

If we alter the function to $f(x) + \mathbf{a}$, then its graph will shift vertically. If **a** is positive the graph will shift **a** units up, and if **a** is negative the graph will shift **a** units down.

If we alter the function to $-f(x)$, then its graph will become a reflection in the x-axis.

And combining these three situations, we therefore know that by altering the function to $-f(x + \mathbf{a}) - \mathbf{b}$, the graph will be reflected in the x-axis (i.e. flipped upside down), shifted **a** units to the left, and shifted **b** units down.

Appendix C

Feedback Questionnaire

Interactive Mathematical Software Questionnaire

1. Did you find the session helpful?

A. Yes, extremely helpful.
B. Yes, quite helpful.
C. Neither helpful nor unhelpful.
D. No, not very helpful.
E. No, most unhelpful.

2. Did you enjoy using the package?

A. Yes, very much.
B. Yes, a little.
C. I didn't like it nor dislike it.
D. No, I didn't really like it.
E. No, I hated it.

3. Having used this software, would you be motivated to use software concerned with other mathematical topics?

A. Yes, very much so.
B. Yes, probably.
C. Maybe, maybe not.
D. No, probably not.
E. No, definitely not.

4. How did this 2-hour computer session compare with 2 hours of traditional mathematics classroom activities?

A. I would much rather learn via the computer.
B. I would probably prefer to learn via the computer.
C. I have no preference.
D. I would probably prefer to do traditional mathematics classroom activities.
E. I would much rather do traditional mathematics classroom activities.

5. Do you think anything could have been included in the package in order to improve it? If so, please comment.

____

____

____

Thank you very much for your time and assistance with this research.

School A Control (39 students)					School A Experimental (44 students)				
Question Number	Number answered correctly		% answered correctly		Question Number	Number answered correctly		% answered correctly	
	Proc.	Vis.	Proc.	Vis.		Proc.	Vis.	Proc.	Vis.
1	29		74.4		1	33		75.0	
2		36		92.3	2		40		90.9
3	15		38.5		3	12		27.3	
4		24		61.5	4		40		90.9
5	29		74.4		5	33		75.0	
6		12		30.8	6		18		40.9
7	34		87.2		7	32		72.7	
8		7		17.9	8		20		45.5
9	19		48.7		9	16		36.4	
10		13		33.3	10		15		34.1
11	12		30.8		11	10		22.7	
12		24		61.5	12		27		61.4
13	28		71.8		13	29		65.9	
14		18		46.2	14		23		52.3
15	27		69.2		15	28		63.6	
16		22		56.4	16		22		50.0
17	18		46.2		17	16		36.4	
18		14		35.9	18		16		36.4
19	24		61.5		19	20		45.5	
20		14		35.9	20		22		50.0
Average			60.3	47.2	Average			52.1	55.2

School B Control (72 students)					School B Experimental (45 students)				
Question Number	Number answered correctly		% answered correctly		Question Number	Number answered correctly		% answered correctly	
	Proc.	Vis.	Proc.	Vis.		Proc.	Vis.	Proc.	Vis.
1	53		73.6		1	23		51.1	
2		64		88.9	2		40		88.9
3	18		25.0		3	6		13.3	
4		50		69.4	4		31		68.9
5	44		61.1		5	28		62.2	
6		25		34.7	6		20		44.4
7	48		66.7		7	21		46.7	
8		3		4.2	8		4		8.9
9	19		26.4		9	16		35.6	
10		31		43.1	10		22		48.9
11	9		12.5		11	3		6.7	
12		39		54.2	12		24		53.3
13	52		72.2		13	19		42.2	
14		31		43.1	14		22		48.9
15	35		48.6		15	22		48.9	
16		38		52.8	16		21		46.7
17	25		34.7		17	9		20.0	
18		13		18.1	18		10		22.2
19	28		38.9		19	22		48.9	
20		24		33.3	20		26		57.8
Average			46.0	44.2	Average			37.6	48.9

Appendix D

Analysis of Test Questions

School C Control (25 students)					School C Experimental (20 students)				
Question Number	Number answered correctly		% answered correctly		Question Number	Number answered correctly		% answered correctly	
	Proc.	Vis.	Proc.	Vis.		Proc.	Vis.	Proc.	Vis.
1	13		52.0		1	15		75.0	
2		20		80.0	2		18		90.0
3	4		16.0		3	4		20.0	
4		20		80.0	4		13		65.0
5	19		76.0		5	11		55.0	
6		12		48.0	6		12		60.0
7	22		88.0		7	12		60.0	
8		5		20.0	8		7		35.0
9	8		32.0		9	0		0.0	
10		2		8.0	10		6		30.0
11	9		36.0		11	2		10.0	
12		17		68.0	12		13		65.0
13	20		80.0		13	11		55.0	
14		14		56.0	14		12		60.0
15	19		76.0		15	8		40.0	
16		14		56.0	16		8		40.0
17	13		52.0		17	5		25.0	
18		3		12.0	18		6		30.0
19	15		60.0		19	6		30.0	
20		6		24.0	20		9		45.0
Average			56.8	45.2	Average			37.0	52.0

Control All (136 students)					Experimental All (109 students)				
Question Number	Number answered correctly		% answered correctly		Question Number	Number answered correctly		% answered correctly	
	Proc.	Vis.	Proc.	Vis.		Proc.	Vis.	Proc.	Vis.
1	95		69.9		1	71		65.1	
2		120		88.2	2		98		89.9
3	37		27.2		3	22		20.2	
4		94		69.1	4		84		77.1
5	92		67.6		5	72		66.1	
6		49		36.0	6		50		45.9
7	104		76.5		7	65		59.6	
8		15		11.0	8		31		28.4
9	46		33.8		9	32		29.4	
10		46		33.8	10		43		39.4
11	30		22.1		11	15		13.8	
12		80		58.8	12		64		58.7
13	100		73.5		13	59		54.1	
14		63		46.3	14		57		52.3
15	81		59.6		15	58		53.2	
16		74		54.4	16		51		46.8
17	56		41.2		17	30		27.5	
18		30		22.1	18		32		29.4
19	67		49.3		19	48		44.0	
20		44		32.4	20		57		52.3
Average			52.1	45.2	Average			43.3	52.0

Appendix E

Experimental Data

		Factor				
		Group	School	Sex	Subject	GCSE
Coding	1	Control	A	Male	Mechanics	A/A*
	2	Experimental	B	Female	Statistics	B/C
	3		C			

Each row below represents one student. For example, the first row represents a student who scored 4 out of 10 on the procedural questions, scored 1 out of 10 on the visual questions, is a member of the control group, studies at School A, is male, studies Mathematics with Mechanics, and has an A* or A grade at GCSE Mathematics.

Proc	vis	group	school	sex	subject	gcse	proc-vis
4	1	1	1	1	1	1	3
7	8	1	1	1	1	1	-1
8	6	1	1	1	1	1	2
8	4	1	1	1	1	1	4
7	3	1	1	1	1	1	4
8	7	1	1	1	1	1	1
6	5	1	1	1	1	1	1
9	3	1	1	1	1	1	6
5	6	1	1	1	1	1	-1
7	4	1	1	1	1	1	3
10	7	1	1	1	1	1	3
5	4	1	1	1	1	1	1
6	7	1	1	1	1	1	-1
6	4	1	1	1	1	1	2
6	6	1	1	1	1	1	0
9	6	1	1	1	2	1	3
7	6	1	1	1	2	1	1
2	3	1	1	1	2	1	-1
7	4	1	1	1	2	1	3
8	8	1	1	1	2	1	0
7	5	1	1	1	2	1	2
7	7	1	1	1	2	1	0
7	6	1	1	1	2	1	1
6	6	1	1	1	2	1	0
6	5	1	1	1	2	1	1
6	2	1	1	1	2	1	4
6	1	1	1	1	2	1	5
8	7	1	1	1	2	1	1
2	4	1	1	1	1	2	-2
6	7	1	1	1	1	2	-1

5	4	1	1	1	2	2	1
4	3	1	1	1	2	2	1
5	6	1	1	2	1	1	-1
2	2	1	1	2	1	1	0
8	6	1	1	2	2	1	2
6	4	1	1	2	2	1	2
6	1	1	1	2	1	2	5
2	4	1	1	2	2	2	-2
1	2	1	1	2	2	2	-1
8	3	1	2	1	1	1	5
2	4	1	2	1	1	1	-2
4	5	1	2	1	1	1	-1
7	5	1	2	1	1	1	2
3	4	1	2	1	1	1	-1
4	5	1	2	1	1	1	-1
4	2	1	2	1	1	1	2
6	5	1	2	1	1	1	1
7	8	1	2	1	1	1	-1
6	8	1	2	1	1	1	-2
4	6	1	2	1	1	1	-2
10	5	1	2	1	1	1	5
6	7	1	2	1	1	1	-1
7	7	1	2	1	1	1	0
8	10	1	2	1	1	1	-2
3	2	1	2	1	1	1	1
6	4	1	2	1	1	1	2
6	6	1	2	1	1	1	0
10	6	1	2	1	1	1	4
5	4	1	2	1	1	1	1
4	6	1	2	1	2	1	-2
3	2	1	2	1	2	1	1
1	6	1	2	1	2	1	-5
8	5	1	2	1	2	1	3
3	5	1	2	1	2	1	-2
4	2	1	2	1	2	1	2
9	4	1	2	1	2	1	5
3	1	1	2	1	2	1	2
8	7	1	2	1	2	1	1
3	3	1	2	1	2	1	0
6	3	1	2	1	1	2	3
1	4	1	2	1	1	2	-3
4	2	1	2	1	1	2	2
4	5	1	2	1	1	2	-1
6	4	1	2	1	2	2	2
4	5	1	2	1	2	2	-1
8	2	1	2	1	2	2	6
4	4	1	2	1	2	2	0
5	3	1	2	1	2	2	2
1	2	1	2	1	2	2	-1

2	7	1	2	1	2	2	-5
6	5	1	2	2	1	1	1
4	3	1	2	2	1	1	1
6	8	1	2	2	1	1	-2
6	5	1	2	2	1	1	1
6	7	1	2	2	1	1	-1
7	4	1	2	2	1	1	3
7	4	1	2	2	2	1	3
1	4	1	2	2	2	1	-3
7	2	1	2	2	2	1	5
2	5	1	2	2	2	1	-3
2	6	1	2	2	2	1	-4
1	3	1	2	2	2	1	-2
3	6	1	2	2	2	1	-3
3	5	1	2	2	2	1	-2
3	4	1	2	2	2	1	-1
1	1	1	2	2	2	1	0
8	6	1	2	2	2	1	2
5	5	1	2	2	2	1	0
9	6	1	2	2	2	1	3
3	5	1	2	2	2	1	-2
2	4	1	2	2	1	2	-2
6	7	1	2	2	1	2	-1
0	3	1	2	2	2	2	-3
0	6	1	2	2	2	2	-6
3	5	1	2	2	2	2	-2
6	1	1	2	2	2	2	5
4	1	1	2	2	2	2	3
4	2	1	2	2	2	2	2
2	2	1	2	2	2	2	0
2	1	1	2	2	2	2	1
6	3	1	2	2	2	2	3
3	2	1	3	2	1	1	1
7	3	1	3	2	1	1	4
5	3	1	3	2	1	1	2
5	5	1	3	2	1	1	0
5	4	1	3	2	1	1	1
7	10	1	3	2	1	1	-3
4	5	1	3	2	1	1	-1
6	5	1	3	2	1	1	1
4	6	1	3	2	1	1	-2
6	2	1	3	2	1	1	4
9	8	1	3	2	1	1	1
5	6	1	3	2	1	1	-1
8	6	1	3	2	2	1	2
6	5	1	3	2	2	1	1
7	3	1	3	2	2	1	4
4	5	1	3	2	2	1	-1
8	5	1	3	2	2	1	3

4	6	1	3	2	2	1	-2
8	6	1	3	2	2	1	2
7	5	1	3	2	2	1	2
5	1	1	3	2	2	1	4
6	3	1	3	2	2	2	3
3	3	1	3	2	2	2	0
5	3	1	3	2	2	2	2
5	3	1	3	2	2	2	2
3	7	2	1	1	1	1	-4
4	3	2	1	1	1	1	1
6	6	2	1	1	1	1	0
6	5	2	1	1	1	1	1
5	9	2	1	1	1	1	-4
4	4	2	1	1	1	1	0
7	5	2	1	1	1	1	2
8	6	2	1	1	1	1	2
6	8	2	1	1	1	1	-2
8	5	2	1	1	1	1	3
5	7	2	1	1	1	1	-2
4	6	2	1	1	1	1	-2
3	5	2	1	1	1	1	-2
1	4	2	1	1	2	1	-3
7	8	2	1	1	2	1	-1
7	5	2	1	1	2	1	2
6	5	2	1	1	2	1	1
6	3	2	1	1	2	1	3
7	6	2	1	1	2	1	1
8	3	2	1	1	2	1	5
9	6	2	1	1	2	1	3
2	6	2	1	1	2	1	-4
6	5	2	1	1	2	1	1
4	3	2	1	1	2	1	1
8	8	2	1	1	2	1	0
1	4	2	1	1	1	2	-3
1	3	2	1	1	1	2	-2
4	6	2	1	1	1	2	-2
4	3	2	1	1	2	2	1
1	4	2	1	1	2	2	-3
4	3	2	1	1	2	2	1
3	5	2	1	1	2	2	-2
5	2	2	1	1	2	2	3
7	6	2	1	2	1	1	1
6	8	2	1	2	1	1	-2
8	4	2	1	2	1	1	4
7	6	2	1	2	2	1	1
7	9	2	1	2	2	1	-2
6	5	2	1	2	2	1	1
5	5	2	1	2	2	1	0
5	8	2	1	2	2	1	-3

6	8	2	1	2	2	1	-2
6	8	2	1	2	2	1	-2
4	7	2	1	2	1	2	-3
3	4	2	2	1	1	1	-1
4	9	2	2	1	1	1	-5
8	8	2	2	1	1	1	0
7	6	2	2	1	1	1	1
3	5	2	2	1	1	1	-2
4	2	2	2	1	1	1	2
3	3	2	2	1	1	1	0
5	5	2	2	1	1	1	0
7	8	2	2	1	1	1	-1
7	4	2	2	1	1	1	3
2	5	2	2	1	1	1	-3
3	3	2	2	1	1	1	0
3	7	2	2	1	2	1	-4
6	4	2	2	1	2	1	2
6	5	2	2	1	2	1	1
5	3	2	2	1	2	1	2
3	7	2	2	1	2	1	-4
3	6	2	2	1	1	2	-3
6	5	2	2	1	1	2	1
2	5	2	2	1	1	2	-3
2	5	2	2	1	1	2	-3
2	2	2	2	1	1	2	0
5	1	2	2	1	1	2	4
1	4	2	2	1	1	2	-3
1	7	2	2	1	1	2	-6
0	3	2	2	1	1	2	-3
4	3	2	2	1	1	2	1
1	8	2	2	1	1	2	-7
1	1	2	2	1	1	2	0
4	5	2	2	1	1	2	-1
3	6	2	2	2	1	1	-3
6	8	2	2	2	1	1	-2
5	8	2	2	2	1	1	-3
5	4	2	2	2	1	1	1
5	6	2	2	2	1	1	-1
7	4	2	2	2	2	1	3
2	2	2	2	2	2	1	0
5	4	2	2	2	2	1	1
7	4	2	2	2	2	1	3
2	6	2	2	2	2	1	-4
4	7	2	2	2	2	1	-3
0	6	2	2	2	1	2	-6
2	2	2	2	2	2	2	0
3	5	2	2	2	2	2	-2
2	4	2	2	2	2	2	-2
9	10	2	3	2	1	1	-1

4	6	2	3	2	1	1	-2
2	2	2	3	2	1	1	0
4	7	2	3	2	1	1	-3
6	8	2	3	2	2	1	-2
2	3	2	3	2	2	1	-1
2	6	2	3	2	2	1	-4
4	4	2	3	2	2	1	0
3	5	2	3	2	2	1	-2
4	5	2	3	2	2	1	-1
3	6	2	3	2	2	1	-3
5	7	2	3	2	2	1	-2
2	6	2	3	2	1	2	-4
4	5	2	3	2	1	2	-1
3	3	2	3	2	1	2	0
3	6	2	3	2	2	2	-3
5	3	2	3	2	2	2	2
4	3	2	3	2	2	2	1
2	6	2	3	2	2	2	-4
3	3	2	3	2	2	2	0

Appendix F

Publications by the Author

Bowles, R., Calay, P., Flores, R., Formosa, J., Goda, D., Malabar, I., Norrington, P., O'Rourke, F., Scott, P., Simpson, A., Street, D. (2002). "Deep and Surface Learning in Undergraduate Mathematics", *Proceedings of the 28th Annual Undergraduate Mathematics Teaching Conference (UMTC)*, University of Birmingham, September, pp 58-68.

Malabar, I., Pountney, D.C. (2002). "Using Technology to Integrate Constructivism and Visualisation in Mathematics Education", *Proceedings of the 2nd International Conference on the Teaching of Mathematics*, University of Crete, Greece, July, CD-ROM.

Malabar, I., Pountney, D.C. (2000). "How do Traditional Examination Questions Fare in the Presence of a Computer Algebra System (CAS)?", *International Journal of Computer Algebra in Mathematics Education*, Vol. 7, No.4, pp. 241-254.

Malabar, I., Pountney, D.C., Townend, M.S. (1998). "Combining Visual and Symbolic Skills in the Teaching and Learning of Mathematics", *Proceedings of the Third International DERIVE and TI-92 Conference*, Gettysburg, Pennsylvania, USA, July, CD-ROM.

Malabar, I., Pountney, D.C. (1998). "Assessing Visualisation Skills in Mathematics Education", *Proceedings of the International Conference on the Teaching of Mathematics*, University of the Aegean, Samos, Greece, July, pp 203-205.

Malabar, I., Pountney, D.C. (1997). "Interactions Between Visualisation and Symbolic Senses", *Proceedings of the Day Conference of the British Society for Research into Learning Mathematics (BSRLM)*, Department of Educational Studies, University of Oxford, June, pp 75-80, and the *MathSkills Newsletter*, February, 1998, pp 9-11.