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A MAD DICTATOR *PARTITIONS* HIS COUNTRY

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This paper has two integrated parts. In the first part, a task is introduced around the concept of equivalence relation. The initial aim of the task is to lead undergraduate students to define an equivalence relation as traditionally defined. The task is tried out on a small sample of students. Based on students' experience of the situation I reveal a change in my perspective in which I learned to consider defining in the realm of organising. The results of this change are reported in the second part of this paper in which I report a certain way of organising the given situation, a 'new' definition of equivalence relations, and consequently a new representation for them, that seems to be overlooked in the literature.

INTRODUCTION

This paper is more than anything else a story of a change in my personal perspective. In the early stages of my doctoral study, when following scholars such as Freudenthal (1973), I was interested in *defining* as a mathematical activity. I devoted myself to devising situations, which on the one hand, could bring familiarity with certain new concepts, and on the other hand, could prompt students to define newly emerged concepts. My attempts in this direction are reported in the first part of this paper.

The Mad Dictator Task (see below) is one of the tasks designed in the early stages of my study. It is based on the notion of *equivalence relation*; when devising the task, *I* had the standard formulation of equivalence relations in mind. The task was originally designed with the intention of leading students to define *symmetry* and *transitivity* which, together with *reflexivity,* constitute *the* defining properties of an equivalence relation. However, while my focus was on defining of certain predetermined concepts, the students' experience of the situation taught me something more overarching, i.e. *organizing*. This change is described at the end of the first part of this paper and the start of the second part, in which organizing plays a central role. The second part is set in a frame based on my *new* reading of Freudenthal, particularly some of his ideas about organizing activities. In the following, I report my journey as I experienced it. Moreover, based on the way that one of the students in the study organizes the given situation, a *new* definition of equivalence relations is introduced.

The Mad Dictator Task

Consider the following task, which concerns a "visiting law" as defined below.

> A country has ten cities. A mad dictator of the country has decided that he wants to introduce a strict law about visiting other people. He calls this 'the visiting law'.

A visiting-city of the city, which you are in, is: A city where you are allowed to visit other people.

A visiting law must obey two conditions to satisfy the mad dictator:

1. When you are in a particular city, you are allowed to visit other people in that city.

2. For each pair of cities, either their visiting-cities are identical or they mustn't have any visiting-cities in common.

The dictator asks different officials to come up with valid visiting laws, which obey both these rules. In order to allow the dictator to compare the different laws, the officials are asked to represent their laws on a grid as figure 1.

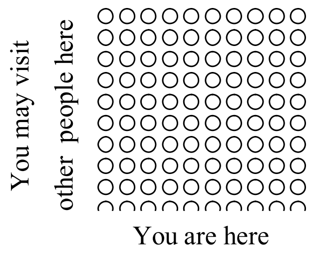


Figure 1: a grid to represent a visiting law

The initial aim of the Mad Dictator Task was to lead 'lay' undergraduate students (i.e. those unfamiliar with the mathematical definition of equivalence relations) to define an equivalence relation in the traditional way, in terms of the three properties of *reflexivity*, *symmetry* and *transitivity*. The *success* of the tasks was determined by the extent to which lay students noticed and defined these predetermined properties. The tasks and the results of this phase of the study are discussed below.

The underlying structure of The Mad Dictator Task

The Mad Dictator Task is based on the notion of equivalence relation. According to the standard mathematical account, an equivalence relation is a relation ~ on a set *S* that has three properties: *reflexivity* ($a \sim a$ for all a in S), *symmetry* (if $a \sim b$ then $b \sim a$), *transitivity* (if $a \sim b$ and $b \sim c$ then $a \sim c$). Suppose ~ is an equivalence relation on S and a is an arbitrary element of S, then the set of all elements of S that are related to a is called the *equivalence class* of a. It follows from reflexivity, symmetry and transitivity that for each pair of elements of S, say a and b, either the equivalence class of a is *equal* to the equivalence class of b or the intersection of the equivalence class of a and the equivalence class of b is the empty set. Given this, and also considering that every element, say a, belongs to an equivalence class, namely the equivalence class of a, it can be seen that an equivalence relation on a set S *partitions* the set into equivalence classes, i.e. the set is divided into mutually exclusive classes. Conversely, suppose there is a set of mutually exclusive subsets of S, such that each element of S belongs to one of them, then the following relation is an equivalence relation:

$$a \sim b \text{ if and only if a and b belong to the same subset.}$$

In the context of the Mad Dictator Task, let a~b if b is a visiting-city of a. Then ~ would be an equivalence relation, providing that we have satisfied the second conditions of a visiting law:

> For each pair of cities, either their visiting-cities are identical or they mustn't have any visiting-cities in common.

It is worth stressing that the first condition of a visiting law guarantees that each city has at least one visiting-city, i.e. that city itself.

The task was originally designed with the intention of leading students to the ideas of symmetry and transitivity through creating their own examples as demanded in the first task. They would then be asked to provide *the minimum amount of information* through the following additional task:

> The minimum amount of information task: The mad dictator decides that the officials are using too much ink in drawing up these laws. He decrees that, on each grid, the officials must give the least amount of information possible so that the dictator (who is an intelligent person and who knows the two rules) could deduce the whole of the official's visiting law. Looking at each of the examples you have created, what is the least amount of information you need to give to enable the dictator deduce the whole of your visiting law.

Before discussing how students' experience of these tasks changed my point of view, I shall explain some of their subtleties.

SUBTLETIES IN THE MAD DICTATOR TASK

An equivalence relation is first and foremost a relation. Thus let me start from relations in general. Set theoretic treatment of relations gives both a *unit* and *plural* character to those elements that relate to each other. This aspect can be implicitly seen in the introductory paragraph of the chapter on relations in Stewart and Tall (2000, p.62):

> The notion of a relation is one that is found throughout mathematics and applies in many situations outside the subject as well. Examples involving numbers include 'greater than', 'less than', 'divides', 'is not equal to', examples from the realms of set theory include 'is a subset of', 'belongs to'; examples from other areas include 'is the brother of', 'is the son of'. What all these have in common is that they refer to two things and the first is either related to the second in the manner described, or not.

Each one of the 'two things' in Stewart and Tall's examples implicitly belongs to a set; therefore, even though, for example, 1 in $2 > 1$ is treated as an individual, being in the set of integers gives an infinite access to it and illuminates its plurality. In general, those 'two things' are not only single individuals, but also something that can fill one of the two sides of a relationship, or more importantly fill both sides of a relationship; they are simultaneously unit and plural.

As a particular relation, an equivalence relation inherits the above peculiarities in a more remarkable way. When we are looking for a concrete example of an equivalence relation, we are apt to define a relation between *two* different things or people, saying, for example, *both* have the same colour, *both* live in the same street. We can check the possession of the given relationship between those two things or people by pointing to them. We can even do this at a more concrete level, what Dienes' (1976, p.9) refers to as 'first order attributes' For example, they are both green, or they both live in Oxford street. However, as Dienes pointed out, the former way of checking, described as 'second order attributes', is more abstract and more difficult than the latter:

> To have the same colour as something else is a much more sophisticated judgement than to say that they are both green. (ibid, p.9)

I should add that considering 'second order attributes' seems inextricable from grasping the reflexive property. To grasp the reflexive property, first we must go one step further, and look at the situation as '…having the same colour as…', '…living in the same street as…', and so on. This step demands, on the one hand, a transfer from unity to plurality in the sense described for relations in general, and on the other hand, a transfer from plurality to unity, i.e. coming from *both* to *each.*

In sum, although bringing plurality and unity together is hardly accessible in the concrete cases, I tried to achieve it, in the designed situation, by giving a definition in which a city is used to refer to the people in the city. That is, "each city is its own visiting-city" (metonymically) stands for "in each city you can visit other people". The former is an expression of the reflexive property that is visible as the main diagonal of the grid.

I now turn to the earlier data where more details of the tasks are revealed in the context of the students' work.

PRELIMINARY STUDY

The preliminary study started with a small opportunistic sample of students comprising two first year undergraduate mathematics students, and two second year undergraduate physics students. All were students in one of the top five ranked universities in the UK. The two mathematics students have been taught equivalence relations a few weeks before the interview in a course entitled 'Foundations' held at the University. The two physics students had no previous *formal* idea about the subject. In the interview with the mathematics students, I aimed at seeing the situation through the eyes of *informed* persons. I planned to revise the tasks (if necessary) for use with lay students like the two physics students.

The students took part in interviews in which they worked on the Mad Dictator Task. The students were invited to be interviewed in pairs, two mathematics students together, and two physics students together. Each interview took about one hour. In the course of the interview, I invited them to think and talk aloud so that I could audiotape their utterances. They were also encouraged to write down their ideas. The interviews had a simple structure; the two tasks (generating an example of a visiting law, and giving the minimum amount of information) were posed in order. As soon as one of the students had made one or two examples agreeable to all (the interviewer and interviewees), we started the second task. I participated in the interview like a teacher, with the aim of teaching certain predetermined concepts. As interviewer and as analyst, I was looking for these concepts in students' utterances.

The interview with the physics students illuminates the nature of the initial interviews. Andy is one of the two physics students that made his own examples, noticed the symmetry of each figure, and gave a reason for that symmetry. He suggested the following information to convey his examples to another student:

Andy: I am going to tell you some groups, each group visits all the other ones in the group and hence it is visited by all the other ones in the group, it visits them and it is visited by them.

Soon afterwards, I helped him to *separate* the given conditions from the symmetry (in the following, we agreed to take the first condition for granted):

Andy: the second one implies the symmetry.

Interviewer: the second one implies the symmetry or the symmetry property implies the second one.

Andy: they imply each other.

Interviewer: But you had an example here (pointing to a non-example symmetrical figure).

Andy: This is symmetrical, oh no, it isn't, and this one doesn't work.

And then, looking for the transitivity, I asked the following direct question:

Interviewer: What condition must you add to have the second one?

While scrutinizing the present examples:

Andy: *I am not sure what sort of answer you are looking for*.

And shortly after that:

Andy: I can't think of any way to say the second law better than it is already said.

I was surprised by Andy's responses. More surprises were to come when I compared the results of the first two interviews.

Hagh and Shah, the two mathematics students, came to talk about symmetry when they were looking for the minimum amount of information:

Hagh: If you just give me a symmetrical half, you can give me half, then I can do it by symmetry.

Hagh then employed this newly noticed property to reject one of Shah's generated figures as an example:

Hagh: I am saying picture has to be symmetric, now what he's got there, it is not symmetric, it's not valid, it doesn't obey the rules, it's not an example.

And soon after, based on the same property, they accepted a symmetric non-example figure as an example. They also gave "half of the information" (the symmetric half) as the minimum amount of information, while considering other properties:

Hagh: I ignore the diagonal; I'll give half of the information.

Interviewer: Does it always work?

Hagh: I guess if you had like a rotating group, if you like, two can go to five, five can go to seven, then,

Shah: two has to go to five,

Hagh: seven has to go to five, and …

While in the course of these events there was no sign of relating the symmetry idea and grouping idea to each other or of separating them from each other. Suddenly, however, they realized the equivalence relation in the situation:

Shah: Basically it's equivalence relation; this is reflexive, symmetrical and transitive.

Not even the successful deduction of these three properties from the two given conditions was of help to them to *separate* the different concepts encountered:

Interviewer: what about symmetry, if one is symmetric, it has necessarily both properties or not.

Shah: if it has symmetry, yes it is.

Hagh: that means if you come up with a symmetric picture it must be an example, symmetry is equal to that two happen (the two given conditions).

In comparison, Andy, one of the two physics students, related the symmetry idea to the other ideas when only two cities were involved:

Andy: because it's gonna be each city that you visit can visit you, it's gonna be symmetrical about that line, that's what implied by this, so it's gonna be *symmetrical* down there, um, three and six are *grouped*

Even so, when more than two cities were involved, still those two ideas appeared to be isolated:

Andy: Each in group visits all the other ones in the group, because one is the same group as five, so then it visits one, five and seven, five is in the same group as one and seven, and five so visit one, five and seven

While in the last excerpt there is no sign of symmetry, in the following there is only a loose reference to symmetry:

Andy: if you go along the columns one by one, and you see, you know symmetry, you can see that two visit nine and ten, you know that they must visit each other, so you have to check that ten visits nine...

In general, he referred to many different ideas without maintaining those ideas from one time to the other, and without necessarily relating those ideas to each other; as the mathematics students did so.

These two interviews radically affected my criterion for success and failure of the situation.

Discussion: success or failure

When I started the preliminary study, my criterion for success was whether students engaging in the situation could spot certain predetermined concepts; in particular, whether they could spot symmetry and transitivity or not. Having that criterion in mind let me scrutinize the last two interviews in terms of success or failure. Both groups of students successfully noticed symmetry, while only the mathematics students who had already been taught equivalence relations noticed transitivity. Therefore, as far as symmetry is concerned, the situation could be taken as a success, and regarding transitivity, as a failure, particularly, since the study would eventually target students who have not been taught the subject.

On the other hand, the students brought to my attention something I had previously ignored, something that was not then a yardstick to assess the success of the situation, but turned out to be as effective as symmetry and transitivity when tackling the requirements of the situation, i.e. the idea of 'grouping'. To be precise, despite the fact that I was aware of the idea of 'grouping' (or in a certain sense, 'partitioning') as a logically closely related concept to equivalence relations, the presence of symmetry and transitivity prevented me from realizing the extent to which these other concepts were involved. Furthermore, beyond these individual concepts, this first study revealed something that certainly played a crucial role in preparing me for relinquishing my criterion for success, i.e. the ways that those individual concepts could be related to each other. While for me these individual concepts had certain relations informed by the formal treatment of them, for students, they were mainly related to each other by their functionality within the situation.

Altogether, considering the data it seemed that my original view of success, which was mainly based on tracing certain predetermined concepts in students' utterances, was very narrow. Nonetheless, there was still one possibility to keep that view by modifying the tasks and/or adding certain new tasks in such a way that they would facilitate (1) the students' grasp of the missing concept of 'transitivity', and (2) the predetermined ways of connecting all the concepts involved in the situation. Furthermore, any measures of success should take account of two criteria: the extent to which those predetermined concepts would be brought up in each interview, and the extent to which their logical relations would be matched with the standard ones. To put it more simply: the more standard the outcome, the more successful the situation. On the other hand, by narrowing the situation in order to make success more likely, the more restricted, involved and artificial it would be. Nonetheless, there would still be no guarantee that it would achieve what it had been designed to achieve. These realizations led to the second part of the study in which constructing a definition is regarded as the last part of an *organizing* activity.

DEFINING IN THE REALM OF ORGANIZING

The realisations I have described above led to a change in my reading of the literature. In this section, I discuss my reading of Freudenthal.

Freudenthal's works has certainly had a great effect on the literature on defining. However, he himself favoured defining only as the finishing touch of an organizing activity:

> Most often definitions are not preconceived but the finishing touch of the organizing activity… In the course of these activities the student learns to define, and he experiences that defining is more than describing, that it is a means of the deductive organization of the properties of an object. (Freudenthal, 1973, p. 417)

Looking at the initial data, the idea of organizing that was then in the background of my reading of Freudenthal came to the foreground. I learnt to see the situation that the students were engaged in as a situation that "begs to be organized" (Freudenthal, 1983, p.32). I should add that I have only partly adopted Freudenthal's plan; I became interested in the ways that students organize the given situation, rather than "teaching them to manipulate any particular means of organizing".

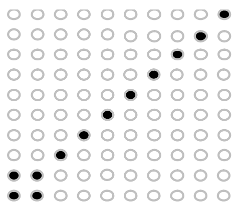
The change in my perspective naturally led to the following methodological change: when interviewing and analysing I started to suspend my understanding of equivalence relations and related concepts as the yardstick for success and failure. In other words, I started to see the situation through students' eyes. These changes are put into practice in the pilot study.

PILOT STUDY

To give a flavour of the pilot study, let me present a snapshot of the data coming from the interview with Tyler who was an undergraduate computer science student. This interview took about one hour. The structure of this interview remained more or less as it was in the preliminary study. However, I was trying to keep an open mind when interviewing and when analysing the data. The verbatim transcribed tape and Tyler' written works were treated as data.

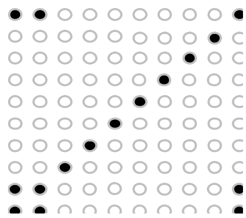
To satisfy the first condition of the given situation, Tyler marks the diagonal and continued as follows:

Tyler: If I am in city one, and we allow to visit city two, how the other things need to change, to keep the rules consistent and see either they are completely the same or completely different, so aha, so city two now have to be able to visit city one…



He then considers two things: "mirroring in y equals x" and "box" (square) and then "to see what was happening" he decides to make city one visit city ten:

Tyler: … and I realised first that, city ten has to visit city one… so that the second law …city ten has to visit city two… now I look at the city two, now I realised they are different from city one…so I copy number one on to number two also just to keep them the same…



As a result, Tyler abandons the "block square", keeps the "mirroring" and *proves* it as a "general pattern of these dots" (if (x, y) then (y, x)). In addition, the way that he proves "mirroring", gives him a new insight, i.e. considering the relationship between any two individual cities:

Tyler: If you allow a city to visit any other city, then it's gonna end up with having the same visiting-rules as that city that's allowed to visit and vice versa...

Having considered several different ideas, he transcends the situation by introducing a new concept with general applicability (the 'box concept'):

Tyler: How do I say that columns must be the same mathematically? (He writes)

If (x_1, y_1) and (x_1, y_2) and (x_2, y_1) then (x_2, y_2)

Interviewer: Could you explain?

Tyler: I think it's a mathematical way of saying …if a column has two dots, and there is another column with a dot in the same row, then that column must also have the second dot in the same row…I take maybe *a box of four dots*…I use the coordinate because that makes it very general, and so if I made that my second law, for a mathematician might be easier to follow.

Has Tyler explicitly generated a new definition? It depends on what we decide to count as an act of defining. For the moment, it is safer to say Tyler has explicitly generated a new concept (and, for me, an unexpected one) in order to *locally organize* this situation. Interestingly, using this new concept (hereafter, the box concept) we can offer a new definition for equivalence relations (see below).

The box concept as a *local* concept

Tyler has explicitly generated the box concept in order to *locally organize* the situation. This idea is inspired by both Tyler, who used the box concept for the first time, and Freudenthal, who developed the ideas of *global organization* and *local organization*. In this section, I will discuss the box concept in the light of the latter ideas.

In the preface of "Mathematics as an educational task", Freudenthal (1973) says how as a mathematician he found it hard to rearrange his old ideas about teaching for his book. He explains that:

> The problem was not the dialectic instead of the deductive style, and the local organization of the subject matter was not a problem either. But the *global organization* was the sore point. I could not use the formal organization of a mathematics course or treatise where the author says, or writes things like "because of theorem… (cp. p. …), applied under the condition of corollary… (p. …), it appears that the definitions of …on

p. …and on p. …are equivalent.” I could not use this method nor could I invent another form of organization. Thus the present book is, from the view point of a mathematician, badly organized. (ibid, p. IX)

The tension between global and local organization starts right from the beginning of the book and goes through to the end. Examples of this tension and of local and global organizations are abundant throughout the book. However, if we look for a *definition* that covers all these examples, we find only certain sparse attempts to give a definition. Even these rare attempts are somehow anchored to the examples or the context that they originated from. One of these attempts is the following, made when Freudenthal discusses the case of (teaching) geometry.

> [The student] learns the global organization, that is organizing not a system internally, but a category of systems by looking from outside- he learns to axiomatize. (ibid, p. 454)

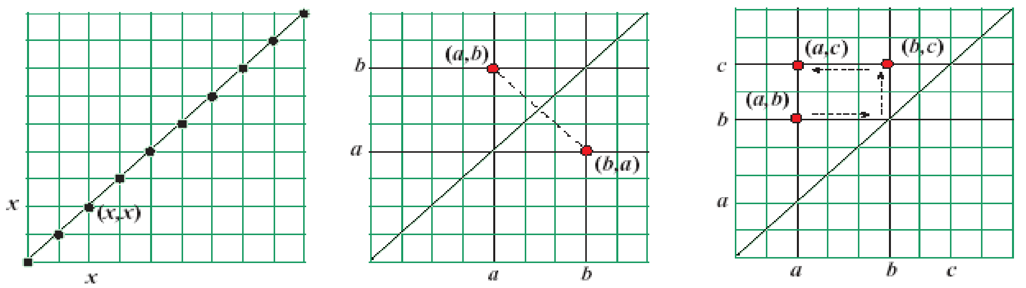
But globally organizing is not axiomatizing. Axiomatization is only an example, an extreme case of a global organization.

Though Freudenthal does not say explicitly what he means by local and global organization, he is explicit about “mathematics as an activity” in which organizing in its different forms plays a vital role. He also explicitly warns us not to underestimate the importance of organizing locally; for “in general, what we do if we create and if we apply mathematics is an activity of local organization” (ibid, p. 461).

In the light of this distinction it is now possible to add the box concept to Freudenthal’s examples. The box concept only gives us a local organization while the standard account of equivalence relations provides us with a global one in which two important types of relations (equivalence relations and order relations) can be seen as particular types of transitive relations. However, it seems that the box concept has one or two interesting consequences, as I will now discuss.

Equivalence relations revisited

To see the importance of the box concept, I have to resort to an even more formal treatment of equivalence relations. To give this formal and *partially pictorial* formulation, let me refer to a research paper that interestingly suggests a “visual representation” for *all* the standard properties of an equivalence relation. According to Chin and Tall (2001, p.245), these visual representations are as follows:

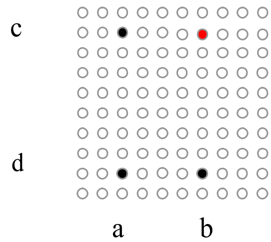


reflexive: $\forall x \in S, (x,x) \in R$ | symmetric: $(a,b) \in R \Rightarrow (b,a) \in R$ | transitive: $(a,b) \in R, (b,c) \in R \Rightarrow (a,c) \in R$

The first two (reflexivity and symmetry) have an easy visual representation than can also be found in the textbooks. But the visual representation of the transitive law is not given in the textbooks. Following the picture it reads:

> The transitive law (a, b), (b, c) ∈ R implies (a, c) ∈ R is a little more sophisticated. (The transitive law moves horizontally from (a, b)—maintaining the second coordinate b—to the diagonal then vertically to the point (b, c), completing the rectangle to give the third point (a, c).) (ibid, p.245)

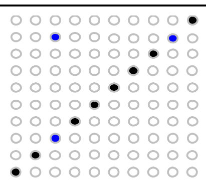
Compare this complexity with the simplicity of the visual representation of the box concept.



If (a, c) and (a, d) and (b, d) then (b, c)

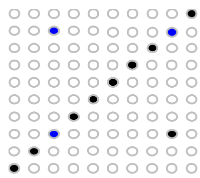
(Box concept)

We only need a few pictures to see how, having reflexivity and the box concept, we can deduce symmetry:



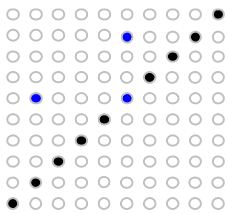
(a, b), (a, a), (b, b)

are three corners of the box



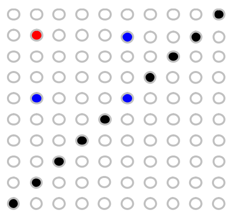
(b, a) is the fourth

and transitivity:



(a, b), (b, b), **(b, c)**

are three corners of the box



(a, c) is the fourth

It can also be seen that having the normative definition of equivalence relations, based on reflexivity, symmetry and transitivity, we can deduce the box concept. Hence, the standard definition of equivalence relations and the definition based on the box concept are *logically equivalent*, but they have two dramatically *different* *representations*. More importantly, having a simpler alternative representation of an equivalence relation calls into question one of Chin and Tall's hypotheses.

Regarding the visual representation of the standard account, Chin and Tall (ibid, p. 245) hypothesize that "the complexity of the visual representation" is a source of a "complete dichotomy between the notion of relation (interpreted as a subset of S × S) represented by pictures and the notion of the equivalence relation which is not". Accordingly, they suspected that that dichotomy inhibits students from grasping the notion of relation encompassing the notion of equivalence relation. However, the above figures show that the alleged source of the dichotomy, to a large extent depends on the standard way of defining equivalence relations. In other words, if we define an equivalence relation as a relation having the reflexive property and the box property, the source of the dichotomy would disappear, although the dichotomy itself remains. It is also worth stressing that it does not seem that using the box concept as one of the defining properties of equivalence relations could result in a better understanding of the subject, particularly because it does not look like there would be any mathematical relation for which the box concept is a convenient encapsulation.

CONCLUSION AND AFTERWORD

In this paper, I have reported two early phases of my doctoral study. As mentioned above, the aim of the first phase was to lead lay students to define certain predetermined concepts, i.e. symmetry and transitivity. The Mad Dictator Task was initially designed as a "problem situation" serving this aim. However, the students' experience of the situation brought some of my unsuspected assumptions to the fore. That is, leading students to define certain predetermined concepts is tantamount to taking it for granted that (1) there are certain fixed concepts that the students are trying to negotiate, and (2) there is a fixed way of relating these concepts to each other. The preliminary study showed, however, that the Mad Dictator Task, despite being designed around an intended concept, fails to guarantee either of these taken-for-granted aspects. This realization led to the second phase of the study (the pilot study) in which constructing a definition is regarded as the last part of an organizing activity.

Organizing was the main theme of the pilot study in which the aim was to investigate the ways that students *organize* the given situation. On reflection, making use of the idea of organizing is a kind of reaction against the thwarted expectations of the first phase of the study in which it was expected that students would see in the situation what I saw in it. As a result, the *different* ideas that students use to organize the situation were given more weight than the ways that they relate these ideas to each other. Thus, some of the ideas that were treated as the means of organizing may remain quite unorganized. For example, it was claimed that Tyler generated the box concept in order to locally organize the situation, but it was not discussed how he himself related this concept to the other concepts that he *experienced* in the situation. In sum, in this phase the idea of organizing was used equivocally to say the least.

These realizations lead to the third phase of the study (see, for example, Asghari and Tall, 2005), in which the idea of organizing is replace by the more neutral idea of *experiencing*, and subsequently the intention of the study is defined anew.

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